Lecture 08: Laplace's Method and the Mean Field Ising Model

In Lecture Notes 05, we introduced the mean-field Ising Model and determined the thermal equilibrium properties of the system by analyzing the system's macrostate. In these notes, we study the same model in terms of *microstates* by computing the partition function. Along the way we derive a useful method (called Laplace's method) for computing exponential integrals

1 A special kind of integral

One of the first physical models we introduced in the class was the mean-field Ising model in which each of the N spins in a lattice interacted with every other spin. Given the methods we had developed at the time, we had to study the model in terms of the macrostate variable average-spin: $m = \sum_{i=1}^{N} s_i/N$. The utility of this macrostate analysis was that we were readily able to obtain the phase behavior of the thermal system by minimizing the Helmholtz free energy. In particular, we found that at thermal equilibrium temperature T, the average spin m had the value $\overline{m}$ which satisfied

$$\overline{m} = \tanh\left(\frac{J\,\overline{m}}{k_B T}\right), \tag{1}$$

where J defined the interaction between spins. Solving Eq.(1) for various parameter choices, we were able to define two phases for this system, one on either side of the critical temperature $T_c = J/k_B$. Now that we have developed the partition function and have more rigorously developed the methods of equilibrium statistical physics, we can analyze the mean-field Ising model in terms of microstates, that is, by applying the Boltzmann distribution. Such an analysis will be more mathematically elegant than that used in the macrostate analysis, but it will lead us to a partition function which seems intractable. For our system with N lattice sites, the partition function of the mean-field Ising model has the form

$$Z_N \propto \int_{-\infty}^{\infty} dx\, e^{-Nf(x)}, \tag{2}$$

where "$\propto$" is the symbol for "proportional to", and $f(x)$ is *not* a quadratic function of x. Were $f(x)$ to be a quadratic function, we could just use the standard formula for Gaussian integrals to evaluate Eq.(2). But, it turns out, finding an approximation for the more general case of arbitrary $f(x)$ is not too far off from analyzing a basic gaussian integral.

In these notes we will develop a method to compute integrals like Eq.(2). Such a method is applicable beyond the context of this problem and we will use it later to analyze a model of single-stranded DNA to double-stranded DNA dimerization. Outside of our specific uses, this method (under various aliases) is constantly applied in quantum field theory and condensed matter theory.

Our framing question is as follows:

Framing Questions

How do we analyze the mean-field Ising model using the Boltzmann distribution and the partition function? And–when we get to it–how do we compute partition functions of the form Eq.(2)?

2 Mean-field Ising Model and its Partition Function

We make our way towards the final form of the partition function of the mean-field Ising model by first writing the partition function in terms of the Boltzmann distribution and then using integration identities to allow us to explicitly compute the summation.

Recalling the starting points of the mean-field Ising model, we have a lattice of N spins labeled $s_1, s_2, \ldots, s_N$, and each spin can take on the value $+1$ or -1. Each spin interacts with every other spin, and between two spins, the interaction energy is proportional to $J/2N$. The resulting total energy for a particular microstate $\{s_i\}$ is given by

$$E(\{s_i\}) = -\frac{J}{2N} \sum_{i,j=1}^{N} s_i s_j. \tag{3}$$

We want to use the partition function to define the equilibrium thermodynamics of the spin system with the energy Eq.(3). To compute the partition function we need to define our summation over states, in addition to defining the energy in Eq.(3) and the microstates. We note that each spin is independent of every other spin. We can therefore sum over all microstates of the system by summing each spin over its two possible values. The partition function is then

$$\begin{aligned} Z_N(\beta J) &= \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \exp\left(-\beta E(\{s_i\})\right) \\ &= \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \exp\left(\frac{\beta J}{2N} \sum_{i,j=1}^{N} s_i s_j\right) \end{aligned} \tag{4}$$

In our previous calculations of the partition function for a spin system, we were able to factor the net Boltzmann factor for an arbitrary microstate into a product of Boltzmann factors for each lattice site. Such a product rendered the partition function soluble. However, no such factoring is possible with the partition function in Eq.(4). Instead, to move the calculation forward, we need to express the partition function in a new form. First, we note that a change of variables in a Gaussian integral yields the identity

$$e^{b^2/4a} = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} dx\, e^{-ax^2+bx}. \tag{5}$$

This identity is true for any real value of b and any positive real value of a. Therefore, we can use the identity to express the partition function of the mean-field Ising model as an integral. We begin by focusing on the exponential in Eq.(4). Using Eq.(5), we obtain

$$\exp\left(\frac{\beta J}{2N} \sum_{i,j=1}^{N} s_i s_j\right) = \exp\left(\frac{1}{4}\frac{2(\beta J)^2}{N\beta J} \sum_{i=1}^{N} s_i \sum_{j=1}^{N} s_j\right) = \exp\left(\frac{1}{4}\frac{2}{N\beta J}\left(\beta J \sum_{i=1}^{N} s_i\right)^2\right). \tag{6}$$

Identifying

$$b = \beta J \sum_{i=1}^{N} s_i, \quad \text{and} \quad a = \frac{N\beta J}{2}, \tag{7}$$

and using Eq.(5), we find

$$\exp\left(\frac{\beta J}{2N} \sum_{i,j=1}^{N} s_i s_j\right) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{N\beta J}{2}\, x^2 + x\, \beta J \sum_{i=1}^{N} s_i\right). \tag{8}$$

The choices for a and b in Eq.(7) might seem arbitrary, and to some extent they are arbitrary; physical results are independent of how exactly we parameterize our integral. However, the choices in Eq.(7) lead to a final result for the order parameter of this system which most closely resembles the result of the macrostate analysis.

Since the x integration in Eq.(5) is independent of the summations over s_k, we can move the integral outside the summation. Applying product identities for the exponential, we then find

$$\begin{aligned} Z_N(\beta J) &= \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{N\beta J}{2}\, x^2 + x\,\beta J \sum_{i=1}^{N} s_i\right) \\ &= \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-N\beta J x^2/2} \prod_{j=1}^{N} \sum_{s_j=\pm 1} e^{x\beta J s_j} \\ &= \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-N\beta J x^2/2}\, 2^N \cosh^N(\beta J x), \end{aligned} \tag{9}$$

where we used $\cosh(x) = (e^x + e^{-x})/2$ in the second line. With the identity $A = e^{\ln A}$, we then have

$$Z_N(\beta J) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-N f(x,\beta J)}, \tag{10}$$

where we defined

$$f(x, \beta J) = \frac{\beta J}{2} x^2 - \ln\left[2\cosh(\beta J x)\right]. \tag{11}$$

Eq.(10) may appear to be a fine form for the partition function, but it is not yet in a form that makes manifest the phase behavior of the system or how to compute the order parameter. For example, given our previous partition function, we might want to compute

$$\langle s \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} \langle s_i \rangle, \tag{12}$$

representing the microstate-averaged spin of the system. By determining the temperature dependence of $\langle s \rangle$ we will be able to determine whether the system exhibits a phase transition. But of course we have already answered this question: We previously used an analysis grounded in *macrostates* to show that the system indeed undergoes phase transitions, so our partition function based calculation of $\langle s \rangle$ should simply affirm this result.

With our original partition function Eq.(4), Eq.(12) can be found from

$$\langle s \rangle = \frac{1}{Z_N(\beta J)} \frac{1}{N} \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \sum_{i=1}^{N} s_i \exp\left(\frac{\beta J}{2N} \sum_{i,j=1}^{N} s_i s_j\right), \tag{13}$$

however this form does not admit an explicit equation which can be solved to yield the temperature dependence of $\langle s \rangle$. Instead, in order to calculate $\langle s \rangle$, we will use Eq.(10) as a starting point, and doing so requires us to find a way to reduce the associated integral to something more analytically tractable. We turn to this task now.

3 Laplace's Method and Evaluating Exponential Integrals

Our objective is to find a way to evaluate the integral in Eq.(10) (if only approximately) and other integrals similar to it. The first such integral we have encountered in this course was the standard Gaussian integral.

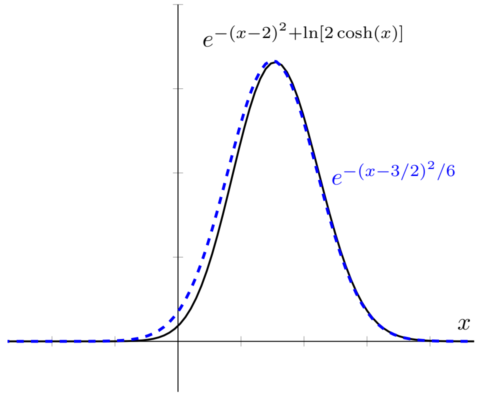

Figure 1: Approximation as a Gaussian. The Gaussian function $e^{-(x-3/2)^2/6}$ is plotted in the blue dashed line and the exponential function $e^{-(x-2)^2+\ln[2\cosh(x)]}$ is plotted in the solid black line. We see that the exponential function can be roughly approximated as a Gaussian function.

In Assignment # 2, we were able to show that

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}. \tag{14}$$

More generally, a change of integration variables, leads us to the result

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{b^2/4a+c}, \tag{15}$$

which is valid for $a \geq 0$ and any real numbers b and c. Now let's consider the integral of the kind that appears in Eq.(10). We define

$$I_N = \int_{-\infty}^{\infty} dx\, e^{-Nf(x)}, \tag{16}$$

for some positive N. Integrals of the form Eq.(16) cannot in general be computed exactly unless they are of the form Eq.(15). However, it turns out that we can use Eq.(15) to compute an approximation of Eq.(16) (presuming the latter is a finite integral).

We will assume that I_N is a finite integral, meaning that it evaluates to a finite number. Then we can make certain assumptions about the integrand. Since the exponential function e^x diverges to infinity as x increases, the only way integrating $e^{-Nf(x)}$ from $x = -\infty$ to $x = +\infty$ could yield a finite result is if $e^{-Nf(x)}$ went to zero as x went to $\pm\infty$. However, because the exponential function is always positive, we should also find that $e^{-Nf(x)}$ is non-zero for some finite portions of its domain. Schematically, then, we could expect $e^{-Nf(x)}$ to have a plot that looks like a hill which flattens as x varies away from the top of the hill.

The key thing to note about this plot is that it *looks like* a Gaussian function (See Fig. 1 for an $N = 1$ example). The essence of our approximation of Eq.(16) uses this fact to consider only terms up to quadratic order in the argument of the exponential, and to thus approximate $e^{-Nf(x)}$ as the Gaussian it appears to be. Namely, let's say $e^{-Nf(x)}$ is highly peaked at some $x = x_1$. If $e^{-Nf(x)}$ has a local maximum at $x = x_1$, then $f(x)$ must have a local *minimum* at $x = x_1$. That is, at x_1, the first derivative of $f(x)$ should be zero, and the second derivative should be positive.

> **Local maximum of $e^{-Nf(x)}$:** If $e^{-Nf(x)}$ has a local maximum at $x = x_1$, then $f(x)$ has a zero first derivative and a positive second derivative at $x = x_1$. That is, $f(x)$ has a local minimum at $x = x_1$.

Given that $e^{-Nf(x)}$ is highly peaked around $x = x_1$, we can assume that the values of x near x_1 are the ones most relevant to the integral. We can capture these values of the integrand $e^{-Nf(x)}$ by expanding $f(x)$ about its local minimum at $x = x_1$. Doing so, we have

$$\begin{aligned} f(x) &= f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2}(x - x_1)^2 f''(x_1) + \mathcal{O}\left((x - x_1)^3\right) \\ &\simeq f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2}(x - x_1)^2 f''(x_1). \end{aligned} \tag{17}$$

In the final line we dropped the terms of order $(x - x_1)^3$ and higher because such terms make sub-leading contributions to the final evaluation of the integral. Now, we recall that since $e^{-Nf(x)}$ has a local maximum at $x = x_1$, then $f(x)$ has a local minimum at the same value of x. Thus, we have $f'(x_1) = 0$, and our approximation of $f(x)$ near $x = x_1$ becomes a quadratic function of x:

$$f(x) \simeq f(x_1) + \frac{1}{2}(x - x_1)^2 f''(x_1). \tag{18}$$

Inserting this representation of the function into the integral Eq.(16), we find

$$I_N \simeq \int_{-\infty}^{\infty} dx \, \exp\left(-Nf(x_1) - \frac{N}{2}(x - x_1)^2 f''(x_1)\right). \tag{19}$$

But now Eq.(19) is in a form we can evaluate. Making the u-substitution $u = x - x_1$, yields

$$\begin{aligned} \int_{-\infty}^{\infty} dx \, \exp\left(-Nf(x_1) - \frac{N}{2}(x - x_1)^2 f''(x_1)\right) &= e^{-Nf(x_1)} \int_{-\infty}^{\infty} du \, \exp\left(-\frac{N}{2} f''(x_1) u^2\right) \\ &= \sqrt{\frac{2\pi}{Nf''(x_1)}} \, e^{-Nf(x_1)}, \end{aligned} \tag{20}$$

so that, finally, we have the approximation

$$\boxed{I_N = \int_{-\infty}^{\infty} dx \, e^{-Nf(x)} \simeq \sqrt{\frac{2\pi}{Nf''(x_1)}} \, e^{-Nf(x_1)} \qquad \text{[Laplace's Method]}.} \tag{21}$$

We note that the form of this approximation already has built into it, the requirement that $f''(x_1)$ be positive (and hence that $e^{-Nf(x)}$ has a local maximum at $x = x_1$) because if $f''(x_1)$ were negative, then taking its square root would yield an imaginary number. But we cannot have an imaginary number on the right hand side of Eq.(21) because we know that the integral (if $f(x)$ is a continuously differentiable function over the entire real axis) is real.

Now, no talk of approximations is permissible without also talking about error. For simplicity, we did not keep track of the errors in this approximation, but any such error arises from the $\mathcal{O}\left((x - x_1)^3\right)$ term we dropped after the Taylor expansion of $f(x)$. Had we kept track of these errors, we would have found $\mathcal{O}\left(N^{-3/2}\right)$ corrections to our approximation. So, rather than Eq.(21) we could write

$$I_N = \sqrt{\frac{2\pi}{Nf''(x_1)}} \, e^{-Nf(x_1)} + \mathcal{O}\left(N^{-3/2}\right). \tag{22}$$

We note that Eq.(22) shows the error term getting smaller as $N \to \infty$. Thus the approximation Eq.(21) gets better and better for large N. For calculational purposes, we will assume we are always working in this "large N limit" and Eq.(21) will suffice.

The approximation Eq.(21) is called **Laplace's method**[1] and it is applied in many areas of physics (like quantum field theory and statistical field theory) where "Gaussian-like" integrals need to be evaluated. With Eq.(21), we can return to our calculation of the partition function Eq.(10) and evaluate it approximately.

4 Return to the partition function

For our mean-field Ising model, we found the partition function

$$Z_N(\beta J) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-Nf(x,\beta J)}, \tag{23}$$

where

$$f(x, \beta J) = \frac{\beta J}{2} x^2 - \ln\left[2\cosh(\beta J x)\right]. \tag{24}$$

Implementing, Laplace's method defined in Eq.(21), we obtain the approximate result

$$Z_N(\beta J) \simeq \sqrt{\frac{N\beta J}{2\pi}} \sqrt{\frac{2\pi}{Nf''(\overline{x}, \beta J)}}\, e^{-Nf(\overline{x},\beta J)} = \sqrt{\frac{\beta J}{f''(\overline{x}, \beta J)}}\, e^{-Nf(\overline{x},\beta J)}, \tag{25}$$

where $\overline{x}$ is defined by

$$\frac{\partial}{\partial x} f(x, \beta J)\Big|_{x=\overline{x}} = 0. \tag{26}$$

Computing Eq.(26), we find

$$\frac{\partial}{\partial x} f(x, \beta J)\Big|_{x=\overline{x}} = \beta J\, \overline{x} - \beta J \tanh(\beta J \overline{x}) = 0, \tag{27}$$

which yields the condition

$$\overline{x} = \tanh(\beta J \overline{x}). \tag{28}$$

Although, it is perhaps not apparent in the relationship between Eq.(23) and Eq.(12), one can show (See Appendix A for the demonstration) that the average spin $\langle s \rangle$ is related to $\overline{x}$ through

$$\langle s \rangle \simeq \tanh(\beta J \overline{x}). \tag{29}$$

Therefore, with Eq.(28), we can identity $\overline{x}$ with $\langle s \rangle$. Writing Eq.(28) in terms of $\langle s \rangle$, we then find that the microstate-averaged spin satisfies

$$\langle s \rangle = \tanh\left(\beta J \langle s \rangle\right), \tag{30}$$

which is identical to the result Eq.(1), if we identify $\overline{m} = \langle s \rangle$. We obtained Eq.(1) through an analysis of the macrostate of the mean-field Ising model, so it is gratifying that we obtained an identical result when we analyzed the system in terms of microstate probabilities.

Similar to our work in Lecture Notes 05 "Free Energy and Order Parameters", we can analyze the solutions to Eq.(30) as we change T and ultimately show that this system exhibits two phases separated by the temperature $T_c = J/k_B$. We forgo describing the phase behavior since it is discusses in Lecture Notes 05.

> **Aside: Second Derivative of $f(x, \beta J)$**
>
> Although, we do not really need it in order to establish the correspondence between Eq.(1) and Eq.(28), we can compute the second derivative of Eq.(24) at $x = \overline{x}$ to complete the

[1] It also falls under the title "method of steepest descent" and "saddle point approximation".

evaluation of the partition function Eq.(23) . We find

$$\begin{aligned} f''(\overline{x},\beta J) &= \beta J - \frac{(\beta J)^2}{\cosh^2(\beta J\overline{x})} \\ &= \beta J\left(1-\frac{\beta J}{\cosh^2(\beta J\overline{x})}\right) \\ &= \beta J\left(1-\beta J(1-\tanh^2(\beta J\overline{x}))\right) \\ &= \beta J\left(1-\beta J(1-\overline{x}^2)\right). \end{aligned} \tag{31}$$

In the first line we used $d\tanh(x)/dx = 1/\cosh^2(x)$; in the third line we used $1-\tanh^2(x) = \cosh^2 x$; and in the final line we used Eq.(28). Requiring, $f''(\overline{x},\beta J)$ to be greater than zero thus yields the condition

$$\overline{x}^2 > 1 - \frac{1}{\beta J}. \tag{32}$$

With $\overline{x}$ identified with $\langle s\rangle$, Eq.(32) becomes

$$\langle s\rangle^2 > 1 - \frac{1}{\beta J}. \tag{33}$$

which reproduces Eq.(42) in Lecture Notes 05.

5 Final Remarks

Although we managed to re-derive the fundamental equation (i.e., Eq.(30)) defining the phase behavior of the mean-field Ising Model, the main purpose of these notes was to introduce a method (i.e., Laplace's method) for evaluating integrals of the form

$$Z_N \propto \int_{-\infty}^{\infty} dx\, e^{-Nf(x)}. \tag{34}$$

Now that we have managed to achieve this objective, we are prepared to apply it to many different problems in statistical physics which result in integral-defined partition functions. In the next lecture notes, we apply this method to studying a model of DNA dimerization.

A Demonstrating relationship between $\langle s\rangle$ and $\overline{x}$

We want to establish the relationship between the average spin $\langle s\rangle$ and the value of x at which $f(x,\beta J)$ is minimized. We begin with the definition of the average spin:

$$\langle s\rangle = \frac{1}{Z_N(\beta J)}\frac{1}{N}\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}\sum_{i=1}^{N} s_i \exp\left(\frac{\beta J}{2N}\sum_{i,j=1}^{N} s_i s_j\right), \tag{35}$$

We can write this definition in terms of the partition function alone by using a different energy function. Let us define the energy

$$E(\{s_i\}) = -\frac{J}{2N}\sum_{i,j=1}^{N} s_i s_j - h\sum_{j=1}^{N} s_j. \tag{36}$$

The partition function associated with this energy is then

$$Z_N(\beta J,\beta h)=\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}\exp\left(\frac{\beta J}{2N}\sum_{i,j=1}^{N}s_is_j+\beta h\sum_{j=1}^{N}s_j\right). \tag{37}$$

Computing the βh partial derivative of this partition function, we find

$$\frac{\partial}{\partial(\beta h)}Z_N(\beta J,\beta h)=\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}\sum_{j=1}^{N}s_j\exp\left(\frac{\beta J}{2N}\sum_{i,j=1}^{N}s_is_j+\beta h\sum_{j=1}^{N}s_j\right), \tag{38}$$

which would be proportional to Eq.(35) if we were to take $h=0$. Therefore, we can write Eq.(35) exactly as

$$\langle s\rangle=\left[\frac{1}{N}\frac{1}{Z_N(\beta J,\beta h)}\frac{\partial}{\partial(\beta h)}Z_N(\beta J,\beta h)\right]_{h=0}=\left[\frac{1}{N}\frac{\partial}{\partial(\beta h)}\ln Z_N(\beta J,\beta h)\right]_{h=0}. \tag{39}$$

Now, going through a similar application of the Gaussian integral identity, we can write Eq.(37) as

$$\begin{aligned}Z_N(\beta J,\beta h)&=\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1}\sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty}dx\,\exp\left(-\frac{N\beta J}{2}\,x^2+(x\,\beta J+\beta h)\sum_{i=1}^{N}s_i\right)\\&=\sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty}dx\,e^{-N\beta Jx^2/2}\prod_{j=1}^{N}\sum_{s_j=\pm1}e^{(x\beta J+\beta h)s_j}\\&=\sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty}dx\,e^{-N\beta Jx^2/2}\,2^N\cosh^N(\beta(xJ+h)),\end{aligned} \tag{40}$$

or as

$$Z_N(\beta J,\beta h)=\sqrt{\frac{N\beta J}{2\pi}}\int_{-\infty}^{\infty}dx\,e^{-Ng(x,\beta J,\beta h)}, \tag{41}$$

where

$$g(x,\beta J,\beta h)=\frac{\beta J}{2}x^2-\ln\left[2\cosh\left(\beta(xJ+h)\right)\right]. \tag{42}$$

In order to approximate Eq.(41) via Laplace's method, we need to find the local minimum of Eq.(42). Computing the first derivative of Eq.(42) and setting the result to zero at $x=x_0$ in a way similar to the calculation in Sec Sec. 4, we find the condition

$$x_0=\tanh\left(\beta(x_0J+h)\right). \tag{43}$$

We see that if we were to set $h=0$ in Eq.(43), then x_0 would reduce to $\overline{x}$ defined in Eq.(28). Now, approximating Eq.(41) by its Laplace's method expression, we have

$$Z_N(\beta J,\beta h)=\sqrt{\frac{\beta J}{Ng''(x_0,\beta J,\beta h)}}e^{-Ng(x_0,\beta J,\beta h)}+\mathcal{O}\left(N^{-3/2}\right), \tag{44}$$

where we used Eq.(22) to state the error exactly. Taking the logarithm of Eq.(44) and differentiating with respect to (βh), we obtain

$$\begin{aligned}\frac{\partial}{\partial(\beta h)}\ln Z_N(\beta J,\beta h)&=\frac{\partial}{\partial(\beta h)}\ln\left[\sqrt{\frac{\beta J}{Ng''(x_0,\beta J,\beta h)}}e^{-Ng(x_0,\beta J,\beta h)}\right]+\mathcal{O}\left(N^{-3/2}\right)\\&=\frac{\partial}{\partial(\beta h)}\left[-Ng(x_0,\beta J,\beta h)+\mathcal{O}(N^0)\right]\end{aligned}$$

$$= -N \frac{\partial}{\partial(\beta h)} \left[\frac{\beta J}{2} x_0^2 - \ln\left[2\cosh\left(\beta(x_0 J + h)\right)\right] + \mathcal{O}(N^{-1}) \right]$$
$$= N \tanh\left(\beta(x_0 J + h)\right) + \mathcal{O}(N^0), \tag{45}$$

where in the second line we represented all terms not proportional to N as at least $\mathcal{O}(N^0)$ and in the final line we used the fact that the coefficient of the $\partial x_0/\partial(\beta h)$ term was zero. Finally, using Eq.(45) in Eq.(39), we find

$$\langle s \rangle = \left[\tanh\left(\beta(x_0 J + h)\right) + \mathcal{O}(N^{-1})\right]_{h=0}. \tag{46}$$

From $\overline{x}$'s definition in Eq.(28), it is clear from Eq.(43) that x_0 becomes $\overline{x}$ when we take $h = 0$. Therefore, taking $h = 0$ in Eq.(46), we obtain

$$\langle s \rangle = \tanh\left(\beta J \overline{x}\right) + \mathcal{O}(N^{-1}), \tag{47}$$

thus proving Eq.(29).