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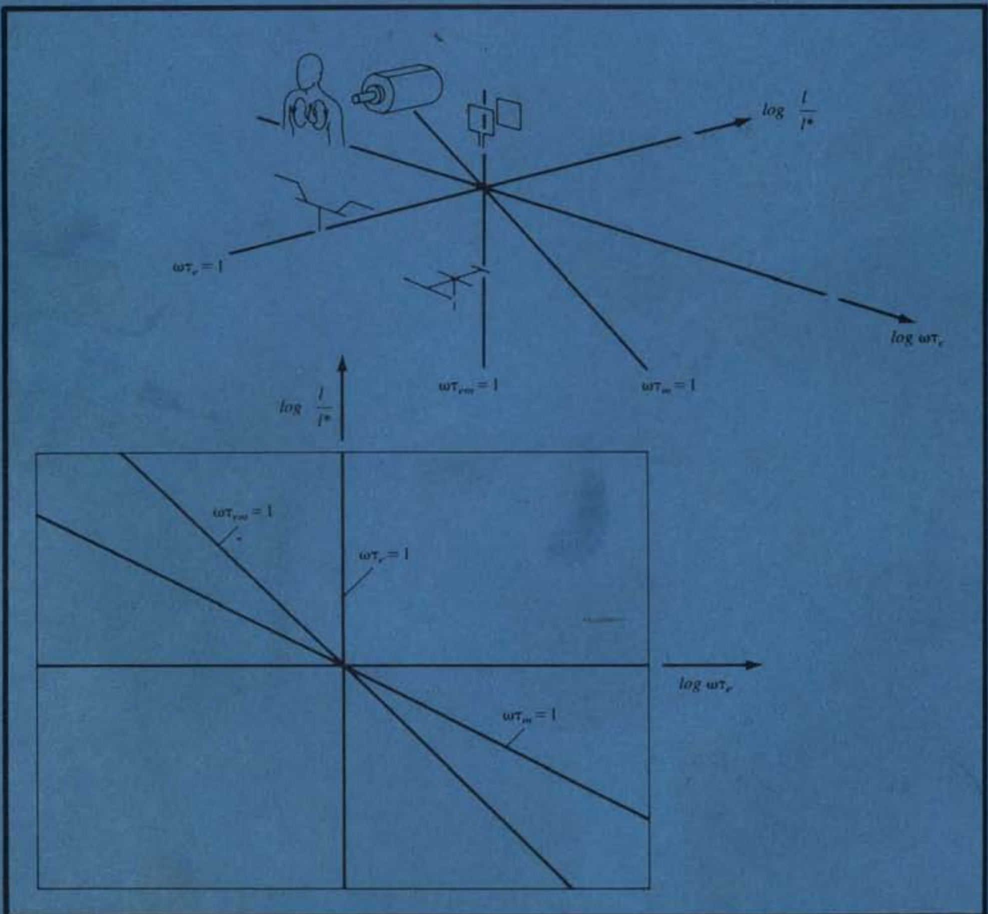
Solutions Manual

Electromagnetic Fields and Energy



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Solutions Manual

Electromagnetic Fields and Energy

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PREFACE TO SOLUTION MANUAL

We are fortunate that electromagnetic aspects of engineering systems are accurately described by remarkably concise and general laws. Yet, a price paid for the generality of Maxwell's equations is the effort required to make these laws of practical use to the engineer who is not only analyzing, but synthesizing and inventing. Key to the maturation of an engineer who hopes to use a basic background in electromagnetic fields for effectively dealing with complex problems is working out examples that strike the right balance among a number of interrelated objectives. First, even in the beginning, the examples should couch the development of skill in using the mathematical language of field theory in physical terms. Second, while being no more mathematically involved then required to make the point, they should collectively give insight into the key phenomena implied by the general laws. This means that they have to be sufficiently realistic to at least be physically demonstrable and at best of practical interest. Third, as the student works out a series of examples, they should form the basis for having an overview of electromagnetics, hopefully helping to achieve an early maturity in applying the general laws.

In teaching this subject at MIT, we have placed a heavy emphasis on working out examples, basing as much as 40 percent of a student's grade on homework solutions. Because new problems must then be generated each term, this emphasis has mandated a continual search and development, stimulated by faculty and graduate student teaching assistant colleagues. Some of these problems have become the "examples," worked out in the text. These have in turn determined the development of the demonstrations, also described in the text (and available on video tape through the authors). The problems given at the ends of chapters in the text and worked out in this manual do not include still other combinations of geometries, models and physical phonemena. These combinations become apparent when the examples and problems from one chapter are compared with those from another. A review of the example summaries given in Chap. 15 will make evident some of these opportunities for problem creation.

After about two decades, the number of faculty and teaching assistants who have made contributions, at least by preparing the official solutions during a given term, probably exceeds 100, so individual recognition is not appropriate. Preliminary versions of solutions for several chapters were prepared by Rayomond H. Kotwal while he was a teaching assistant. However, finally, the authors shared responsibility for writing up the solutions. Corrections to the inevitable errors would be appreciated.

Our view that an apprenticeship of problem solving is essential to learning field theory is reflected in the care which has been taken in preparing this solution manual. This was only possible because Ms. Cindy Kopf not only "Tex't" the manual (as she did the text itself) while taking major responsibility for the art-work, but organized and produced the camera-ready copy as well. The "Tex macros" were written by Ms. Amy Hendrickson.

SOLUTIONS TO CHAPTER 1

1.1 THE LORENTZ LAW IN FREE SPACE

1.1.1 For $v_i = 0$, (7) gives

$$h = -\frac{1}{2}\frac{e}{m}E_x(t-t_i)^2 \Rightarrow t - t_i = \sqrt{\frac{-2hm}{\epsilon E_x}} \tag{1}$$

and from (8)

$$v = \sqrt{\frac{-2h\epsilon E_x}{m}} \tag{2}$$

so

$$v = \sqrt{\frac{2(1\times 10^{-2})(1.602\times 10^{-19})(10^{-2})}{(9.106\times 10^{-31})}} = 5.9\times 10^{31}\text{m/s} \tag{3}$$

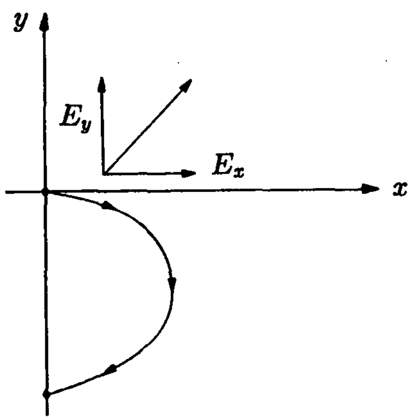


Figure S1.1.2

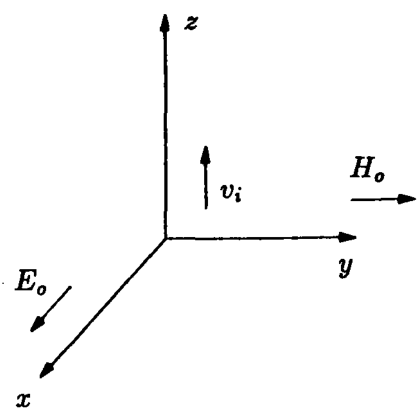


Figure S1.1.3

1.1.2 (a) In two-dimensions, (4) gives

$$\frac{md^2\xi_x}{dt^2} = -eE_x \tag{1}$$

$$\frac{md^2\xi_y}{dt^2} = -eE_y \tag{2}$$

so, because $v_x(0) = v_i$, while $v_y(0) = 0$,

$$\frac{d\xi_x}{dt} = -\frac{e}{m}E_x t + v_i \tag{3}$$

$$\frac{d\xi_y}{dt} = -\frac{e}{m}E_y t \tag{4}$$

To make $\xi_x(0) = 0$ and $\xi_y(0) = 0$

$$\xi_x = -\frac{e}{2m}E_x t^2 + v_i t \tag{5}$$

$$\xi_y = -\frac{e}{2m}E_y t^2 \tag{6}$$

(b) From (5), $\xi_x = 0$ when

$$t = \frac{v_i 2m}{eE_x} \tag{7}$$

and at this time

$$\xi_y = -\frac{e}{2m}E_y\left(\frac{v_i 2m}{eE_x}\right)^2 \tag{8}$$

1.1.3 The force is

$$\mathbf{f} = q[\mathbf{E} + \mathbf{v} \times \mu_o \mathbf{H}] = -e[E_o \mathbf{i_x} - v_z \mu_o H_o \mathbf{i_x}] \tag{1}$$

so, $\mathbf{f} = 0$ if $E_o = v_i \mu_o H_o$. Thus,

$$\frac{dv_x}{dt} = 0, \qquad \frac{dv_y}{dt} = 0, \qquad \frac{dv_z}{dt} = 0 \tag{2}$$

and v_x, v_y and v_z are constants. Because initial velocities in x and y directions are zero, $v_x = v_y = 0$ and $\mathbf{v} = v_i \mathbf{i_z}$.

1.1.4 The force is

$$\mathbf{f} = -e(\mathbf{E} + \mathbf{v} \times \mu_o \mathbf{H}) = -e(E_o \mathbf{i_y} + v_x \mu_o H_o \mathbf{i_z} - v_z \mu_o H_o \mathbf{i_x}) \tag{1}$$

so

$$\frac{md^2\xi_y}{dt^2} = -eE_o \Rightarrow \xi_y = -\frac{eE_o}{2m}t^2 \tag{2}$$

and

$$\frac{mdv_x}{dt} = ev_z \mu_o H_o \Rightarrow \frac{dv_x}{dt} = \omega_c v_z \tag{3}$$

$$\frac{mdv_z}{dt} = -ev_x \mu_o H_o \Rightarrow \frac{dv_z}{dt} = -\omega_c v_x \tag{4}$$

where $\omega_c = e\mu_o H_o/m$. Substitution of (3) into (4) gives

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0 \tag{5}$$

Solutions are $\sin \omega_c t$ and $\cos \omega_c t$. To satisfy the initial conditions on the velocity,

$$v_x = v_o \cos \omega_c t = \frac{d\xi_x}{dt} \tag{6}$$

in which case (3) gives:

$$v_z = -v_o \sin \omega_c t = \frac{d\xi_z}{dt} \tag{7}$$

Further integration and the initial conditions on $\bar{\xi}$ gives

$$\xi_x = \frac{v_o}{\omega_c} \sin \omega_c t \tag{8}$$

$$\xi_z = \frac{v_o}{\omega_c} \cos \omega_c t - \frac{v_o}{\omega_c} + z_o \tag{9}$$

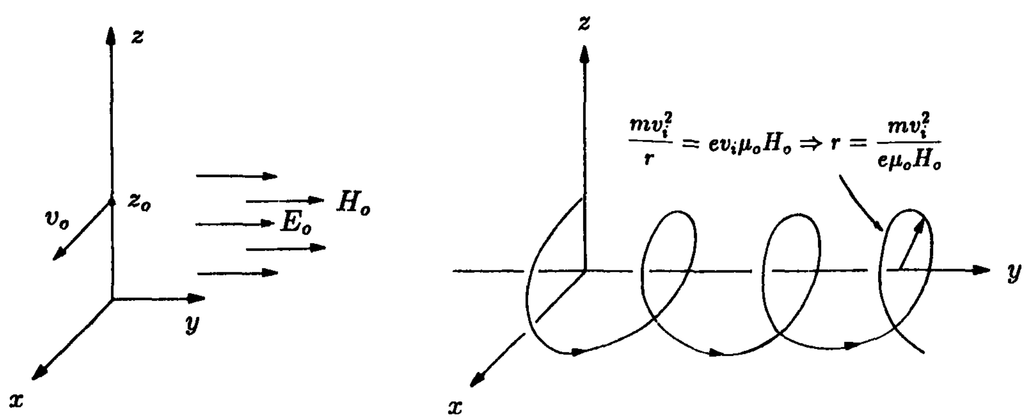


Figure S1.1.4

1.2 CHARGE AND CURRENT DENSITIES

1.2.1 The total charge is

$$q = \int_o^R \rho 4\pi r^2 dr = \int_o^R \frac{4\pi\rho_o r^3}{R} dr = \pi \rho_o R^3 \tag{1}$$

1.2.2 Integration of the density over the given volume gives the total charge

$$\begin{aligned} q &= \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}\frac{\rho_o}{a^2}[x^2+y^2+z^2]dxdydz \\ &= \frac{2\rho_o}{a^2}\int_{-a}^{a}\int_{-a}^{a}(\frac{a^3}{3}+ay^2+az^2)dydz \end{aligned} \tag{1}$$

Two further integrations give

$$q = \frac{4\rho_o}{a^2}\int_{-a}^{a}(\frac{a^4}{3}+\frac{a^4}{3}+a^2z^2)dz = \frac{8\rho_o}{a^2}[\frac{2a^5}{3}+\frac{a^5}{3}] = 8\rho_o a^3 \tag{2}$$

1.2.3 The normal to the surface is $\mathbf{i_x}$, so

$$\begin{aligned} i &= \int_{-a}^{a}\int_{-a}^{a}\mathbf{J}\cdot\mathbf{n}dydz = \frac{J_o}{a^2}\int_{-a}^{a}\int_{-a}^{a}(y^2+z^2)dydz \\ &= \frac{2J_o}{a^2}\int_{-a}^{a}(\frac{1}{3}a^3+az^2)dz = \frac{8J_oa^2}{3} \end{aligned} \tag{1}$$

1.2.4 The net current is

$$i = \int_0^a J_o(\frac{r^2}{a^2})2\pi r dr = 2\pi\frac{J_o}{a^2}\frac{1}{4}r^4|_0^a = \frac{\pi J_o a^2}{2} \tag{1}$$

1.2.5 (a) From Newton's second law

$$\frac{mdv_r}{dt} = -\frac{eE_ob}{\xi_r} \tag{1}$$

where

$$v_r = \frac{d\xi_r}{dt} \tag{2}$$

(b) On multiplying (1) by v_r,

$$mv_r\frac{dv_r}{dr} = -\frac{eE_ob}{\xi_r}v_r$$

and using (2), we obtain

$$mv_r\frac{dv_r}{dt} + \frac{eE_ob}{\xi_r}\frac{d\xi_r}{dt} = \frac{d}{dt}[\frac{1}{2}mv_r^2 + eE_ob\ln\xi_r] = 0 \tag{3}$$

(c) Integrating (3) with respect to t gives

$$\frac{1}{2}mv_r^2 + eE_o b ln \xi_r = c_1 \tag{4}$$

When $t = 0, v_r = 0, \xi_r = b$ so $c_1 = eE_o b ln b$ and

$$\frac{1}{2}mv_r^2 + eE_o b ln \frac{\xi_r}{b} = 0 \tag{5}$$

Thus,

$$v_r(r) = \sqrt{\frac{2e}{m}E_o b ln \frac{b}{r}} \tag{6}$$

(d) The current density is

$$J_r = \rho(r)v_r(r) \Rightarrow \rho(r) = \frac{J_r}{v_r(r)} \tag{7}$$

The total current, i, must be independent of r, so

$$J_r = \frac{i}{2\pi r l} \tag{8}$$

and it follows from (6) and (7) that

$$\rho(r) = \frac{i}{2\pi r l}\sqrt{\frac{m}{2eE_o b ln(b/r)}} \tag{9}$$

1.3 GAUSS' INTEGRAL LAW OF ELECTRIC FIELD INTENSITY

1.3.1 (a) The unit vectors perpendicular to the 5 surfaces are as shown in Fig. S1.3.1. The given area elements follow from the same construction.

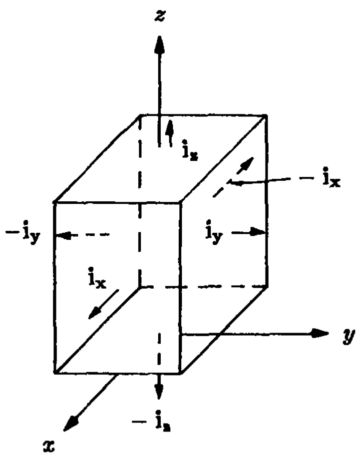
(b) From Fig. S1.3.1,

$$|\mathbf{i_r}|_x = \cos\phi = \frac{x}{\sqrt{x^2+y^2}}; \quad |\mathbf{i_r}|_y = \sin\phi = \frac{y}{\sqrt{x^2+y^2}} \tag{1}$$

$$r = \sqrt{x^2+y^2} \tag{2}$$

Thus, the conversion from polar to Cartesian coordinates gives

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = \frac{\lambda_l}{2\pi\epsilon_o}\frac{1}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{i_y}\right) \tag{3}$$



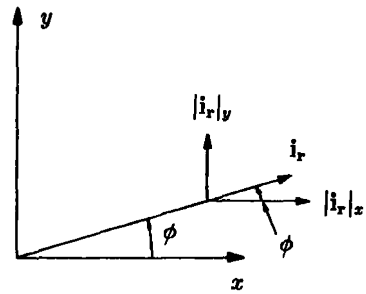


Figure S1.3.1

(c) On the given surface, the normal vector is $\mathbf{i}_x$ and so the integral is of the x component of (3) evaluated at $x = a$.

$$\int \epsilon_o \mathbf{E} \cdot d\mathbf{a}\Big|_{x=a} = \frac{\epsilon_o \lambda_l}{2\pi\epsilon_o}\int_0^1 \int_{-a}^{a} \frac{a}{a^2+y^2}dydz$$
$$= \frac{\lambda_l}{2\pi}\tan^{-1}\left(\frac{y}{a}\right)\Big|_{-a}^{a} = \frac{\lambda_l}{2\pi}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\lambda_l}{4} \tag{4}$$

Integration over the surface at $x = -a$ reverses both the sign of E_x and of the normal and so is also given by (4). Integrations over the surfaces at $y = a$ and $y = -a$ are respectively the same as given by (4), with the roles of x and y reversed. Integrations over the top and bottom surfaces make no contribution because there is no normal component of $\mathbf{E}$ on these surfaces. Thus, the total surface integration is four times that given by (4), which is indeed the charge enclosed, λ_l.

1.3.2 On the respective surfaces,

$$\mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_o}\begin{cases} 1/a^2 \\ 0 \\ 1/b^2 \end{cases} \tag{1}$$

On the two surfaces where these integrands are finite, they are also constant, so integration amounts to multiplication by the respective areas.

$$\oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} = \frac{q\epsilon_o}{4\pi\epsilon_o}\left[\frac{1}{a^2}(2\pi a^2) + \frac{1}{b^2}(2\pi b^2)\right] = q \tag{2}$$

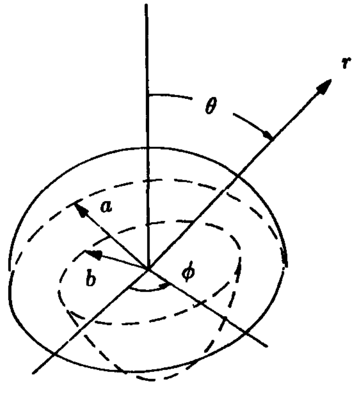


Figure S1.3.2

1.3.3 (a) Because of the axial symmetry, the electric field must be radial. Thus, integration of E_r over the surface at $r = r$ amounts to a multiplication by the area. For $r < b$, Gauss' integral law therefore gives

$$2\pi r l \epsilon_o E_r = \int_0^l \int_0^{2\pi} \int_0^r \rho dr r d\phi dz = 2\pi l \int_0^r \frac{\rho_o r^3}{b^2} dr \tag{1}$$

$$E_r = \frac{\rho_o r^3}{4\epsilon_o b^2}; \quad r < b$$

For $b < r < a$, the integral on the right stops at $r = b$.

$$E_r = \frac{\rho_o b^2}{4\epsilon_o b^2}; \quad b < r < a \tag{2}$$

(b) From (17)

$$\sigma_s = \mathbf{i}_r \cdot (\epsilon_o \mathbf{E}^a - \epsilon_o \mathbf{E}^b) = -\epsilon_o E_r(r = a) = -\frac{\rho_o b^2}{4a} \tag{3}$$

(c) Because it is uniform there, integration of the surface charge density given by (3) over the surface $r = a$ amounts to a multiplication by the surface area.

$$\int \sigma_s da = \sigma_s 2\pi a l = -\rho_o b^2 \pi l/2 \tag{4}$$

That this is the negative of the net charge within is confirmed by integrating over the enclosed charge density.

$$\int_V \rho dV = \int_0^l \int_0^{2\pi} \int_0^b \rho_o \Big(\frac{r}{b}\Big)^2 r d\phi dz = \frac{\pi l \rho_o b^2}{2} \tag{5}$$

(d) As shown in the solution to Prob. 1.3.1,

$$\mathbf{i}_r = (x\mathbf{i}_x + y\mathbf{i}_y)/\sqrt{x^2+y^2}; \quad r = \sqrt{x^2+y^2} \tag{6}$$

and substitution into

$$\mathbf{E} = \frac{\rho_o}{4\epsilon_o}\begin{cases}(r^3/b^2)\mathbf{i}_r; & r<b\\ (b^2/r)\mathbf{i}_r; & b<r<a\end{cases} \tag{7}$$

indeed results in the given field distribution.

(e) For the surfaces at $x = \pm c$,

$$d\mathbf{a} = \pm\mathbf{i}_x dydz; \quad \mathbf{E}\cdot\mathbf{n} = E_x(x=\pm c) \tag{8}$$

while for those at $y = \pm c$,

$$d\mathbf{a} = \pm\mathbf{i}_y dxdz; \quad \mathbf{E}\cdot\mathbf{n} = E_y(y=\pm c) \tag{9}$$

The four terms in the given surface integral are the integrations over the respective surfaces using the field given by (d) evaluated in accordance with (8) and (9). According to (1), this integral must give the same answer as found by integrating the charge density over the enclosed volume. This has already been done and is given by (5).

1.3.4 (a) For $r < b$, (1) gives

$$4\pi r^2\epsilon_o E_r = \int_0^r \rho_b 4\pi r^2 dr = \frac{4\pi\rho_b r^3}{3} \tag{1}$$

Thus,

$$E_r = \frac{\rho_o r}{3\epsilon_o}; \qquad r<b \tag{2}$$

Similarly, for $b < r < a$

$$4\pi r^2\epsilon_o E_r = \frac{4}{3}\pi\rho_b b^3 + \int_b^r \frac{4\pi\rho_a r^2}{3}dr = \frac{4}{3}\pi[b^3\rho_b + (r^3-b^3)\rho_a] \tag{3}$$

so that

$$E_r = \frac{1}{3\epsilon_o}\left[\frac{b^3\rho_b}{r^2} + \left(r - \frac{b^3}{r^2}\right)\rho_a\right]; \qquad b<r<a \tag{4}$$

(b) At $r = a$, (17) can be evaluated with $\mathbf{n} = \mathbf{i}_r, \mathbf{E}^a = 0$ and $\mathbf{E}^b$ given by (4)

$$\sigma_s = -\frac{1}{3}\left[\frac{b^3\rho_b}{a^2} + \left(a - \frac{b^3}{a^2}\right)\rho_a\right] \tag{5}$$

(c) For $r < b, E_r$ is still given by (2), while for $b < r < a$, (3) has an additional term on the right $4\pi b^2\sigma_o$. Thus,

$$E_r = \frac{1}{3\epsilon_o}\left[\frac{b^3\rho_b}{r^2} + \left(r - \frac{b^3}{r^2}\right)\rho_a\right] + \frac{b^2\sigma_o}{\epsilon_o r^2}; \qquad b<r<a \tag{6}$$

Then, instead of (5) we have

$$\sigma_s = -\frac{1}{3}\left[\frac{b^3\rho_b}{a^2} + \left(a - \frac{b^3}{a^2}\right)\rho_a\right] - \frac{b^2\sigma_o}{a^2} \tag{7}$$

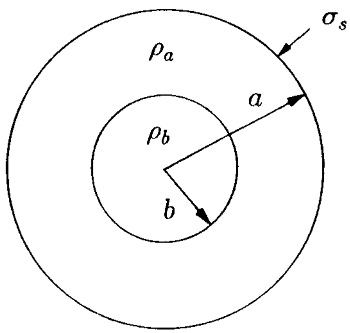


Figure S1.3.4

1.3.5 Using the volume described in Example 1.3.2, with the upper surface between the sheets, there is a contribution to the charge enclosed from both the lower sheet and the volume between that sheet and the position, z, of the upper surface. Thus, from (1)

$$\epsilon_o E_z(z) - \epsilon_o E_o = \sigma_o + \int_{-s/2}^{z} \rho dz = \sigma_o + \frac{2\rho_o z^2}{2s}\Big|_{-s/2}^{z} \tag{1}$$

and the solution for E_z gives

$$E_z = E_o + \frac{\sigma_o}{\epsilon_o} + \frac{\rho_o}{\epsilon_o s}\left[z^2 - (s/2)^2\right] \tag{2}$$

Note that the charge density is an odd function of z. Thus, there is no net charge between the sheets. With the surface above the upper sheet, the field given by (1) with the integration terminated at $z = s/2$ is just what it was below the lower sheet, E_o.

1.3.6 With the understanding that the charge distribution extends to infinity in the y and z directions, it follows from arguments already given that the electric field is independent of y and z and that that part of it due to the charge sheets can result only in a z directed electric field. It then follows from (1) that if the regions above and below the charge sustain no electric field intensity, then the net charge from the three layers must be zero. Thus, not only is

$$\sigma_a = 2\sigma_b \tag{1}$$

but also,

$$\sigma_a + \sigma_b + \sigma_o = 0 \tag{2}$$

From these relations, it follows that

$$\sigma_b = -\sigma_o/3; \qquad \sigma_a = -2\sigma_o/3 \tag{3}$$

1.3.7 The gravitational force has a component in the ξ direction, $-Mg\sin\alpha$. Thus, the sum of the forces acting on the upper particle in the ξ direction is

$$\frac{QQ_o}{4\pi\epsilon_o\xi^2} - Mg\sin\alpha = 0 \tag{1}$$

It follows that, for the particle to be in static equilibrium,

$$\xi = \sqrt{\frac{QQ_o}{4\pi\epsilon_o Mg\sin\alpha}} \tag{2}$$

1.4 AMPERE'S INTEGRAL LAW

1.4.1 Evaluation of (1) is carried out for a contour having the constant radius, r, on which symmetry requires that the magnetic field intensity be constant and in the ϕ direction. Because the fields are static, the last term on the right makes no contribution. Thus,

$$2\pi rH_\phi = \int_0^r J_r 2\pi r dr = \int_0^r 2\pi r J_o e^{-r/a} dr \tag{1}$$

Solving this expression for H_ϕ and carrying out the integration then gives

$$H_\phi = \frac{J_o}{r}\int_0^r re^{-r/a}dr = \frac{J_o}{r}a^2\big[1 - e^{-r/a}\big(1 + \frac{r}{a}\big)\big] \tag{2}$$

1.4.2 (a) The net current carried by the wire in the $+z$ direction must be returned in the $-z$ direction on the surface at $r = a$. Thus,

$$\pi b^2 J_o + 2\pi a K_z = 0 \Rightarrow K_z = -\frac{b^2 J_o}{2a} \tag{1}$$

(b) For a contour at the constant radius, r, (1) is evaluated (with the last term on the right zero because the fields are static), first for $r < b$ and then for $b < r < a$.

$$2\pi rH_\phi = \int_0^r J_o 2\pi r dr = \pi r^2 J_o \Rightarrow H_\phi = \frac{J_o r}{2}; \qquad r < b \tag{2}$$

$$2\pi rH_\phi = \pi b^2 J_o \Rightarrow H_\phi = \frac{J_o b^2}{2r}; \qquad b < r < a \tag{3}$$

(c) From (1.4.16),

$$H_\phi^a - H_\phi^b = K_z \Rightarrow H_\phi^a = K_z + H_\phi^b \tag{4}$$

This expression can be evaluated using (1) and (3).

$$H_\phi^a = -\frac{b^2 J_o}{2a} + \frac{J_o b^2}{2a} = 0 \tag{5}$$

(d) In Cartesian coordinates,

$$H_x = -H_\phi \sin\phi = -H_\phi \frac{y}{\sqrt{x^2+y^2}}; \qquad H_y = H_\phi \cos\phi = H_\phi \frac{x}{\sqrt{x^2+y^2}} \tag{6}$$

Thus, with $r = \sqrt{x^2+y^2}$, evaluation of this expression using (2) and (3) gives

$$\mathbf{H} = \begin{cases} \frac{J_o}{2}\sqrt{x^2+y^2}(-y\mathbf{i}_x + x\mathbf{i}_y)/\sqrt{x^2+y^2} = \frac{J_o}{2}(-y\mathbf{i}_x + x\mathbf{i}_y) \\ \frac{J_o b^2}{2\sqrt{x^2+y^2}}\frac{(-y\mathbf{i}_x + x\mathbf{i}_y)}{\sqrt{x^2+y^2}} = \frac{J_o b^2}{2(x^2+y^2)}(-y\mathbf{i}_x + x\mathbf{i}_y) \end{cases} \tag{7}$$

(e) On $x = \pm c, \mathbf{H}\cdot d\mathbf{s} = \pm\mathbf{H}\cdot\mathbf{i}_y$ while on $y = \pm c, \mathbf{H}\cdot d\mathbf{s} = \mp\mathbf{H}\cdot\mathbf{i}_x$ so evaluation of (1) on the square contour gives

$$\begin{aligned} \int_{-c}^{c} [H_x(x,-c) - H_x(x,c)]dx &+ \int_{-c}^{c} [H_y(c,y) - H_y(-c,y)]dy \\ &= \int_{-c}^{c} \frac{J_o b^2}{2}\left[\frac{c}{x^2+c^2} - \frac{(-c)}{x^2+y^2}\right]dx \\ &+ \int_{-c}^{c} \frac{J_o b^2}{2}\left[\frac{c}{c^2+y^2} - \frac{(-c)}{c^2+y^2}\right]dy \end{aligned} \tag{8}$$

The result of carrying out this integration must be equal to what is obtained by carrying out the surface integral on the right in (1).

$$\oint_C \mathbf{H}\cdot d\mathbf{s} = \int_S \mathbf{J}\cdot d\mathbf{a} = \pi b^2 J_o \tag{9}$$

1.4.3 (a) The total current in the $+z$ direction through the shell between $r = a$ and $r = b$ must equal that in the $-z$ direction through the wire at the center. Because the current density is uniform, it is then simply the total current divided by the cross-sectional area of the shell.

$$I = J_z[\pi(a^2 - b^2)] \Rightarrow J_z = I/\pi(a^2 - b^2) \tag{1}$$

(b) Ampère's integral law is written for a contour that circulates around the z axis at the constant radius r. The fields are constant, so the last term in (1.4.1) is zero. Symmetry arguments can be used to argue that $\mathbf{H}$ is ϕ directed and uniform on this contour, thus

$$2\pi r H_\phi = -I \Rightarrow H_\phi = -I/2\pi r; \qquad 0 < r < b \tag{2}$$

$$2\pi r H_\phi = -I + \frac{I}{\pi(a^2-b^2)}\pi(r^2-b^2) \Rightarrow H_\phi = I\left[-\frac{1}{2\pi r} + \frac{(r^2-b^2)}{a^2-b^2}\frac{1}{2\pi r}\right] \tag{3}$$

(c) Analysis of the ϕ directed H-field into Cartesian coordinates gives

$$\begin{aligned} H_x &= -H_\phi \sin\phi = -H_\phi y/\sqrt{x^2+y^2} \\ H_y &= -H_\phi \cos\phi = H_\phi x/\sqrt{x^2+y^2} \end{aligned} \tag{4}$$

where $r = \sqrt{x^2+y^2}$. Thus, from (2) and (3),

$$\mathbf{H} = \frac{I(y\mathbf{i_x} - x\mathbf{i_y})}{2\pi(x^2+y^2)}\begin{cases} 1; & 0 < \sqrt{x^2+y^2} < b \\ 1 - \frac{(x^2+y^2-b^2)}{a^2-b^2}; & b < r < a \end{cases} \tag{5}$$

(d) In evaluating the line integral on the four segments of the square contour, on $x = \pm c, d\mathbf{s} = \pm\mathbf{i}_y dy$ and $\mathbf{H}\cdot d\mathbf{s} = \pm H_y(\pm c, y)dy$ while on $y = \pm c, d\mathbf{s} = \mp\mathbf{i_x}dx$ and $\mathbf{H}\cdot d\mathbf{s} = \mp H_x(x, \mp c)dx$. Thus,

$$\begin{aligned} \oint_C \mathbf{H}\cdot d\mathbf{s} &= \int_{-c}^{c} H_y(c,y)dy + \int_{-c}^{c} -H_x(x,-c)dx \\ &\quad + \int_{-c}^{c} -H_y(-c,y)dy + \int_{-c}^{c} H_x(x,c)dx \end{aligned} \tag{6}$$

This integral must be equal to the right hand side of (1.4.1), which can be evaluated in accordance with whether the contour stays within the region $r < b$ or is closed within the shell. In the latter case, the integration over the area of the shell enclosed by the contour is accomplished by simply multiplying the current density by the area of the square minus that of region inside the radius $r = b$.

$$\oint_S \mathbf{J}\cdot d\mathbf{a} =$$

$$\begin{cases} -I; & c < b/\sqrt{2} \\ -I + \frac{I}{\pi(a^2-b^2)}[(2c)^2 - \pi b^2 + 4(\alpha b^2 - c\sqrt{b^2-c^2})]; & b/\sqrt{2} < c < b \\ -I + \frac{I}{\pi(a^2-b^2)}[(2c)^2 - \pi b^2]; & b < c < a/\sqrt{2} \end{cases} \tag{7}$$

where $\alpha = \cos^{-1}(c/b)$. The range $b/\sqrt{2} < c < b$ is complicated by the fact that the square contour overlaps the circle $r = b$. Thus, the area over which the return current in the shell passes through the square contour is the area of the square $(2c)^2$, minus the area of the region inside the radius b (as in the last case where there is no overlap of the square contour and the surface at $r = b$) plus the area where the circle $r = b$ extends beyond the square, which should not have been subtracted away.

1.4.4 (a) The net current passing through any plane of constant z must be zero. Thus,

$$2\pi a K_{za} + 2\pi b K_{zb} = I \tag{1}$$

and we are given that

$$K_{za} = 2K_{zb} \tag{2}$$

Solution of these expressions gives the desired surface current densities

$$K_{za} = \frac{I}{\pi(2a+b)}; \qquad K_{zb} = \frac{I}{2\pi(2a+b)} \tag{3}$$

(b) For $r < b$, Ampère's integral law, (1.4.1), applied to the region $r < b$ where the only current enclosed by the contour is due to that on the z axis, gives

$$2\pi r H_\phi = -I \Rightarrow H_\phi = \frac{-I}{2\pi r}; \qquad r < b \tag{4}$$

In the region $b < r < a$, the contour encloses the inner of the two surface current densities as well. Because it is in the z direction, its contribution is of opposite sign to that of I.

$$2\pi r H_\phi = -I + 2\pi b K_{zb} = -\left(\frac{2a}{2a+b}\right)I \tag{5}$$

Thus,

$$H_\phi = -\frac{I}{2\pi r}\left(\frac{2a}{2a+b}\right); \qquad b < r < a \tag{6}$$

Note that if Ampère's law is applied where $a < r$, the net current enclosed is zero and hence the magnetic field intensity is zero.

1.4.5 Symmetry arguments can be used to show that $\mathbf{H}$ depends only on z. Ampère's integral law is used with a contour that is in a plane of constant y, so that it encloses the given surface and volume currents. With z taken to be in the vertical direction, the area enclosed by this contour has unit length in the x direction, its lower edge in the field free region $x < -s/2$ and its upper edge at the location z. Then, (1.4.1) becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = H_x(z) = -K_o + \int_{-s/2}^{z} J_y dz \tag{1}$$

and for $-s/2 < z < s/2$,

$$H_x = -K_o + \int_{-s/2}^{z} \frac{2J_o z}{s} dz = -K_o + \frac{J_o}{s}\left[z^2 - (s/2)^2\right] \tag{2}$$

while for $s/2 < z$,

$$H_x = 0 \tag{3}$$

1.5 CHARGE CONSERVATION IN INTEGRAL FORM

1.5.1 Because of the radial symmetry, a spherical volume having its center at the origin and a radius r is used to evaluate 1.5.2. Because the charge density is uniform, the volume integral is evaluated by simply multiplying the volume by the charge density. Thus,

$$4\pi r^2 J_r + \frac{d}{dt}\Big[\frac{4}{3}\pi r^3 \rho_o(t)\Big] = 0 \Rightarrow J_r = -\frac{r}{3}\frac{d\rho_o}{dt} \tag{1}$$

1.5.2 Equation 1.5.2 is evaluated for a volume enclosed by surfaces having area A in the planes $x = x$ and $x = 0$. Because the the current density is x directed, contributions to the surface integral over the other surfaces, which have normals that are perpendicular to the x axis, are zero. Thus, (1.5.2) becomes

$$A[J_x(x) - J_x(0)] + \frac{d}{dt}[Ax\rho_o(t)] = 0 \Rightarrow J_x = -x\frac{d\rho_o}{dt} \tag{1}$$

1.5.3 From (12),

$$\frac{\partial \sigma_s}{\partial t} = -\mathbf{n}\cdot(\mathbf{J}^a - \mathbf{J}^b) = -(0) + J_z^b(z=0) = J_o(x,y)\cos(\omega t) \tag{1}$$

Integration of this expression on time gives

$$\sigma_s = \frac{J_o(x,y)}{\omega}\sin\omega t \tag{2}$$

where the integration function of (x, y) is zero because, at every point on the surface, the surface charge density is initially zero.

1.5.4 The charge conservation continuity condition is applied to the surface at $r = R$, where $\mathbf{J}^b = 0$ and $\mathbf{n} = \mathbf{i}_r$. Thus,

$$J_o(\phi, z)\sin\omega t + \frac{\partial \sigma_s}{\partial t} = 0 \tag{1}$$

and it follows that

$$\sigma_s = -\int_0^t J_o(\phi, z)\sin\omega t dt = \frac{J_o(\phi, z)}{\omega}\cos\omega t \tag{2}$$

1.6 FARADAY'S INTEGRAL LAW

1.6.1 (a) On the contour $y = sx/g$,

$$d\mathbf{s} = dx\mathbf{i_x} + dy\mathbf{i_y} = dx\big(\mathbf{i_x} + \frac{dy}{dx}\mathbf{i_y}\big) = dx\big(\mathbf{i_x} + \frac{s}{g}\mathbf{i_y}\big) \tag{1}$$

(b) On this contour,

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^b E_o\mathbf{i_y}\cdot\big(\mathbf{i_x} + \frac{s}{g}\mathbf{i_y}\big)dx = \int_0^g \frac{E_o s}{g}dx = E_o s \tag{2}$$

while the line integral from $(x, y) = (g, s)$ [from $b \to c$] to $(0, s)$ along $y = s$ is zero because $\mathbf{E}\cdot d\mathbf{s} = 0$. The integral over the third segment, $[c \to a]$, is

$$\int_c^a \mathbf{E}\cdot d\mathbf{s} = \int_0^s E_o\mathbf{i_y}\cdot(-\mathbf{i_y}\,dy) = -E_o s \tag{3}$$

so that

$$\oint \mathbf{E}\cdot d\mathbf{s} = E_o s - E_o s = 0 \tag{4}$$

and the circulation is indeed zero.

1.6.2 (a) The solution is as in Prob. 1.6.1 except that $dy/dx = 2sx/g^2$. Thus, the first line integral gives the same answer.

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^g E_o\mathbf{i_y}\cdot\big(\mathbf{i_x} + \frac{2s}{g^2}x\mathbf{i_y}\big)dx = \int_0^g \frac{E_o 2sx}{g^2}dx = E_o s \tag{1}$$

Because the other contours are the same as in Prob. 1.6.1, their contributions are also the same and the net circulation is again found to be zero.

(b) The first integral is as in (b) of Prob. 1.6.2 except that the differential line element is described as in (1) and the field has the given dependence on x.

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^g \big(\frac{E_o x}{g}\big)\big(\frac{2sx}{g^2}\big)dx = \frac{2}{3}\frac{E_o s}{g^3}x^2\Big|_0^g = \frac{2}{3}E_o s \tag{2}$$

(Note that we would now get a different answer, $E_o s/2$, if we carried out this integral using this field but the straight-line contour of Prob. 1.6.1.) From $b \to c$ there is again no contribution because $\mathbf{E}\cdot d\mathbf{s} = 0$ while from $c \to a$, the integral is

$$\int_0^s -E_o\frac{x}{g}\Big|_{x=0}dy = -\frac{E_o xy}{g}\Big|_{x=0} = 0 \tag{3}$$

which makes no contribution because the contour is at $x = 0$. Thus, the net contribution to the closed integral, the circulation, is given by (2).

1.6.3 (a) The conversion to cylindrical coordinates of (1.3.13) follows from the arguments given with the solution to Prob. 1.3.1.

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = \frac{\lambda_l}{2\pi\epsilon_o\sqrt{x^2+y^2}}\left[\frac{x}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{i_y}\right] \tag{1}$$

(b) Evaluation of the line integral amounts to recognizing that on the four segments,

$$d\mathbf{s} = \mathbf{i_x}dx, \quad \mathbf{i_y}dy, \quad -\mathbf{i_x}dx, \quad -\mathbf{i_y}dy \tag{2}$$

respectively. Note that care is taken to take the endpoint of the integrals as being in the direction of an increasing coordinate. This avoids taking double account of the sign implied by the dot product $\mathbf{E}\cdot d\mathbf{s}$.

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \frac{\lambda_l}{2\pi\epsilon_o}\Big[&\int_k^g E_x(x,0)dx \\ &+ \int_0^h E_y(g,y)dy + \int_k^g E_x(x,h)(-dy) + \int_0^h E_y(k,y)(-dy)\end{aligned} \tag{3}$$

These integrals become

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \Big\{&ln(g/k) + \frac{1}{2}ln(g^2+h^2) - \frac{1}{2}ln g^2 - \frac{1}{2}ln(h^2+g^2) \\ &+\frac{1}{2}ln(h^2+k^2) - \frac{1}{2}ln(k^2+h^2) + \frac{1}{2}ln k^2\Big\} = 0\end{aligned} \tag{4}$$

and it follows that the sum of these contributions is indeed zero.

1.6.4 Starting at $(x,y)=(s,0)$, the line integral is

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = &\int_s^d E_x(x,0)dx + \int_0^d E_y(d,y)dy - \int_0^d E_x(x,d)dx \\ &- \int_s^d E_y(0,y)dy + \int_0^s E_x(x,s)dx - \int_0^s E_y(s,y)dy\end{aligned} \tag{1}$$

This expression is evaluated using $\mathbf{E}$ as given by (a) of Prob. 1.6.3 and becomes

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \frac{\lambda_l}{2\pi\epsilon_o}\Big[&\int_s^d \frac{dx}{x} + \int_0^d \frac{y}{d^2+y^2}dy - \int_0^d \frac{x}{x^2+d^2}dx \\ &- \int_s^d \frac{dy}{y} + \int_0^s \frac{x}{x^2+s^2}dx - \int_0^s \frac{y}{s^2+y^2}dy\Big] = 0\end{aligned} \tag{3}$$

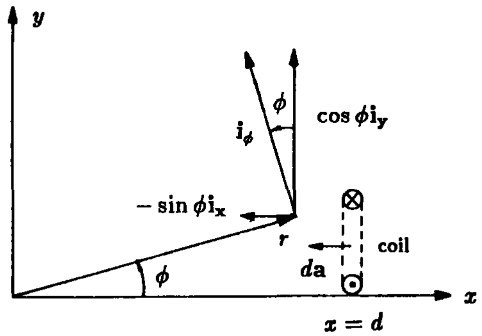


Figure S1.6.5

1.6.5 (a) In view of Fig. S1.6.5, the magnetic field given by (1.4.10)

$$\mathbf{H} = \mathbf{i}_\phi\left(\frac{i}{2\pi r}\right) \tag{1}$$

is converted to Cartesian coordinates by recognizing that

$$\mathbf{i}_\phi = -\sin\phi\mathbf{i_x} + \cos\phi\mathbf{i_y} = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{i_y}; \quad r = \sqrt{x^2+y^2} \tag{2}$$

so that (1) becomes

$$\mathbf{H} = \frac{i}{2\pi}\left[\frac{-y}{x^2+y^2}\mathbf{i_x} + \frac{x}{x^2+y^2}\mathbf{i_y}\right] \tag{3}$$

(b) The surface of Fig. 1.7.2a, shown in terms of the $x - y$ coordinates by Fig. S1.6.5, can be used to evaluate the net flux as follows.

$$\begin{aligned}\lambda_f &= \int_S \mu_o \mathbf{H}\cdot d\mathbf{a} = l\int_0^{\sqrt{R^2-d^2}} -\mu_o H_x(d,y)dy \\ &= -\frac{l\mu_o i}{2\pi}\int_0^{\sqrt{R^2-d^2}} \frac{(-y)}{d^2+y^2}dy = \frac{\mu_o l i}{2\pi}ln(R/d)\end{aligned} \tag{4}$$

This result agrees with (1.7.5), where the flux is evaluated using a different surface. Just why the flux is the same, regardless of surface, is the point of Sec. 1.7.

(c) The circulation follows from Faraday's law, (1.6.1),

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = -\frac{d\lambda_f}{dt} = -\frac{\mu_o l}{2\pi}ln(R/d)\frac{di}{dt} \tag{5}$$

(d) This flux will be linked N times by an N turn coil. Thus, the EMF at the terminals of the coil follows from (8) as

$$\mathcal{E}_{ab} = \frac{\mu_o l N}{2\pi}ln(R/d)\frac{di}{dt} \tag{6}$$

1.6.6 The left hand side of (1.6.1) is the desired circulation of $\mathbf{E}$, found by determining the right hand side, where $d\mathbf{s} = \mathbf{i}_y dxdz$.

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{s} &= -\frac{d}{dt}\int_S \mu_o \mathbf{H} \cdot d\mathbf{s} \\ &= -\frac{d}{dt}\int_{-l/2}^{l/2}\int_0^w \mu_o H_y(x,0,z)dxdz \\ &= -\mu_o wl\frac{dH_o}{dt}\end{aligned} \quad (1)$$

1.6.7 From (12), the tangential component of $\mathbf{E}$ must be continuous, so

$$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0 \Rightarrow E_x^a - E_1 = 0 \Rightarrow E_y^a = E_1 \quad (1)$$

From (1.3.17),

$$\epsilon_o E_y^a - \epsilon_o E_2 = \sigma_o \Rightarrow E_y^a = \frac{\sigma_o}{\epsilon_o} + E_2 \quad (2)$$

These are components of the given electric field just above the $y = 0$ surface.

1.6.8 In polar coordinates,

$$\mathbf{E} = E_o(\sin\phi \mathbf{i}_r + \cos\phi \mathbf{i}_\phi) \quad (1)$$

The tangential component follows from (1.6.12)

$$E_\phi(r = R^+) = E_\phi(r = R^-) = E_o \cos\phi \quad (2)$$

while the normal is given by using (1.3.17)

$$E_r(r = R^+) = \frac{\sigma_o}{\epsilon_o}\cos\phi + E_o \sin\phi \quad (3)$$

1.7 GAUSS' INTEGRAL LAW OF MAGNETIC FLUX

1.7.1 (a) In analyzing the z directed field, note that it is perpendicular to the ϕ axis and, for $0 < \theta < \pi/2$, in the negative θ direction.

$$\mathbf{H} = H_o(\cos\theta \mathbf{i}_r - \sin\theta \mathbf{i}_\theta) \quad (1)$$

(b) Faraday's law, (1.6.1), gives the required circulation in terms of the surface integral on the right. This integral is carried out for the given surface by simply multiplying the z component of $\mathbf{H}$ by the area. The result is as given.

(c) For the hemispherical surface with its edge the same as in part (b), the normal is in the radial direction and it follows from (1) that

$$\mu_o \mathbf{H} \cdot d\mathbf{s} = (\mu_o H_o \cos\theta) r \sin\theta d\theta r d\phi \tag{2}$$

Thus, the surface integral becomes

$$\begin{aligned}\int_S \mu_o \mathbf{H} \cdot d\mathbf{a} &= \int_0^{2\pi}\int_0^{\pi/2} \mu_o H_o R^2 \cos\theta \sin\theta d\theta d\phi \\ &= \mu_o H_o R^2 (2\pi)(1/2)\end{aligned} \tag{3}$$

so that Faraday's law again gives

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\mu_o \pi R^2 \frac{dH_o}{dt} \tag{4}$$

1.7.2 The first only has contributions on the right and left surfaces, where it is of the same magnitude. Because the normals are oppositely directed on these surfaces, these integrals cancel. Thus, (a) satisfies (1.7.1).

The contributions of (b) are to the top and bottom surfaces. Because $\mathbf{H}$ differs on these two surfaces ($x = x$ on the upper surface while $x = 0$ on the lower one), this $\mathbf{H}$ has a net flux.

$$\oint_S \mathbf{H} \cdot d\mathbf{s} = \frac{AH_o x}{d} \tag{1}$$

As for (b), the top and bottom surfaces are where the only contributions can be made. This time, however, there is no net contribution because $\mathbf{H}$ does not depend on x. Thus, at each location y on the upper surface where there is a positive contribution, there is one at the same location y on the lower surface that makes a contribution of the opposite sign.

1.7.3 Continuity of the normal flux density,(1.7.6), requires that

$$\mu_0 H_y^a - \mu_o H_1 = 0 \Rightarrow H_y^a = H_1 \tag{1}$$

while Ampère's continuity condition, (1.4.16) requires that the jump in tangential $\mathbf{H}$ be equal to the given current density. Using the right hand rule,

$$H_z^a - H_2 = K_o \Rightarrow H_z^a = K_o + H_2 \tag{2}$$

These are the components of the given $\mathbf{H}$ just above the surface.

1.7.4 Given that the tangential component of $\mathbf{H}$ is zero inside the cylinder, it follows from Ampère's continuity condition, (1.4.16), that

$$H_\phi(r = R_+) = K_o \tag{1}$$

According to (1.7.6), the normal component of $\mu_o\mathbf{H}$ is continuous. Thus,

$$\mu_o H_r(r = R_+) = \mu_o H_r(r = R_-) = H_1 \tag{2}$$

SOLUTIONS TO CHAPTER 2

2.1 THE DIVERGENCE OPERATOR

2.1.1 From (2.1.5)

$$\begin{aligned} \mathrm{Div}\mathbf{A} &= \frac{\partial(A_x)}{\partial x} + \frac{\partial(A_y)}{\partial y} + \frac{\partial(A_z)}{\partial z} \\ &= \frac{A_o}{d^2}\Big[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)\Big] && (1) \\ &= \frac{2A_o}{d^2}(x+y+z) && (2) \end{aligned}$$

2.1.2 (a) From (2.1.5), operating on each vector

$$\nabla\cdot\mathbf{A} = \frac{A_o}{d}\Big[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x)\Big] = 0 \tag{1}$$

$$\nabla\cdot\mathbf{A} = \frac{A_o}{d}\Big[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y)\Big] = 0 \tag{2}$$

$$\begin{aligned} \nabla\cdot\mathbf{A} &= A_o\Big[\frac{\partial}{\partial x}(e^{-ky}\cos kx) - \frac{\partial}{\partial y}(e^{-ky}\sin kx)\Big] \\ &= A_o\big[-ke^{-ky}\sin kx + ke^{-ky}\sin kx\big] = 0 \end{aligned} \tag{3}$$

(b) All vectors having only one Cartesian component, a (non-constant) function of the coordinate corresonding to that component. For example, $\mathbf{A} = \mathbf{i_x} f(x)$ or $\mathbf{A} = \mathbf{i_y} g(y)$ where $f(x)$ and $g(y)$ are not constants. The example of Prob. 2.1.1 is a superposition of these possibilities.

2.1.3 From Table I

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{1}$$

Thus, for (a)

$$\begin{aligned} \nabla\cdot\mathbf{A} &= \frac{A_o}{d}\Big[\frac{1}{r}\frac{\partial}{\partial r}(r^2\cos 2\phi) - \frac{\partial}{\partial \phi}(\sin 2\phi)\Big] \\ &= \frac{A_o}{d}[2\cos 2\phi - 2\cos 2\phi] = 0 \end{aligned} \tag{2}$$

for (b)

$$\nabla\cdot\mathbf{A} = A_o\Big[\frac{1}{r}\frac{\partial}{\partial r} r\cos\phi - \frac{1}{r}\frac{\partial}{\partial \phi}\sin\phi\Big] = 0 \tag{3}$$

while for (c)

$$\nabla\cdot\mathbf{A} = \frac{A_o}{d^2}\frac{1}{r}\frac{\partial}{\partial r}r^3 = \frac{A_o}{d^2}3r \tag{4}$$

2.1.4 From (2),

$$\text{Div}\mathbf{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot d\mathbf{a} \tag{1}$$

Following steps like (2.1.3)-(2.1.5)

$$\begin{aligned}\oint_S \mathbf{A} \cdot d\mathbf{a} \simeq & \Delta\phi\Delta z\big[(r + \frac{\delta r}{2})A_r(r + \frac{\delta r}{2}, \phi, z)\big] \\ & - \Delta\phi\Delta z\big[(r - \frac{\Delta r}{2})A_r(r - \frac{\Delta r}{2}, \phi, z)\big] \\ & + \Delta r\Delta z\big[A_\phi(r, \phi + \frac{\Delta\phi}{2}, z) - A_\phi(r, \phi - \frac{\Delta\phi}{2}, z)\big] \\ & + r\Delta\phi\Delta r\big[A_z(r, \phi, z + \frac{\Delta z}{2}) - A_z(r, \phi, z - \frac{\Delta z}{2})\big]\end{aligned} \tag{2}$$

Thus, the limit

$$\begin{aligned}\text{Div}\mathbf{A} = & \lim_{r\Delta\phi\Delta z \to 0} \\ & \Big\{r\Delta\phi\Delta z \frac{\big[(r + \Delta r)A_r(r + \frac{\Delta r}{2}, \phi, z) - (r - \frac{\Delta r}{2})A_r(r - \frac{\Delta r}{2}, \phi, z)\big]}{r\Delta\phi\Delta z\Delta r} \\ & + \frac{\big[A_\phi(r, \phi + \frac{\Delta\phi}{2}, z) - A_\phi(r, \phi - \frac{\Delta\phi}{2}, z)\big]}{r\Delta\phi} \\ & + \frac{\big[A_z(r, \phi, z + \frac{\Delta z}{2}) - A_z(r, \phi, z - \frac{\Delta z}{2})\big]}{\Delta z}\Big\}\end{aligned} \tag{3}$$

gives the result summarized in Table I.

2.1.5 From Table I,

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1}$$

For (a)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^3}\big[\frac{1}{r^2}\frac{\partial}{\partial r}(r^5)\big] = \frac{A_o}{d^3}(5r^2) \tag{2}$$

for (b)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2) = 0 \tag{3}$$

and for (c)

$$\begin{aligned}\nabla \cdot \mathbf{A} &= A_o\big[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cos\theta) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\big] \\ &= A_o\big[\frac{2}{r}\cos\theta - \frac{2\sin\theta\cos\theta}{r\sin\theta}\big] = 0\end{aligned} \tag{4}$$

2.1.6 Starting with (2) and using the volume element shown in Fig. S2.1.6,

$$\begin{aligned}\oint_S \mathbf{A}\cdot d\mathbf{a} = \lim_{(\Delta r)(r\Delta\theta)(r\sin\theta\Delta\phi)\to 0} \Big\{ & (r+\frac{\Delta r}{2})\Delta\theta(r+\frac{\Delta r}{2})\sin\theta\Delta\phi A_r(r+\frac{\Delta r}{2},\theta,\phi) \\ & -(r-\frac{\Delta r}{2})\Delta\theta(r-\frac{\Delta r}{2})\sin\theta\Delta\phi A_r(r-\frac{\Delta r}{2},\theta,\phi) \\ & +\Delta r r\Delta\phi\big[\sin(\theta+\frac{\Delta\theta}{2})A_\phi(r,\theta+\frac{\Delta\theta}{2},\phi) \\ & -\sin(\theta-\frac{\Delta\theta}{2})A_\phi(r,\theta-\frac{\Delta\theta}{2},\phi)\big] \\ & +r\Delta\theta\Delta r\big[A_\phi(r,\theta,\phi+\frac{\Delta\phi}{2})-A_\phi(r,\theta,\phi-\frac{\Delta\phi}{2})\big]\Big\}\end{aligned} \tag{1}$$

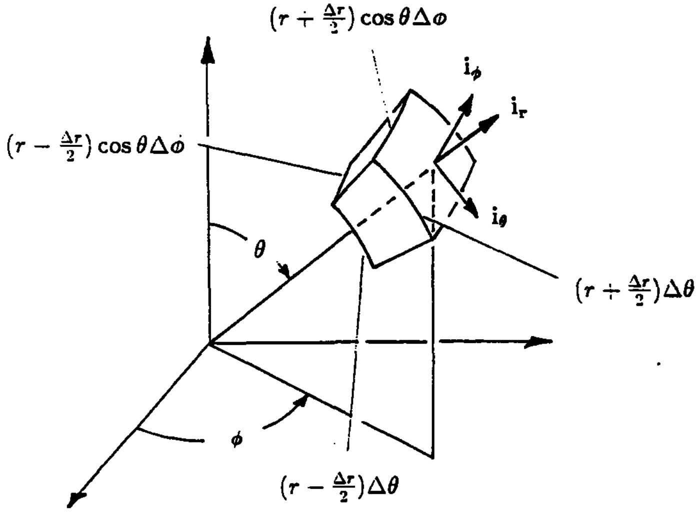


Figure S2.1.6

Thus,

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{\oint_S \mathbf{A}\cdot d\mathbf{a}}{(\Delta r)(r\Delta\theta)(r\sin\theta\Delta\phi)} \\ &= \lim_{\Delta r\to 0}\Big\{\frac{1}{r^2}\frac{\big[(r+\frac{\Delta r}{2})^2A_r(r+\frac{\Delta r}{2},\theta,\phi)-(r-\frac{\Delta r}{2})^2A_r(r-\frac{\Delta r}{2},\theta,\phi)\big]}{\Delta r} \\ &+\lim_{\Delta\theta\to 0}\sin\theta\frac{1}{r}\frac{\big[\sin(\theta+\frac{\Delta\theta}{2})A_\theta(r,\theta+\frac{\Delta\theta}{2},\phi)-\sin(\theta-\frac{\Delta\theta}{2})A_\theta(r,\theta-\frac{\Delta\theta}{2},\phi)\big]}{\Delta\theta} \\ &+\lim_{\Delta\phi\to 0}\frac{1}{r\sin\theta}\frac{\big[A_\phi(r,\theta,\phi+\frac{\Delta\phi}{2})-A_\phi(r,\theta,\phi-\frac{\Delta\phi}{2})\big]}{\Delta\phi}\Big\}\end{aligned} \tag{2}$$

In the limit

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{3}$$

2.2 GAUSS' INTEGRAL THEOREM

2.2.1

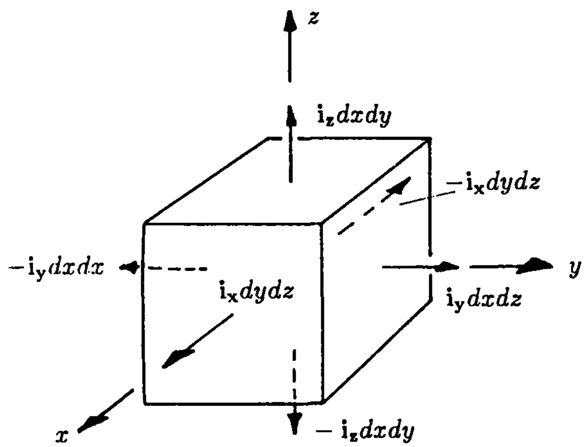


Figure S2.2.1

(a) The vector surface elements are shown in Fig. S2.2.1.

(b) There is no z contribution, so there are only $x = \pm d$ surfaces, $A_x = (A_o/d)(\pm d)$ and $\mathbf{n} = \pm\mathbf{i}_x dydz$. Hence, the first two integrals. The second and third are similar.

(c) From (2.1.5)

$$\nabla\cdot\mathbf{A} = \frac{A_o}{d}\Big[\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\Big] = \frac{2A_o}{d} \tag{1}$$

Thus, because $\nabla\cdot\mathbf{A}$ is constant over the volume

$$\int_V \nabla\cdot\mathbf{A}dV = \frac{2A_o}{d}(2d)^3 = 16A_o d^2 \tag{2}$$

2.2.2 The surface integration is

$$\oint_S \mathbf{A}\cdot d\mathbf{a} = \frac{A_o}{d^3}\Big[\int_{-d}^{d}\int_{-d}^{d} dy^2dydz - \int_{-d}^{d}\int_{-d}^{d}(-d)y^2dydz + \int_{-d}^{d}\int_{-d}^{d} dx^2dxdz - \int_{-d}^{d}\int_{-d}^{d}(-d)x^2dxdz \tag{1}$$

From the first integral

$$= \frac{A_o}{d^3}(2d^2)\left(\frac{2}{3}d^3\right) \tag{2}$$

The others give the same contribution, so

$$= \frac{4A_o}{d^3}\frac{4d^5}{3} = \frac{16A_o d^2}{3} \tag{3}$$

To evaluate the right hand side of (2.2.4)

$$\nabla \cdot \mathbf{A} = \frac{A_o}{d^3}\left[\frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}x^2 y\right] = \frac{A_o}{d^3}(y^2 + x^2) \tag{4}$$

So, indeed

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{-d}^{d}\int_{-d}^{d}\int_{-d}^{d} \frac{A_o}{d^3}(y^2 + x^2)dydxdz = \frac{16}{3}A_o d^2 \tag{5}$$

2.3 GAUSS' LAW, MAGNETIC FLUX CONTINUITY AND CHARGE CONSERVATION

2.3.1 (a) From Prob. 1.3.1

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\left[\frac{x}{x^2+y^2}\mathbf{i_x} + \frac{y}{x^2+y^2}\mathbf{i_y}\right] \tag{1}$$

From (2.1.5)

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] \\ &= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right] \\ &= \frac{\lambda}{2\pi\epsilon_o}\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0 \end{aligned} \tag{2}$$

except where $x^2 + y^2 = 0$ (on the z-axis).

(b) In cylindrical coordinates

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_o}\frac{1}{r}\mathbf{i_r} \tag{3}$$

Thus, from Table I,

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\lambda}{2\pi\epsilon_o}\right) = 0 \tag{4}$$

2.3.2 From Table I in cylindrical coordinates with $\partial(\ \ )/\partial\phi$ and $\partial(\ \ )/\partial z = 0$,

$$\nabla \cdot \epsilon_o \mathbf{E} = \frac{\epsilon_o}{r}\frac{\partial}{\partial r}(rE_r) \tag{1}$$

so

$$\nabla \cdot \epsilon_o \mathbf{E} = \frac{\epsilon_o}{r}\frac{\partial}{\partial r}\begin{cases} \rho_o r^4/4\epsilon_o b^2; & r < b \\ \rho_o b^2/4\epsilon_o; & b < r < a \end{cases} \tag{2}$$

$$= \begin{cases} \rho_o r^2/b^2; & r < b \\ 0; & b < r < a \end{cases} \tag{3}$$

2.3.3 Using $\mathbf{H} = H_o(\mathbf{i_x} + \mathbf{i_y})$ in (2.1.5),

$$\nabla \cdot \mu_o \mathbf{H} = \mu_o H_o\Big[\frac{\partial(1)}{\partial x} + \frac{\partial(1)}{\partial y}\Big] = 0 \tag{1}$$

2.3.4 In cylindrical coordinates (Table I):

$$\nabla \cdot \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\phi}{\partial \phi} + \frac{\partial H_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial \phi}\Big(\frac{i}{2\pi r}\Big) = 0 \tag{1}$$

2.3.5 If $\nabla \cdot \mu_o \mathbf{H} = 0$ *everywhere* then the integral of its normal over an arbitrary closed surface in that region will be zero and

(a)

$$\nabla \mu_o \mathbf{H} = 0$$

(b)

$$\nabla \cdot \mu_o \mathbf{H} = \frac{H_o}{a}\frac{\partial x}{\partial x} = \frac{H_o}{a}$$

(c)

$$\nabla \cdot \mu_o \mathbf{H} = \frac{H_o}{a}\frac{\partial y}{\partial x} = 0$$

Thus, only (b) will not satisfy (1.7.1)

2.3.6 Evaluation using (2.1.5) gives

$$\rho = \nabla \cdot \epsilon_o \mathbf{E} = \epsilon_o \frac{\partial E_z}{\partial z} = \frac{2\rho_o}{s} z$$

which is the given charge density.

2.3.7 Using $\nabla \cdot \mathbf{F}$ in spherical coordinates from Table I with $\partial/\partial\theta$ and $\partial/\partial\phi = 0$,

$$\nabla \cdot \mathbf{J} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{3}\frac{d\rho_o}{dt}\right) = -\frac{d\rho_o}{dt}$$

which, since ρ_o is independent of r, checks with (2.3.3).

2.4 THE CURL OPERATOR

2.4.1 All cases have only x and y components, independent of z.

$$\nabla \times \mathbf{A} = \begin{bmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ A_x & A_y & 0 \end{bmatrix} = \mathbf{i}_z\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

Thus
(a)

$$\nabla \times \mathbf{A} = \frac{A_o}{d}[1-1] = 0 \tag{1}$$

(b)

$$\nabla \times \mathbf{A} = \frac{A_o}{d}[0-0] = 0 \tag{2}$$

(c)

$$\nabla \times \mathbf{A} = A_o[-e^{-ky}\cos kx + ke^{-ky}\cos kx] = 0 \tag{3}$$

To make a finite curl make a single component having any dependence on a coordinate perpendicular to the vector.

$$A_y = f(x), \qquad A_x = 0,\ A_z = 0 \tag{4}$$

Say,

$$f(x) = x, x^2, x^3 \Rightarrow \nabla \times \mathbf{A} = \frac{\partial f(x)}{\partial x} = 1, 2x, 3x^2 \tag{5}$$

2.4.2 In all cases $A_z = 0$ and $\partial/\partial z = 0$, so from Table I,

$$\nabla \times \mathbf{A} = \mathbf{i}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right] \tag{1}$$

(a) Thus

$$\begin{aligned}(a) \Rightarrow \nabla \times \mathbf{A} &= \mathbf{i}_z\frac{A_o}{d}\left[\frac{1}{r}\frac{\partial}{\partial r}(-r^2\sin 2\phi) - \frac{1}{r}\frac{\partial}{\partial \phi}(r\cos 2\phi)\right] \\ &= \mathbf{i}_z\frac{A_o}{d}[-2\sin 2\phi + 2\sin 2\phi] = 0\end{aligned} \tag{2}$$

$$(b) \Rightarrow \nabla \times \mathbf{A} = \mathbf{i}_z A_o\left[\frac{1}{r}\frac{\partial}{\partial r}(-r\sin\phi) - \frac{1}{r}\frac{\partial}{\partial \phi}\cos\phi\right] = \mathbf{i}_z A_o\left[-\frac{\sin\phi}{r} + \frac{\sin\phi}{r}\right] = 0 \quad (3)$$

$$(c) \Rightarrow \nabla \times \mathbf{A} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{A_o r^3}{d^2}\right) = \mathbf{i}_z\left(\frac{3A_o r}{d^2}\right) \quad (4)$$

(b) Possible vector functions having a curl make $\mathbf{A} = A_\phi \mathbf{i}_\phi$ where $rA_\phi = f(r)$ is not a constant. For example $f(r) = r, r^2, r^3$, in which case

$$\nabla \times \mathbf{A} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}(r^n) = nr^{n-2} \quad (5)$$

2.4.3 From (2)

$$(\text{curl}\mathbf{A})_n = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \oint_C \mathbf{A} \cdot d\mathbf{s} \quad (1)$$

Using contour of Fig. P2.4.3a,

$$(\nabla \times \mathbf{A})_r = \lim_{r\Delta\phi\Delta z \to 0} \left\{ \frac{\left[\Delta z A_z(r, \phi + \frac{\Delta\phi}{2}, z) - \Delta z A_z(r, \phi - \frac{\Delta\phi}{2}, z)\right]}{r\Delta\phi\Delta z} - \frac{\left[r\Delta\phi A_\phi(r, \phi, z + \frac{\Delta z}{2}) - r\Delta\phi A_\phi(r, \phi, z - \frac{\Delta z}{2})\right]}{r\Delta\phi\Delta z} \right\} = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \quad (2)$$

Using the contour of Fig. P2.4.3b

$$(\nabla \times \mathbf{A})_\phi = \lim_{\Delta r\Delta z \to 0} \left\{ \frac{\left[\Delta r A_r(r, \phi, z + \frac{\Delta z}{2}) - \Delta r A_r(r, \phi, z - \frac{\Delta z}{2})\right]}{\Delta r\Delta z} - \frac{\left[\Delta z A_z(r + \frac{\Delta r}{2}, \phi, z) - \Delta z A_z(r - \frac{\Delta r}{2}, \phi, z)\right]}{\Delta r\Delta z} \right\} = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \quad (3)$$

$$(\nabla \times \mathbf{A})_z = \lim_{\Delta r r\Delta\phi \to 0} \left\{ \frac{\left[(r + \frac{\Delta r}{2})\Delta\phi A_\phi(r + \frac{\Delta r}{2}, \phi, z) - (r - \frac{\Delta r}{2})\Delta\phi A_\phi(r - \frac{\Delta r}{2}, \phi, z)\right]}{\Delta r r\Delta\phi} - \frac{\left[\Delta r A_r(r, \phi + \frac{\Delta\phi}{2}, z) - \Delta r A_r(r, \phi - \frac{\Delta\phi}{2}, z)\right]}{\Delta r r\Delta\phi} \right\} = \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \quad (4)$$

2.4.4

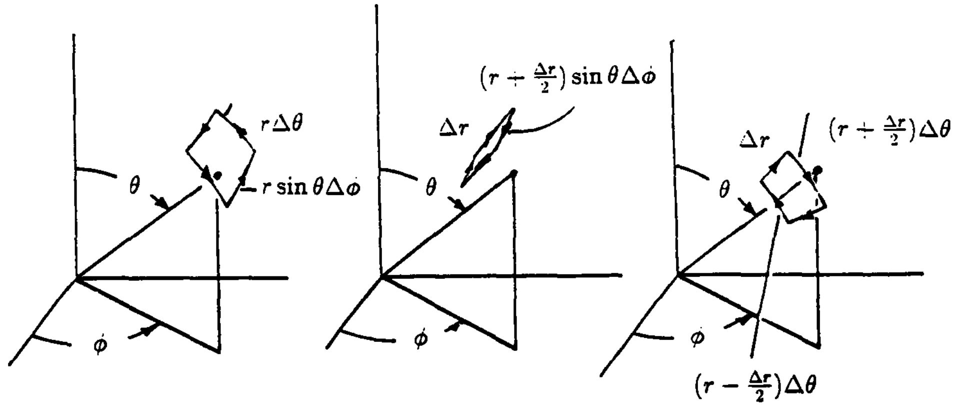


Figure S2.4.4

From (2)

$$(\nabla\times\mathbf{A})_r = \lim_{r\Delta\theta r\sin\theta\Delta\phi\to 0}\left\{-\frac{\left[r\Delta\theta A_\theta\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)-r\Delta\theta A_\theta\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right]}{r\Delta\theta r\sin\theta\Delta\phi}\right.$$
$$\left.+\frac{\left[r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\phi A_\phi\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)-r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\phi A_\phi\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\right]}{r\Delta\theta r\sin\theta\Delta\phi}\right\}$$
$$=-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} \tag{1}$$

$$(\nabla\times\mathbf{A})_\theta = \lim_{\Delta r\sin\theta r\Delta\phi\to 0}\left\{\frac{\left[\Delta r A_r\left(r,\theta,\phi+\frac{\Delta\phi}{2}\right)-\Delta r A_r\left(r,\theta,\phi-\frac{\Delta\phi}{2}\right)\right]}{\Delta r\sin\theta r\Delta\phi}\right.$$
$$\left.-\frac{\left[\Delta\phi\sin\theta\left(r+\frac{\Delta r}{2}\right)A_\phi\left(r+\frac{\Delta r}{2},\theta,\phi\right)-\Delta\phi\sin\phi\left(r-\frac{\Delta r}{2}\right)A_\phi\left(r-\frac{\Delta r}{2},\theta,\phi\right)\right]}{\Delta r\sin\theta r\Delta\phi}\right\}$$
$$=\frac{1}{r(\sin\theta)}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} \tag{2}$$

$$(\nabla\times\mathbf{A})_\phi = \lim_{r\Delta\theta\Delta r\to 0}\left\{\frac{\left[\Delta\theta\left(r+\frac{\Delta r}{2}\right)A_\theta\left(r+\frac{\Delta r}{2},\theta,\phi\right)-\Delta\theta\left(r-\frac{\Delta r}{2}\right)A_\theta\left(r-\frac{\Delta r}{2},\theta,\phi\right)\right]}{r\Delta\theta\Delta r}\right.$$
$$\left.-\frac{\left[\Delta r A_r\left(r,\theta+\frac{\Delta\theta}{2},\phi\right)-\Delta r A_r\left(r,\theta-\frac{\Delta\theta}{2},\phi\right)\right]}{r\Delta\theta\Delta r}\right\}$$
$$=\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta} \tag{3}$$

2.4.5 (a) Stokes' integral theorem, (2.4.1) is

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} \tag{1}$$

With S a closed surface, $C \to 0$, so

$$\oint_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = 0 = \int_V \nabla \cdot (\nabla \times \mathbf{A}) dV \tag{2}$$

Because V is arbitrary, the integrand of this volume integral must be zero.

(b) Carrying out the operations gives

$$\nabla \cdot (\nabla \times \mathbf{A}) = \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] = 0 \tag{3}$$

2.5 STOKES' INTEGRAL THEOREM

2.5.1

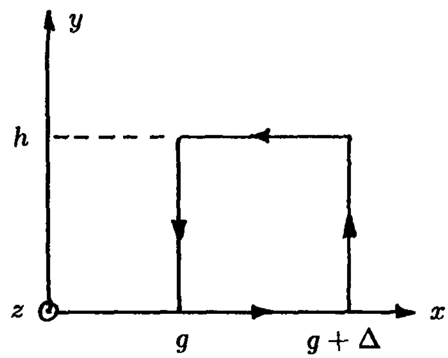


Figure S2.5.1

(a) Using Fig. S2.5.1 to construct $\mathbf{A} \cdot d\mathbf{s}$,

$$\begin{aligned}\oint_C \mathbf{A} \cdot d\mathbf{s} &= \int_g^{g+\Delta} A_x(x,0)dx + \int_0^h A_y(g+\Delta, y)dy \\ &- \int_g^{g+\Delta} A_x(x,h)dx - \int_0^h A_y(g,y)dy \\ &= \int_g^{g+\Delta} (0)dx + \int_0^h \frac{A_o}{d^2}(g+\Delta)^2 dy \\ &- \int_g^{g+\Delta} (0)dx - \int_0^h \frac{A_o}{d^2} g^2 dy \\ &= \frac{A_o}{d^2}[(g+\Delta)^2 h - g^2 h]\end{aligned} \tag{1}$$

(b) The integrand of the surface integral is

$$\nabla\times\mathbf{A} = \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \partial/\partial x & 0 & 0 \\ 0 & A_y & 0 \end{bmatrix} = \mathbf{i_z}\frac{\partial A_y}{\partial x} = \mathbf{i_z}\frac{2A_o x}{d^2}$$

Thus

$$\int_S \nabla\times\mathbf{A}\cdot d\mathbf{a} = \int_0^h \int_g^{g+\Delta} \frac{2A_o x}{d^2} dxdy = \frac{A_o}{d^2}[(g+\Delta)^2 - g^2]h \qquad (2)$$

2.5.2 (a) Using the contour shown in Fig. S2.5.1,

$$\oint_C \mathbf{A}\cdot d\mathbf{s} = \frac{A_o}{d}[\int_g^{g+\Delta}(0)dx + \int_0^h (g+\Delta)dy - \int_g^{g+\Delta}(-h)dx - \int_0^h gdy] \qquad (1)$$

$$= \frac{A_o}{d}[(g+\Delta)h + h\Delta - gh] = \frac{2A_o h\Delta}{d}$$

(b) To get the same result carrying out the surface integral,

$$\nabla\times\mathbf{A} = \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \partial/\partial x & \partial/\partial y & 0 \\ A_x & A_y & 0 \end{bmatrix} = \mathbf{i_z}[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}]$$
$$= \frac{A_o}{d}[1+1] = \frac{2A_o}{d}$$

and hence

$$\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \frac{2A_o}{d}(\Delta h) \qquad (2)$$

2.6 DIFFERENTIAL LAWS OF AMPERE AND FARADAY

2.6.1 From Prob. 1.4.2

$$\mathbf{H} = \frac{J_o}{2}\begin{cases} -y\mathbf{i_x} + x\mathbf{i_y}; & r < b \\ -b^2 y(x^2+y^2)^{-1}\mathbf{i_x} + b^2 x(x^2+y^2)^{-1}\mathbf{i_y}; & b < r < a \end{cases} \qquad (1)$$

Thus,

$$\nabla\times\mathbf{H} = \mathbf{i_z}[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}]$$
$$= \frac{J_o}{2}\begin{cases} 1-(-1) = 2 & r < b \\ \frac{b^2}{x^2+y^2} - \frac{2b^2x^2}{(x^2+y^2)^2} - \frac{(-b^2)}{(x^2+y^2)} + \frac{2(-b^2)y^2}{(x^2+y^2)} = 0; & b < r < a \end{cases} \qquad (2)$$

Thus, $\nabla\times\mathbf{H} = \mathbf{J}$ at each point, r.

2.6.2 Ampére's differential law is written in cylindrical coordinates using the expression for $\nabla \times \mathbf{H}$ from Table I with $\partial/\partial\phi$ and $\partial/\partial z = 0$ and $H_r = 0$, $H_z = 0$. Thus

$$\nabla \times \mathbf{H} = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = \mathbf{i}_z \frac{1}{r}\frac{\partial}{\partial r}\{J_o a^2\big[1 - e^{-r/a}\big(1 + \frac{r}{a}\big)\big]\} = J_o e^{-r/a}\mathbf{i}_z \tag{1}$$

2.7 VISUALIZATION OF FIELDS AND THE DIVERGENCE AND CURL

2.7.1 (a) For ρ and $\mathbf{E}$ given by

$$\rho = \frac{2\rho_o z}{s}$$
$$E_z = \frac{\rho_o}{\epsilon_o s}\big[z^2 - \big(\frac{s}{2}\big)^2\big] \tag{1}$$

the sketch is shown in Fig. S2.7.1

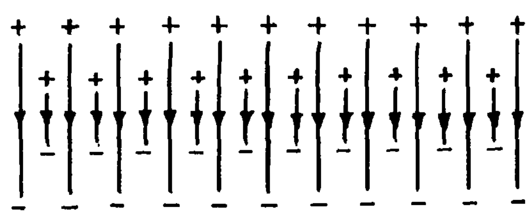


Figure S2.7.1

(b)

$$\nabla \times \mathbf{E} = \begin{bmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ 0 & 0 & \partial/\partial z \\ 0 & 0 & E_z \end{bmatrix} = 0 \tag{2}$$

(c) The density of field lines does not vary in the direction perpendicular to lines.

2.7.2 (a) From Prob. 1.4.1,

$$J_z = J_o e^{-r/a}; \quad H_\phi = \frac{J_o a^2}{r}\big[1 - e^{-r/a}\big(1 + \frac{r}{a}\big)\big] \tag{1}$$

and the field and current plot is as shown in cross-section by Fig. S2.7.2.

(b) From Prob. 1.4.4, the currents are a line current at the origin returned as two surface currents.

$$K_z = \begin{cases} I/\pi(2a + b); & r = a \\ \frac{1}{2}I/\pi(2a + b); & r = b \end{cases} \tag{2}$$

In the annular regions,

$$H_\phi = -\frac{I}{2\pi}\begin{cases} 1/r; & 0 < r < b \\ 2a/r(2a+b); & b < r < a \end{cases} \tag{3}$$

This distribution of current density and magnetic field intensity is shown in cross-section by Fig. S2.7.2.

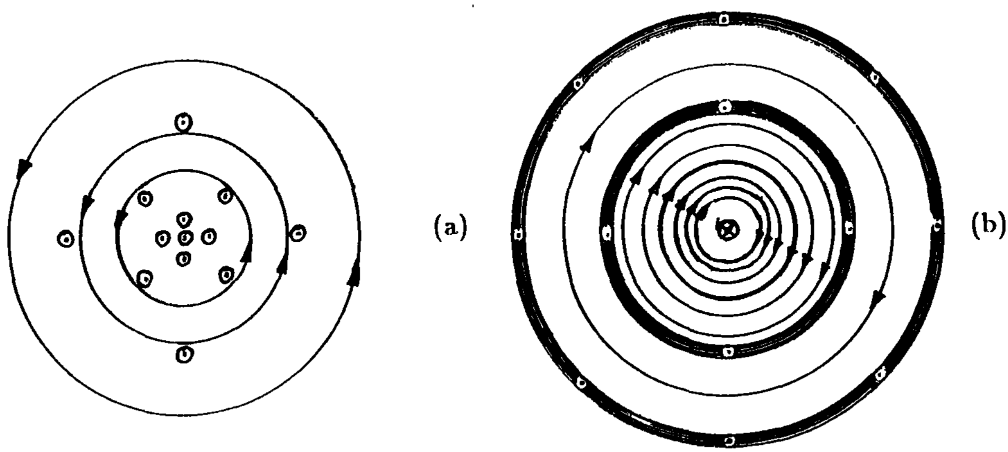


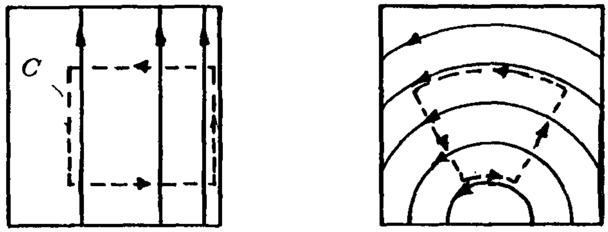
Figure S2.7.2

(c) Because **H** has no ϕ dependence with its only component in the ϕ direction, it must be solenoidal. To check that this is so, note that $\partial/\partial\phi = 0$ and $\partial/\partial z = 0$ and that (from Table I)

$$\nabla \cdot \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) = 0 \tag{4}$$

(d) See (c).

2.7.3 (a) The only irrotational field is (b), where the lines are uniform in the direction perpendicular to their direction. In (a), the line integral of the field around a contour such as that shown in Fig. S2.7.3a must be finite. Similarly, because the field intensity is independent of radius in case (c), the line integral shown in Fig. S2.7.3b must be finite.



(a) (b)

Figure S2.7.3

2.7.4 The respective fields are

$$\mathbf{E} = \frac{\sigma_o}{\epsilon_o}\mathbf{i_x} + \frac{2\sigma_o}{\epsilon_o}\mathbf{i_y} \tag{1}$$

$$\mathbf{E} = \frac{\sigma_o}{\epsilon_o}\mathbf{i_x} + \frac{\sigma_o}{\epsilon_o}\mathbf{i_y} \tag{2}$$

and the field plot is as shown in Fig. S2.7.4. Note that the spacing between lines is lesser above to reflect the greater intensity of the field there.

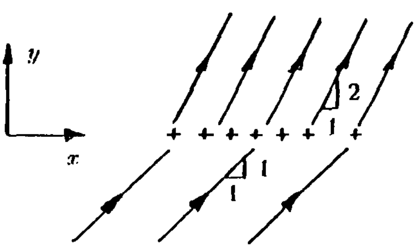


Figure S2.7.4

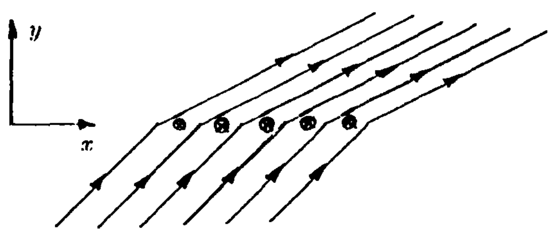


Figure S2.7.5

2.7.5 The respective fields are

$$\mathbf{H} = K_o\mathbf{i_y} + 2K_o\mathbf{i_z} \tag{1}$$

$$\mathbf{H} = K_o\mathbf{i_y} + K_o\mathbf{i_z} \tag{2}$$

and the field plot is as shown in Fig. S2.7.5. Note that, because the field is solenoidal, the number of field lines above and below can be the same while having their spacing reflect the field intensity.

2.7.6 (a) The tangential **E** must be continuous, as shown in Fig. S2.7.6a, so the normal **E** on top must be larger. Because there is than a net flux of **E** out of the interface, it follows from Gauss' integral law [continuity condition (1.3.17)] that the surface charge density is positive.

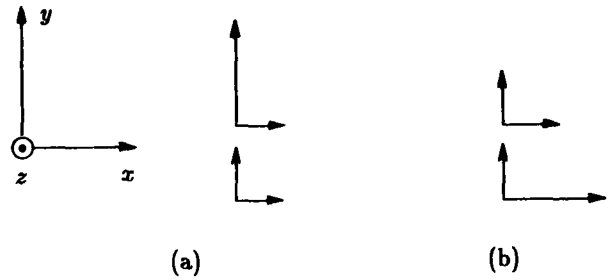


Figure S2.7.6

(b) The normal component of the flux density $\mu_o\mathbf{H}$ is continuous, as shown in Fig. S2.7.6b, so the tangential component on the bottom is largest. From Ampére's integral law [the continuity condition (1.4.16)] it follows that $K_z > 0$.

SOLUTIONS TO CHAPTER 3

3.1 TEMPORAL EVOLUATION OF WORLD GOVERNED BY LAWS OF MAXWELL, LORENTZ, AND NEWTON

3.1.1 (a) Replace z by $z - ct$. Thus

$$\mathbf{E} = E_o \mathbf{i_x} e^{-(z-ct)^2/2a^2}; \quad \mathbf{H} = \sqrt{\frac{\epsilon_o}{\mu_o}} E_o \mathbf{i_y} e^{-(z-ct)^2/2a^2} \tag{1}$$

(b) Because $\partial(\quad)/\partial x = \partial(\quad)/\partial y = 0$ and there are only single components of each field, Maxwell's equations reduce to

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_o}\frac{\partial E_x}{\partial z}; \quad \frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_o}\frac{\partial H_y}{\partial z} \tag{2}$$

Note that we could pick these expressions out of the six components of the laws of Faraday and Ampére by first writing the left hand sides of 3.1.1-2. Thus, these are respectively the y and x components of these laws. In Cartesian coordinates, the divergence equations are automatically satisfied by any vector that only depends on a coordinate perpendicular to its direction. Substitution of (1) into (2a) and into (2b) gives

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \tag{3}$$

which is the velocity of light, in agreement with (3.1.16).

(c) For an observer having the location $z = ct+$ constant, whose position increases linearly with time at the rate c m/s and who therefore has the constant velocity $c, z - ct =$ constant. Thus, the fields given by (1) are constant.

3.1.2 With the given substitution in (3.1.1-4), (with $\mathbf{J} = 0$ and $\rho = 0$)

$$-\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{\epsilon_o}\nabla \times \mathbf{H} \tag{1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_o}\nabla \times \mathbf{E} \tag{2}$$

$$0 = \nabla \cdot \mu_o \mathbf{H} \tag{3}$$

$$0 = -\nabla \cdot \epsilon_o \mathbf{E} \tag{4}$$

Although reordered, the expressions are the same as the original relations.

3.1.3 Note that the direction of wave propagation is obtained by crossing **E** into **H**. Because it would reverse the direction of this cross product, a good guess is to reverse the sign of one or the other of the fields. In that case, the steps followed in Prob. 3.1.1 lead to the requirement that $c = -1/\sqrt{\mu_o\epsilon_o}$. We define c as being positive and so write the solutions with $z-ct$ replaced by $z-(-c)t = z+ct$. Following the same arguments as in part (c) of Prob. 3.1.1, this solution is therefore traveling in the $-z$ direction.

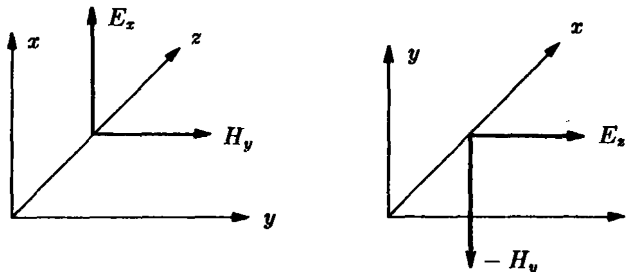


Figure S3.1.4

3.1.4 The role played by z is now taken by x, as shown in Fig. S3.1.4. With the understanding that the z dependence is now replaced by the given x dependence, the magnetic and electric fields are written so that they have the same ratio as in (1) of Prob. 3.1.1. Further, in order to preserve the vector relation between **E**, **H** and the direction of propagation, the sign of **H** is reversed. Thus,

$$\mathbf{E} = E_o\mathbf{i}_z \cos\beta(x-ct); \qquad \mathbf{H} = -\sqrt{\frac{\epsilon_o}{\mu_o}}E_o\mathbf{i}_y \cos\beta(x-ct) \tag{1}$$

3.2 QUASISTATIC LAWS

3.2.1 (a) These fields are transverse to the coordinate, x, upon which they depend. Therefore, the divergence conditions are automatically satisfied. From the direction of the vectors, we know that the x and y components respectively of the laws of Ampére and Faraday will apply.

$$-\frac{\partial H_y}{\partial z} = \frac{\partial \epsilon_o E_x}{\partial t} \tag{1}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial \mu_o H_y}{\partial t} \tag{2}$$

The other four components of these equations are automatically satisfied because $\partial(\ \ )/\partial y = \partial(\ \ )/\partial z = 0$. Substitution of (a) and (b) then gives

$$\beta = \omega\sqrt{\mu_o\epsilon_o} \equiv \frac{\omega}{c} \tag{3}$$

in each case.

(b) The appropriate identities are

$$\cos \beta z \cos \omega t = \frac{1}{2}\left[\cos \beta\left(z - \frac{\omega}{\beta}t\right) + \cos \beta\left(z + \frac{\omega}{\beta}t\right)\right] \tag{4}$$

$$\sin \beta z \sin \omega t = \frac{1}{2}\left[\cos \beta\left(z - \frac{\omega}{\beta}t\right) - \cos \beta\left(z + \frac{\omega}{\beta}t\right)\right] \tag{5}$$

Thus, in view of (3), the fields indeed take the form of the sum of waves traveling in the $+z$ and $-z$ directions with the speed c.

(c) In view of (a), this condition can be written as

$$\beta l = \omega\sqrt{\mu_o \epsilon_o} l = \omega l/c \ll 1 \tag{6}$$

Thus, the condition is equivalent to having the electromagnetic delay time $\tau_{em} \equiv l/c$ short compared to the time $1/\omega$ required for $1/2\pi$ of a cycle.

(d) In the limit of (c), $\cos \beta z \rightarrow 1$ and $\sin \beta z \rightarrow \beta z$ and (a) and (b) become the given fields.

(e) The electric field of (c) is irrotational and hence satisfies (3.2.1a) but not (3.2.1b) while the magnetic field has curl and indeed satisfies (3.2.2a) but not (3.2.2b). Therefore, in the limit of having the frequency low enough to satisfy (6), the system is EQS.

3.2.2 (a) See part (a) of solution to Prob. 3.2.1.

(b) The appropriate identities are

$$\sin(\beta z) \sin(\omega t) = \frac{1}{2}\left[\cos \beta\left(z - \frac{\omega}{\beta}t\right) + \cos \beta\left(z + \frac{\omega}{\beta}t\right)\right] \tag{1}$$

$$\cos(\beta z) \cos(\omega t) = \frac{1}{2}\left[\cos \beta\left(z - \frac{\omega}{\beta}t\right) - \cos \beta\left(z + \frac{\omega}{\beta}t\right)\right] \tag{2}$$

Thus, because $\omega/\beta = c$, the fields indeed take the form of the sum of waves traveling in the $+z$ and $-z$ directions with the speed c.

(c) See (c) of solution to Prob. 3.2.1.

(d) In the limit where $|\beta l| \ll 1$, the given fields become

$$\mathbf{E} \simeq \omega \mu_o H_o z \sin \omega t \mathbf{i_x} \tag{3}$$

$$\mathbf{H} \simeq H_o \cos \omega t \mathbf{i_y} \tag{4}$$

Thus, the magnetic field is uniform while the electric field varies linearly between the source and the "short" at $z = 0$, where it is zero.

(e) The magnetic field of (4) is irrotational and hence satisfies (3.2.2b) with $\mathbf{J} = 0$ but not (3.2.2a). The electric field of (3) does have a curl and hence does not satisfy (3.2.1a) but does satisfy (3.2.1b). Thus, the system is magnetoquasistatic.

3.3 CONDITIONS FOR FIELDS TO BE QUASISTATIC

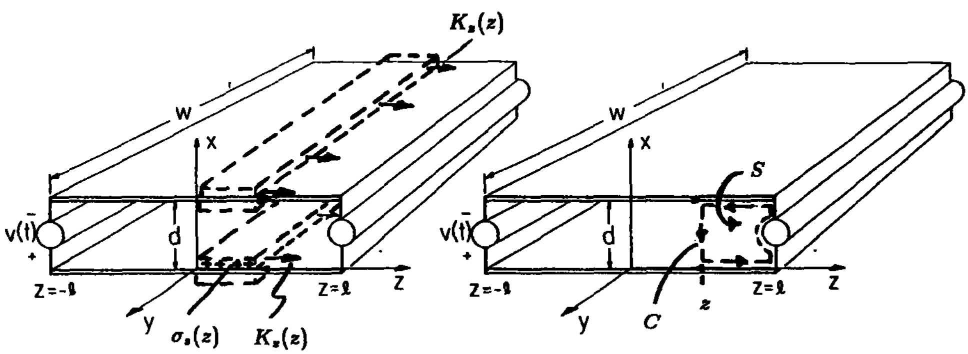
3.3.1 (a) Except that it is in the x direction rather than the z direction, the quasistatic electric field between the plates is, as in Example 3.3.1, uniform. To satisfy the requirement of (a), this field is

$$\mathbf{E} = [v(t)/d]\mathbf{i}_x \tag{1}$$

The surface charge density on the plates follows from Gauss' integral law applied to the plates, much as in (3.3.7).

$$\sigma_s = \begin{cases} -\epsilon_o E_x(x = d) = -\epsilon_o v/a; & x = d \\ \epsilon_o E_x(x = 0) = \epsilon_o v/d; & x = 0 \end{cases} \tag{2}$$

Thus, the quasistatic surface charge density on the interior surfaces of each plate is uniform.



(a) (b)

Figure S3.3.1

(b) The integral form of charge conservation is applied to the lower and upper electrodes using the volume shown in Fig. S3.3.1a. Thus, using symmetry to argue that $K_z = 0$ at $z = 0$, for the lower plate

$$w[K_z(z) - K_z(0)] + \frac{\partial \sigma_s z w}{\partial t} = 0 \Rightarrow K_z(z) = -\frac{z\epsilon_o}{d}\frac{dv}{dt} \tag{3}$$

and we conclude that the surface current density increases linearly from the center toward the edges. At any location z, it is that current required to change the charge on the fraction of "capacitor" at a lesser value of z.

(c) The magnetic field is found using Ampére's integral law, (3.3.9), with the surface $da = \mathbf{i}_x da$ having edges at $z = 0$ and $z = z$. By symmetry, $H_y = 0$ at $z = 0$, so

$$w[-H_y(z) + H_y(0)] = w\int_0^z \epsilon_o E_x dz = \frac{\epsilon_o w}{d} z \frac{dv}{dt} \Rightarrow H_y(z) = \frac{\epsilon_o z}{d}\frac{dv}{dt} \tag{4}$$

Note that, with this field and the surface current density of (3), Ampère's continuity condition, 1.4.16, is satisfied on the upper and lower plates. We could just as well think of the magnetic field as being induced by the surface current of (3) as by the displacement current of (3.3.9).

(d) To determine the correction electric field, use Faraday's integral law with the surface and contour shown in Fig. S3.3.1b, assuming that $\mathbf{E}$ is independent of x.

$$d[E_x(0) - E_x(z)] = -\mu_o d\frac{\partial}{\partial t}\int_z^l H_y dz = \frac{-\mu_o\epsilon_o d}{2d}(l^2 - z^2)\frac{d^2v}{dt^2} \tag{5}$$

Because of (a), it follows that the corrected field is

$$E_x(z) = \frac{v}{d} + \frac{\mu_o\epsilon_o}{2d}(l^2 - z^2)\frac{d^2v}{dt^2} \tag{6}$$

(e) With the second term in (6) called the "correction field," it follows that for the given sinusoidally varying voltage, the ratio of the correction field to the quasistatic field at at most

$$\frac{E_{\text{correction}}}{v/d} = \frac{\mu_o\epsilon_o l^2}{2}\frac{1}{|v|}\left|\frac{d^2v}{dt^2}\right| = \frac{\mu_o\epsilon_o l^2\omega^2}{2} \tag{7}$$

Thus, because $c = 1/\sqrt{\mu_o\epsilon_o}$, the error is negligible if

$$\frac{1}{2}\left[\frac{l}{c}\omega\right] \ll 1 \tag{8}$$

3.3.2 (a) With the understanding that the magnetic field outside the structure is zero, Ampèr'es continuity condition, (1.4.16), requires that

$$0 - H_y = K_y = K \qquad \text{top plate}$$

$$H_y - 0 = K_y = -K \qquad \text{bottom plate} \tag{1}$$

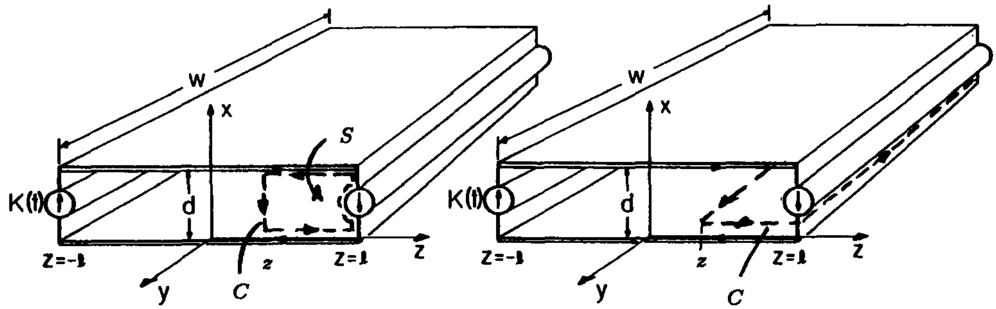
where it is recognized that if the current is essentially steady, the surface current densities must be of equal magnitude $K(t)$ and opposite directions in the top and bottom plates. These boundary conditions also require that

$$\mathbf{H} = -\mathbf{i}_y K(t) \tag{2}$$

at the surface current density sources at the left and right as well. Thus, provided $K(t)$ is essentially steady, (2) is taken as holding everywhere between the plates. Note that this uniform distribution of field not only satisfies the boundary conditions, but also has no curl and hence satisfies the steady form of Ampère's law, (3.2.2b), in the region between the plates where $J = 0$.

(b) The integral form of Faraday's law is used to compute the electric field caused by the time variation of $K(t)$.

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_S \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{3}$$



(a) (b)

Figure S3.3.2

So that it links the magnetic flux, the surface is chosen to be in the $x - z$ plane, as shown in Fig. S3.3.2a. The upper and lower edges are adjacent to the perfect conductor and therefore do not contribute to the line integral of $\mathbf{E}$. The left edge is at $z = 0$ while the right edge is at some arbitrary position z. Thus, with the assumption that E_y is independent of x,

$$d[E_x(z) - E_x(0)] = -\mu_o z d \frac{\partial H_y}{\partial t} = \mu_o z d \frac{dK}{dt} \tag{4}$$

Thus the electric field is $E_x(0)$ plus an odd function of z. Symmetry requires that $E_x(0) = 0$ so that the desired electric field induced through Faraday's law by the time varying magnetic field is

$$E_x(z) = \mu_o z \frac{dK}{dt} \tag{5}$$

Note that the fields given by (2) and (5) satisfy the MQS field laws in the region between the plates.

(c) To compute the correction to $\mathbf{H}$ that results because of the displacement current, we use the integral form of Ampère's law with the surface shown in Fig. S3.3.2. The right edge is at the surface of the current source, where Ampère's continuity condition requires that $H_y(l) = -K(t)$, and the left edge is at the arbitrary location z. Thus,

$$w[-H_y(l) + H_y(z)] = w\epsilon_o \frac{\partial}{\partial t} \int_z^l E_x dz \tag{6}$$

and so, from this first order correction, we have found that the field is

$$H_y = -K(t) + \frac{w\epsilon_o\mu_o}{w}\frac{(l^2 - z^2)}{2}\frac{d^2K}{dt^2} \tag{7}$$

(d) The second term in (7) is the correction field, so, at worst where $z = 0$,

$$\frac{|H_{\text{corrected}}|}{|K|} = \frac{\epsilon_o\mu_o l^2}{2}\frac{1}{|K|}\left|\frac{d^2K}{dt^2}\right| \tag{8}$$

and, for the sinusoidal excitation, we have a negligible correction if

$$\frac{\epsilon_o\mu_o l^2\omega^2}{2} = \frac{1}{2}\left(\frac{l}{c}\omega\right)^2 \tag{9}$$

Thus, the correction can be ignored (and hence the MQS approximation is justified) if the electromagnetic transit time l/c is short compared to the typical time $1/\omega$.

3.4 QUASISTATIC SYSTEMS

3.4.1 (a) Using Ampère's integral law, (3.4.2), with the contour and surface shown in Fig. 3.4.2c gives

$$2\pi r H_\phi = 2\pi b K_o(t) \Rightarrow H_\phi = \frac{b}{r}K_o(t) \tag{1}$$

(b) For essentially steady currents, the net current in the z direction through the inner distributed surface current source must equal that radially outward at any radius r in the upper surface, must equal that in the $-z$ direction in the outer wall and must equal that in the $-r$ direction at any radius r in the lower wall. Thus,

$$\begin{aligned} 2\pi b K_o &= 2\pi r K_r(z = h) = -2\pi a K_z(r = a) = -2\pi r K_r(z = 0) \\ &\Rightarrow K_r(z = h) = \frac{b}{r}K_o;\, K_z(r = a) = \frac{b}{a}K_o;\, K_r(z = 0) = \frac{b}{r}K_o \end{aligned} \tag{2}$$

Note that these surface current densities are what is called for in Ampère's continuity condition, (1.4.16), if the magnetic field given by (1) is to be confined to the annular region.

(c) Faraday's integral law

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_S \mu_o\mathbf{H}\cdot d\mathbf{a} \tag{3}$$

applied to the surface S of Fig. P3.4.2 gives

$$h[E_z(r) - E_z(r = a)] = -\mu_o h \int_r^a H_\phi dr = -\mu_o h b ln(a/r) \frac{dK_o}{dt} \tag{4}$$

Because $E_z(r = a) = 0$, the magnetoquasistatic electric field that goes with (2) in the annular region is therefore

$$E_z = -\mu_o b ln(a/r) \frac{dK_o}{dt} \tag{5}$$

(d) Again, using Ampère's integral law with the contour of Fig. 3.4.2, but this time including the displacement current associated with the time varying electric field of (5), gives

$$2\pi r H_\phi = 2\pi b K_o(t) + \epsilon_o \frac{\partial}{\partial t} \int_b^r E_z 2\pi r dr \tag{6}$$

Note that the first contribution on the right is due to the integral of J associated with the distributed surface current source while the second is due to the displacement current density. Solving (6) for the magnetic field with E_z given by (5) now gives

$$H_\phi = \frac{b}{r} K_o(t) + \frac{\epsilon_o \mu_o b a^2}{r} \Big\{ \Big(\frac{r}{a}\Big)^2 \Big[\frac{1}{2} ln\Big(\frac{r}{a}\Big) - \frac{1}{4}\Big] - \Big(\frac{b}{a}\Big)^2 \Big[\frac{1}{2} ln\Big(\frac{b}{a}\Big) - \frac{1}{4}\Big] \Big\} \frac{d^2 K_o}{dt^2} \tag{7}$$

The last term is the correction to the magnetoquasistatic approximation. Thus, the MQS approximation is appropriate provided that at $r = a$

$$\frac{H_{\text{correction}}}{(b/a)|K_o|} = \epsilon_o \mu_o a^2 \Big\{ \frac{1}{4}\Big[\Big(\frac{b}{a}\Big)^2 - 1\Big] - \frac{1}{2}\Big(\frac{b}{a}\Big)^2 ln\Big(\frac{b}{a}\Big) \Big\} \frac{1}{|K_o|} \Big| \frac{d^2 K_o}{dt^2} \tag{8}$$

(e) In the sinusoidal steady state, (8) becomes

$$\frac{H_{\text{correction}}}{M_{\text{MQS}}} = \Big(\frac{a}{c}\Big)^2 \Big| \frac{1}{4}\Big[\Big(\frac{b}{a}\Big)^2 - 1\Big] - \frac{1}{2}\Big(\frac{b}{a}\Big)^2 ln\Big(\frac{b}{a}\Big) \Big| \omega^2 \ll 1 \tag{9}$$

The term in | | is of the order of unity or smaller. Thus, the MQS approximation holds if the electromagnetic delay time a/c is short compared to the reciprocal typical time $1/\omega$.

SOLUTIONS TO CHAPTER 4

4.1 IRROTATIONAL FIELD REPRESENTED BY SCALAR POTENTIAL: THE GRADIENT OPERATOR AND GRADIENT INTEGRAL THEOREM

4.1.1 (a) For the potential

$$\Phi = \frac{V_o}{a^2}(x^2 + y^2 + z^2) \tag{1}$$

$$\text{grad } \Phi = \frac{2V_o}{a^2}(x\hat{i}_x + y\hat{i}_y + z\hat{i}_z) \tag{2}$$

(b) The unit normal is

$$\mathbf{n} = \frac{\nabla\Phi}{|\nabla\phi|} = \frac{x\hat{i}_x + y\hat{i}_y + z\hat{i}_z}{\sqrt{x^2 + y^2 + z^2}} = \mathbf{i_r} \tag{3}$$

4.1.2 For $\Phi = \frac{V_o}{a^2} xy$, we have

$$\mathbf{E} = -\nabla\Phi = -\frac{V_o}{a^2}(y\mathbf{i_x} + x\mathbf{i_y}) \tag{1}$$

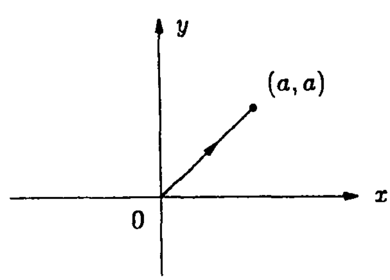


Figure S4.1.2

Integration on the path shown in Fig. S4.1.2 can be accomplished using t as a parameter, where for this curve $x = t$ and $y = d$ so that in

$$d\mathbf{s} = \mathbf{i_x}dx + \mathbf{i_y}dy \tag{2}$$

we can replace $dx = dt, dy = dt$. Thus,

$$\int_{(0,0)}^{(a,a)} \mathbf{E}\cdot d\mathbf{s} = \int_{t=0}^{a} -\frac{V_o}{a^2}(\mathbf{i_x} + \mathbf{i_y})\cdot(\mathbf{i_x} + \mathbf{i_y})dt = -V_o \tag{3}$$

Alternatively, $\Phi(0,0) = 0$ and $\Phi(a,a) = V_o$ and so $\Phi(0,0) - \Phi(a,a) = -V_o$.

4.1.3 (a) The three electric fields are respectively, $\mathbf{E} = -\nabla\Phi$,

$$\mathbf{E} = -(V_o/a)\mathbf{i_x} \tag{1}$$

$$\mathbf{E} = -(V_o/a)\mathbf{i_y} \tag{2}$$

$$\mathbf{E} = -\frac{2V_o}{a^2}(x\mathbf{i_x} - y\mathbf{i_y}) \tag{3}$$

(b) The respective equipotentials and lines of electric field intensity are sketched in the $x - y$ plane in Figs. S4.1.3a-c.

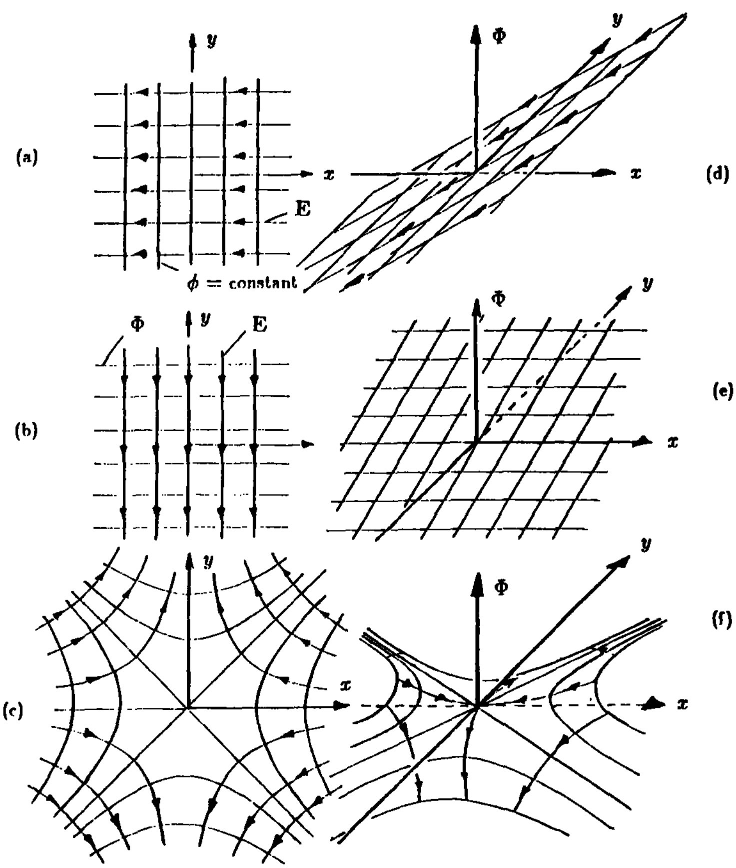


Figure S4.1.3

(c) Alternatively, the vertical axis of a three dimensional plot is used to represent the potential as shown in Figs. S4.1.3d-f.

4.1.4 (a) In Cartesian coordinates, the grad operator is given by (4.1.12). With Φ defined by (a), the desired field is

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \Phi}{\partial x}\mathbf{i_x} - \frac{\partial \Phi}{\partial y}\mathbf{i_y} \\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}\left[\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\mathbf{i_x} + \frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\mathbf{i_y}\right] \end{aligned} \tag{1}$$

(b) Evaluation of the curl gives

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix} = \mathbf{i_z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right] \\ &= \left[\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - \frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\right] \\ &= 0 \end{aligned} \tag{2}$$

so that the field is indeed irrotational.

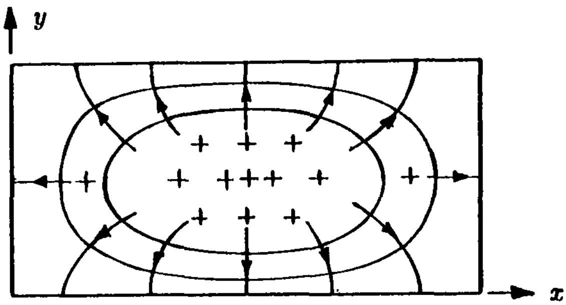


Figure S4.1.4

(c) From Gauss' law, the charge density is given by taking the divergence of (1).

$$\begin{aligned} \rho &= \nabla \cdot \epsilon_o \mathbf{E} = \epsilon_o\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) \\ &= \frac{-\rho_o}{[(\pi/a)^2 + (\pi/b)^2]}\left[-(\pi/a)^2\sin\frac{\pi x}{b} - (\pi/b)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right] \end{aligned} \tag{3}$$

(d) Evalvuation of the tantential component from (1) on each boundary gives; at

$$\begin{aligned} x = 0, E_y = 0; \quad x = a, E_y = 0 \\ y = 0, E_x = 0; \quad y = a, E_x = 0 \end{aligned} \tag{4}$$

(e) A sketch of the potential, the charge density and hence of $\mathbf{E}$ is shown in Fig. S4.1.5.

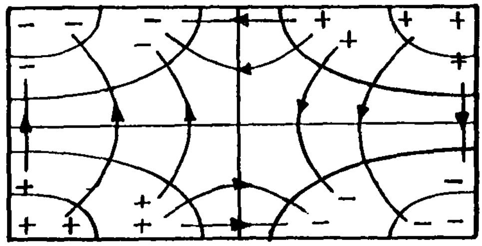


Figure S4.1.5

(f) The integration of $\mathbf{E}$ between points (a) and (b) in FIg. P4.1.5 should be the same as the difference between the potentials evaluated at these end points because of the gradient integral theorem, (16). In this particular case, let $x = t, y = (b/a)t$ so that $dx = dt$ and $dy = (b/a)dt$.

$$\begin{aligned}\int_a^b \mathbf{E}\cdot d\mathbf{s} &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\Big[\frac{\pi}{a}\cos\frac{\pi t}{a}\sin\frac{\pi t}{a}dt \\ &\quad + \frac{\pi}{a}\sin\frac{\pi t}{a}\cos\frac{\pi t}{a}\Big]dt \\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\frac{\pi}{a}\sin\frac{2\pi t}{a}dt \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\end{aligned} \tag{5}$$

The same result is obtained by taking the difference between the potentials.

$$\Phi\Big(\frac{a}{2},\frac{b}{2},t\Big) - \Phi(a,b,t) = \frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]} \tag{6}$$

(g) The net charge follows by integrating the charge density given by (c) over the given volume.

$$Q = \int_V \rho dv = \int_0^d\int_0^b\int_0^a \rho_o \sin(\pi x/a)\sin(\pi y/b)dxdydz = \frac{4\rho_o abd}{\pi^2} \tag{7}$$

From Gauss' integral law, it also follows by integrating the flux density $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the surface enclosing this volume.

$$\begin{aligned}Q = \oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da &= \frac{-\rho_o d}{[(\pi/a)^2+(\pi/b)^2]}\Big\{\int_0^a \frac{\pi}{b}\sin(\pi x/a)\cos\pi dx \\ &\quad - \int_0^a \frac{\pi}{b}\sin(\pi x/a)dx + \int_0^b \frac{\pi}{a}\cos\pi\sin\frac{\pi y}{b}dy \\ &\quad - \int_0^b \frac{\pi}{a}\sin\frac{\pi y}{b}dy\Big\} = \frac{4\rho_o abd}{\pi^2}\end{aligned} \tag{8}$$

(h) The surface charge density on the electrode follows from using the normal electric field as given by (1).

$$\sigma_s = \epsilon_o E_y(y=0) = \frac{-\rho_o}{[(\pi/a)^2 + (\pi/b)^2]} \frac{\pi}{b} \sin \frac{\pi x}{a} \tag{9}$$

Thus, the net charge on this electrode is

$$q = \int_0^d \int_{a/4}^{3a/4} \frac{-\rho_o}{[(\pi/a)^2 + (\pi/b)^2]} \frac{\pi}{b} \sin \frac{\pi x}{a} dx dz = \frac{-\sqrt{2}(a/b) d\rho_o}{(\pi/a)^2 + (\pi/b)^2} \tag{10}$$

(i) The current $i(t)$ then follows from conservation of charge for a surface S that encloses the electrode.

$$\oint_S \mathbf{J} \cdot \mathbf{n} da + \frac{d}{dt} \int_V \rho dv \Rightarrow i + \frac{dq}{dt} = 0 \tag{11}$$

Thus, from (10),

$$i = \frac{\sqrt{2}(a/b)d}{(\pi/a)^2 + (\pi/b)^2} \frac{d\rho_o}{dt} \tag{12}$$

4.1.5 (a) In Cartesian coordinates, the grad operator is given by (4.1.12). With Φ defined by (a), the desired field is

$$\begin{aligned} \mathbf{E} &= -[\frac{\partial \Phi}{\partial x}\mathbf{i_x} + \frac{\partial \Phi}{\partial y}\mathbf{i_y}] \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}[\frac{\pi}{a} \sin \frac{\pi}{a}x \cos \frac{\pi}{b}y\mathbf{i_x} + \frac{\pi}{b} \cos \frac{\pi}{a}x \sin \frac{\pi}{b}y\mathbf{i_y}] \end{aligned} \tag{1}$$

(b) Evaluation of the curl gives

$$\begin{aligned} \nabla \times \mathbf{E} &= \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix} = \mathbf{i_z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}[-\frac{\pi^2}{ab} \sin \frac{\pi}{a}x \sin \frac{\pi}{b}y \\ &+ \frac{\pi^2}{ab} \sin \frac{\pi}{a}x \sin \frac{\pi}{b}y] = 0 \end{aligned} \tag{2}$$

so that the field is indeed irrotational.

(c) From Gauss' law, the charge density is given by taking the divergence of (1).

$$\rho = \nabla \cdot \epsilon_o \mathbf{E} = \epsilon_o(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}) = \rho_o \cos \frac{\pi}{a}x \cos \frac{\pi}{b}y \tag{3}$$

(d) The electric field $\mathbf{E}$ is tangential to the boundaries only if it has no normal component there.

$$E_x(0,y)=0, \quad E_x(a,y)=0$$
$$E_y(x,0)=0, \quad E_y(x,b)=0 \tag{4}$$

(e) A sketch of the potential, the charge density and hence of $\mathbf{E}$ is shown in Fig. S4.1.4.

(f) The integration of $\mathbf{E}$ between points (a) and (b) in Fig. P4.1.4 should be the same as the difference between the potentials evaluated at these end points because of the gradient integral theorem, (16). In this particular case, where $y=(b/a)x$ on C and hence $dy=(b/a)dx$

$$\begin{aligned}\int_{(a)}^{(b)} \mathbf{E}\cdot d\mathbf{s} &= \int_{a/2}^{a}\left\{E_x\left(x,\frac{b}{a}x\right)dx + E_y\left(x,\frac{b}{a}x\right)(b/a)dx\right\}\\ &= \frac{\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\int_{a/2}^{a}\frac{2\pi}{a}\sin\frac{\pi}{a}x\cos\frac{\pi}{a}x dx\\ &= \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]}\end{aligned}$$

The same result is obtained by taking the difference between the potentials.

$$\int_{(a)}^{(b)} \mathbf{E}\cdot d\mathbf{s} = \Phi(a)-\Phi(b) = \frac{-\rho_o}{\epsilon_o[(\pi/a)^2+(\pi/b)^2]} \tag{6}$$

(g) The net charge follows by integrating the charge density over the given volume. However, we can see from the function itself that the positive charge is balanced by the negative charge, so

$$Q=\int_V \rho dV = 0 \tag{7}$$

From Gauss' integral law, the net charge also follows by integrating the flux density $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the surface enclosing this volume. From (d) this normal flux is zero, so that the net integral is certainly also zero.

$$Q=\oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da = 0 \tag{8}$$

The surface charge density on the electrode follows from integrating $\epsilon_o\mathbf{E}\cdot\mathbf{n}$ over the "electrode" surface. Thus, the net charge on the "electrode" is

$$q=\oint_S \epsilon_o\mathbf{E}\cdot\mathbf{n}da = 0 \tag{9}$$

4.1.6 (a) From (4.1.2)

$$\begin{aligned}\mathbf{E} &= -\left(\frac{\partial\Phi}{\partial x}\mathbf{i_x} + \frac{\partial\Phi}{\partial y}\mathbf{i_y}\right) \\ &= -A[m\cosh mx\sin k_y y\sin k_z z\mathbf{i_x} \\ &+ \sinh mx k_y\cos k_y y\sin k_z z\mathbf{i_y} \\ &+ k_z\sinh mx\sin k_y y\cos k_z z\mathbf{i_z}]\sin\omega t\end{aligned} \tag{1}$$

(b) Evaluation using (1) gives

$$\begin{aligned}\bar{\nabla}\times\mathbf{E} &= \begin{vmatrix}\mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ E_x & E_y & E_z\end{vmatrix} \\ &= \mathbf{i_x}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \mathbf{i_y}\left[\frac{\partial E_x}{\partial z} \right. \\ &\left. - \frac{\partial E_z}{\partial x}\right] + \mathbf{i_z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\end{aligned} \tag{2}$$

$$\begin{aligned}&= -A\sin\omega t\{\mathbf{i_x}(k_y k_z\sinh mx\cos k_y y\cos k_z z - k_y k_z\sinh mx\cos k_y y\cos k_z z) \\ &\quad + \mathbf{i_y}(mk_z\cosh mx\sin k_y y\cos k_z z - k_z m\cosh mx\sin k_y y\cos k_z z) \\ &\quad + \mathbf{i_z}(mk_y\cosh mx\cos k_y y\sin k_z z - mk_y\cosh mx\cos k_y y\sin k_z z) \\ &= 0\end{aligned} \tag{3}$$

(c) From Gauss' law, (4.0.2)

$$\rho = \nabla\cdot\epsilon_o\mathbf{E} = -\epsilon_o A(m^2 - k_y^2 - k_z^2)\sinh mx\sin k_y y\sin k_z z\sin\omega t \tag{5}$$

(d) No. The gradient of vector or divergence of scalar are not defined.

(e) For $\rho = 0$ everywhere, make the coefficient in (5) be zero.

$$m^2 = k_y^2 + k_z^2 \tag{6}$$

4.1.7 (a) The wall in the first quadrant is on the surface defined by

$$y = a - x \tag{1}$$

Substitution of this value of y into the given potential shows that on this surface, the potential is a linear function of x and hence the desired linear function of distance along the surface

$$\Phi = Aa(2x - a) \tag{2}$$

To make this potential assume the correct values at the end points, where $x = 0$ and Φ must be $-V$ and where $x = a$ and Φ must be V, make $A = V/a^2$ and hence

$$\Phi = \frac{V}{a^2}(x^2 - y^2) \tag{3}$$

On the remaining surfaces, respectively in the second, third and fourth quadrants

$$y = x + a; \quad y = -a - x; \quad y = x - a \tag{4}$$

Substitution of these functions into (3) also gives linear functions of x which respectively satisfy the conditions on the potentials at the end points.

(b) Using (4.1.12),

$$\mathbf{E} = -\left(\frac{\partial \Phi}{\partial x}\mathbf{i}_x + \frac{\partial \Phi}{\partial y}\mathbf{i}_y\right) = -\frac{V}{a^2}(2x\mathbf{i}_x - 2y\mathbf{i}_y) \tag{5}$$

From Gauss' law, (4.0.2), the charge density is

$$\rho = \epsilon_o\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = -\frac{\epsilon_o V}{a^2}(2 - 2) = 0 \tag{6}$$

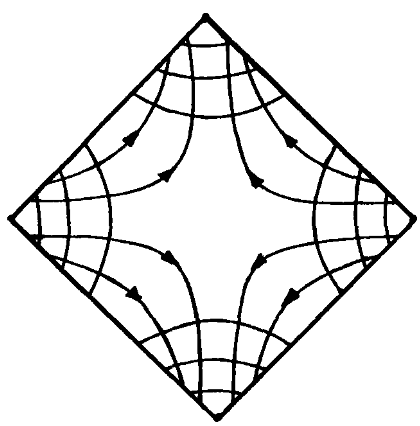


Figure S4.1.7

(c) The equipotentials and lines of $\mathbf{E}$ are shown in Fig. S4.1.7.

4.1.8 (a) For the given $\mathbf{E}$,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \partial/\partial x & \partial/\partial y & 0 \\ Cx & -Cy & 0 \end{vmatrix} = \mathbf{i}_z\left[\frac{\partial}{\partial x}(-Cy) - \frac{\partial}{\partial y}(Cx)\right] = 0 \tag{1}$$

so $\mathbf{E}$ is irrotational. To evaluate C, remember that the vector differential distance $d\mathbf{s} = \mathbf{i}_x dx + \mathbf{i}_y dy$. For ths contour, $d\mathbf{s} = \mathbf{i}_y dy$. To let the integral take account of the sign naturally, the integration is carried out from the origin to (a) (rather than the reverse) and set equal to $\Phi(0,0) - \Phi(0,h) = -V$.

$$-V = \int_0^h -Cy\,dy = -\frac{1}{2}Ch^2 \tag{2}$$

Thus, $C = 2V/h^2$.

(b) To find the potential, observe from $\mathbf{E} = -\nabla\Phi$ that

$$\frac{\partial \Phi}{\partial x} = -Cx; \qquad \frac{\partial \Phi}{\partial y} = Cy \tag{3}$$

Integration of (3a) with respect to x gives

$$\Phi = -\frac{1}{2}Cx^2 + f(y) \tag{4}$$

Differentiation of this expression with respect to y and comparison to (3b) then shows that

$$\frac{\partial \Phi}{\partial y} = \frac{df}{dy} = Cy \Rightarrow f = \frac{1}{2}y^2 + D \tag{5}$$

Because $\Phi(0,0) = 0, D = 0$ so that

$$\Phi = -\frac{1}{2}C(x^2 - y^2) \tag{6}$$

and, because $\Phi(0,h) = V$, it follows that

$$\Phi = -\frac{1}{2}C(0^2 - h^2) \tag{7}$$

so that once again, $C = 2V/h^2$.

(c) The potential and $\mathbf{E}$ are sketched in Fig. S4.1.8a.

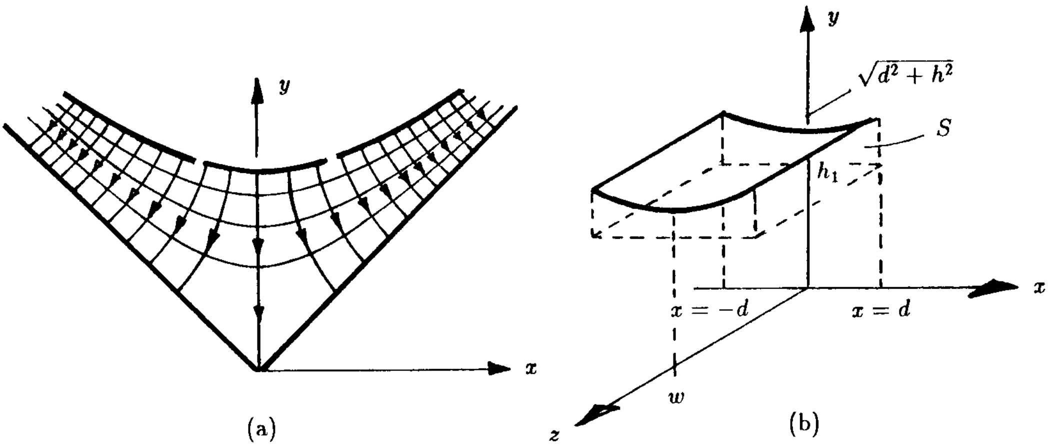


Figure S4.1.8

(d) Gauss' integral law is used to compute the charge on the electrode using the surface shown in Fig. S4.1.8b to enclose the electrode. There are six surfaces possibly contributing to the surface integration.

$$\oint_S \epsilon_o \mathbf{E} \cdot \mathbf{n} da = q \tag{8}$$

On the two having normals in the z direction, $\epsilon_o \mathbf{E} \cdot \mathbf{n} = 0$. In the region above the electrode the field is zero, so there is no contribution there either. On the two side surfaces and the bottom surface, the integrals are

$$\begin{aligned} q =& \epsilon_o \int_0^w \int_{h_1}^{\sqrt{d^2+h^2}} \mathbf{E}(d, y) \cdot \mathbf{i_x} dy dz \\ &+ \epsilon_o \int_0^w \int_{h_1}^{\sqrt{d^2+h^2}} \mathbf{E}(-d, y) \cdot (-\mathbf{i_x}) dy dz \\ &+ \epsilon_o \int_0^w \int_{-d}^{d} \mathbf{E}(x, h_1) \cdot (-\mathbf{i_y}) dx dz \end{aligned} \tag{9}$$

Completion of the integrals gives

$$q = \frac{4wd\epsilon_o\sqrt{d^2+h^2}}{h^2} \tag{10}$$

4.1.9 By definition,

$$\Delta\Phi = \text{grad}\,(\Phi) \cdot \Delta\mathbf{r} \tag{1}$$

In cylindrical coordinates,

$$\Delta\mathbf{r} = \Delta r \mathbf{i_r} + r\Delta\phi \mathbf{i_\phi} + \Delta z \mathbf{i_z} \tag{2}$$

and

$$\begin{aligned} \Delta\phi &= \Phi(r + \Delta r, \Phi + \Delta\Phi, z + \Delta z) - \Phi(r, \phi, z) \\ &= \frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\phi}\Delta\phi + \frac{\partial\Phi}{\partial z}\Delta z \end{aligned} \tag{3}$$

Thus,

$$\frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\phi}\Delta\phi + \frac{\partial\Phi}{\partial z}\Delta z = \text{grad}\,\Phi \cdot (\Delta r \mathbf{i_r} + r\Delta\phi \mathbf{i_\phi} + \Delta z \mathbf{i_z}) \tag{4}$$

and it follows that the gradient operation in cylindrical coordinates is,

$$\text{grad}\,(\Phi) = \frac{\partial\Phi}{\partial r}\mathbf{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\mathbf{i_\phi} + \frac{\partial\Phi}{\partial z}\mathbf{i_z} \tag{5}$$

4.1.10 By definition,

$$\Delta\Phi = \text{grad}\,(\Phi)\cdot\Delta\mathbf{r} \tag{1}$$

In spherical coordinates,

$$\Delta\mathbf{r} = \Delta r\mathbf{i_r} + r\Delta\theta\mathbf{i}_\theta + r\sin\theta\Delta\phi\mathbf{i}_\phi \tag{2}$$

and

$$\begin{aligned}\Delta\Phi &= \Phi(r+\Delta r,\theta+\Delta\theta,\phi+\Delta\phi) - \Phi(r,\theta,\phi)\\ &= \frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\theta}\Delta\theta + \frac{\partial\Phi}{\partial\phi}\Delta\phi\end{aligned} \tag{3}$$

Thus,

$$\frac{\partial\Phi}{\partial r}\Delta r + \frac{\partial\Phi}{\partial\theta}\Delta\theta + \frac{\partial\Phi}{\partial\phi}\Delta\phi = \text{grad}\,(\Phi)\cdot(\Delta r\mathbf{i_r} + r\Delta\theta\mathbf{i}_\theta + r\sin\theta\Delta\phi\mathbf{i}_\phi) \tag{4}$$

and it follows that the gradient operation in spherical coordinates is,

$$\text{grad}(\Phi) = \frac{\partial\Phi}{\partial r}\mathbf{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{i}_\phi \tag{5}$$

4.2 POISSON'S EQUATION

4.2.1 In Cartesian coordinates, Poisson's equation requires that

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_o} \Rightarrow \rho = -\epsilon_o\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) \tag{1}$$

Substitution of the potential

$$\Phi = \frac{\rho_o(t)}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y \tag{2}$$

then gives the charge density

$$\begin{aligned}\rho = &-\frac{\rho_o(t)}{[(\pi/a)^2+(\pi/b)^2]}[-(\pi/a)^2\sin\frac{\pi}{a}x\sin\frac{\pi y}{b}\\ &-(\pi/b)^2\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y]\\ &= \rho_o(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\end{aligned} \tag{3}$$

4.2.2 In Cartesian coordinates, Poisson's equation requires that

$$\rho = -\epsilon_o\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) \tag{1}$$

Substitution of the potential

$$\Phi = \frac{\rho_o}{\epsilon_o[(\pi/a)^2 + (\pi/b)^2]}\cos\frac{\pi}{a}x\cos\frac{\pi}{b}y \tag{2}$$

then gives the charge density

$$\rho = \rho_o\cos\frac{\pi}{a}x\cos\frac{\pi}{b}y \tag{3}$$

4.2.3 In cylindrical coordinates, the divergence and gradient are given in Table I as

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{1}$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial u}{\partial\phi}\mathbf{i}_\phi + \frac{\partial u}{\partial z}\mathbf{i}_z \tag{2}$$

By definition,

$$\nabla^2 u = \nabla\cdot\nabla u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial u}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) \tag{3}$$

which becomes the expression also summarized in Table I.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{4}$$

4.2.4 In spherical coordinates, the divergence and gradient are given in Table I as

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{1}$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\mathbf{i}_\theta \tag{2}$$

By definition,

$$\begin{aligned}\nabla^2 u = \nabla\cdot(\nabla u) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\sin\theta\right)\\ &+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\end{aligned} \tag{3}$$

which becomes the expression also summarized in Table I.

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \tag{4}$$

4.3 SUPERPOSITION PRINCIPLE

4.3.1 The circuit is shown in Fig. S4.3.1. Alternative solutions v_a and v_b must each satisfy the respective equations

$$C\frac{dv_a}{dt} + \frac{v_a}{R} = I_a(t); \tag{1}$$

$$C\frac{dv_b}{dt} + \frac{v_b}{R} = I_b(t) \tag{2}$$

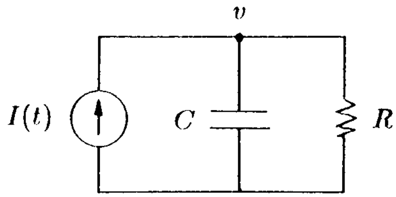


Figure S4.3.1

Addition of these two expressions gives

$$C\Big[\frac{dv_a}{dt} + \frac{dv_b}{dt}\Big] + \frac{1}{R}[v_a + v_b] = I_a + I_b \tag{3}$$

which, by dint of the linear nature of the derivative operator, becomes

$$C\frac{d}{dt}(v_a + v_b) + \frac{1}{R}(v_a + v_b) = I_a + I_b \tag{4}$$

Thus, if $I_a \Rightarrow v_a$ and $I_b \Rightarrow v_b$ then $I_a + I_b \Rightarrow v_a + v_b$.

4.4 FIELDS ASSOCIATED WITH CHARGE SINGULARITIES

4.4.1 (a) The electric field intensity for a line charge having linear density λ_l is

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = -\frac{\partial\Phi}{\partial r}\mathbf{i_r} \tag{1}$$

Integration gives

$$\Phi = -\frac{\lambda_l}{2\pi\epsilon_o} ln(r/r_o) \tag{2}$$

where r_o is the position at which the potential is defined to be zero.

(b) In terms of the distances defined in Fig. S4.4.1, the potential for the pair of line charges is

$$\Phi = -\frac{\lambda_l}{2\pi\epsilon_o} ln\left(\frac{r_+}{r_o}\right) + \frac{\lambda_l}{2\pi\epsilon_o} ln\left(\frac{r_-}{r_o}\right) = \frac{\lambda_l}{2\pi\epsilon_o} ln\left(\frac{r_-}{r_+}\right) \tag{3}$$

where

$$r_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\phi$$

Thus,

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left[\frac{1 + (d/2r)^2 + \frac{d}{r}\cos\phi}{1 + (d/2r)^2 - \frac{d}{r}\cos\phi}\right] \tag{4}$$

For $d \ll r$, this is expanded in a Taylor series

$$ln\left(\frac{1+x}{1+y}\right) = ln(1+x) - ln(1+y) \approx x - y \tag{5}$$

to obtain the standard form of a two-dimensional dipole potential.

$$\Phi \to \frac{\lambda d}{2\pi\epsilon_o}\frac{\cos\phi}{r} \tag{6}$$

4.4.2 From the solution to Prob. 4.4.1, the potential of the pair of line charges is

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left[\frac{1 + (2r/d)^2 + \frac{4r}{d}\cos\phi}{1 + (2r/d)^2 - \frac{4r}{d}\cos\phi}\right] \tag{1}$$

For a spacing that goes to infinity, $r/d \ll 1$ and it is appropriate to use the first term of a Taylor's expansion

$$ln\left(\frac{1+x}{1+y}\right) \simeq x - y \tag{2}$$

Thus, (1) becomes

$$\Phi = \frac{2\lambda}{\pi\epsilon_o d} r\cos\phi \tag{3}$$

In Cartesian coordinates, $x = r\cos\phi$, and (3) becomes

$$\Phi = \frac{2\lambda}{\pi\epsilon_o d} x \tag{4}$$

which is the potential of a uniform electric field.

$$\mathbf{E} = \frac{-2\lambda}{\pi\epsilon_o d}\mathbf{i_x} \tag{5}$$

4.4.3 The potential due to a line charge is

$$\Phi = \frac{\lambda}{2\pi\epsilon_o} ln\frac{r_o}{r} \tag{1}$$

where r_o is some reference. For the quadrapole,

$$\Phi = \frac{\lambda}{2\pi\epsilon_o}\Big[ln\frac{r_o}{r_1} - ln\frac{r_o}{r_2} + ln\frac{r_o}{r_3} - ln\frac{r_o}{r_4}\Big] = \frac{\lambda}{2\pi\epsilon_o}\Big[ln\frac{r_2 r_4}{r_1 r_3}\Big] \tag{2}$$

where, from Fig. P4.4.3,

$$r_1^2 = r^2[1 + (d/2r)^2 - (d/r)\sin\phi]$$

$$r_2^2 = r^2[1 + (d/2r)^2 + (d/r)\cos\phi]$$

$$r_3^2 = r^2[1 + (d/2r)^2 + (d/r)\sin\phi]$$

$$r_4^2 = r^2[1 + (d/2r)^2 - (d/r)\cos\phi]$$

With terms in $(d/2r)^2$ neglected, (2) therefore becomes

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{1-(d/r)^2\cos^2\phi}{1-(d/r)^2\sin^2\phi}\right\} \tag{3}$$

for $d \ll r$.

Now $ln(1+x) \simeq x$ for small x so $ln[(1+x)/(1+y)] \simeq x - y$. Thus, (3) is approximately

$$\begin{aligned}\Phi &= \frac{\lambda}{4\pi\epsilon_o}\big[-(d/r)^2\cos^2\phi + (d/r)^2\sin^2\phi\big] \\ &= \frac{-\lambda d^2}{4\pi\epsilon_o r^2}[\cos^2\phi - \sin^2\phi] \\ &= \frac{-\lambda d^2}{4\pi\epsilon_o r^2}\cos 2\phi\end{aligned} \tag{4}$$

This is of the form $A\cos 2\phi/\gamma^n$ with

$$A = \frac{-\lambda d}{4\pi\epsilon_o}, \qquad n = 2 \tag{5}$$

4.4.4 (a) For $r \ll d$, we rewrite the distance functions as

$$r_1^2 = (d/2)^2\left[\left(\frac{2r}{d}\right)^2 + 1 - \frac{4r}{d}\sin\phi\right] \tag{1a}$$

$$r_2^2 = (d/2)^2\left[\left(\frac{2r}{d}\right)^2 + 1 + \frac{4r}{d}\cos\phi\right] \tag{1b}$$

$$r_3^2 = (d/2)^2\left[\left(\frac{2r}{d}\right)^2 + 1 + \frac{4r}{d}\sin\phi\right] \tag{1c}$$

$$r_4^2 = (d/2)^2\left[\left(\frac{2r}{d}\right)^2 + 1 - \frac{4r}{d}\cos\phi\right] \tag{1a}$$

With the terms $(2r/d)^2$ neglected, at follows that

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{1-(4r/d)^2\cos^2\phi}{1-(4r/d)^2\sin^2\phi}\right\} \tag{2}$$

Because $ln(1+x) \simeq x$ for $x \ll 1, \ln[(1+x)/(1+y)] \simeq x - y$ and (2) is approximately

$$\Phi = -\frac{\lambda}{4\pi\epsilon_o}\left(\frac{4r}{d}\right)^2[\cos^2\phi - \sin^2\phi] = -\frac{4\lambda r^2}{\pi\epsilon_o d^2}\cos 2\phi \tag{3}$$

This potential is seen again in Sec. 5.7. With the objective of writing it in Cartesian coordinates, (3) is written as

$$\begin{aligned}\Phi &= -\frac{4\lambda}{\pi\epsilon_o d^2}[r(\cos\phi + \sin\phi)r(\cos\phi - \sin\phi)] \\ &= \frac{-4\lambda}{\pi\epsilon_o d^2}[(x+y)(x-y)] = \frac{-4\lambda}{\pi\epsilon_o d^2}(x^2 - y^2)\end{aligned} \tag{4}$$

(b) Rotate the quadrapole by 45°.

4.5 SOLUTION OF POISSON'S EQUATION FOR SPECIFIED CHARGE DISTRIBUTIONS

4.5.1 (a) With $|\mathbf{r} - \mathbf{r}'| = \sqrt{x'^2 + y'^2 + z'^2}$, (4.5.5) becomes

$$\Phi = \int_{y'=-a}^{a}\int_{x'=-a}^{a} \frac{\sigma_s(x', y')dx'dy'}{4\pi\epsilon_o\sqrt{x'^2 + y'^2 + z^2}} \tag{1}$$

(b) For the particular charge distribution,

$$\begin{aligned}\Phi &= \frac{\sigma_o}{a^2\pi\epsilon_o}\int_{y'=0}^{a}\int_{x'=0}^{a}\frac{x'y'dx'dy'}{\sqrt{x'^2+y'^2+z^2}}\\ &= \frac{\sigma_o}{a^2\pi\epsilon_o}\int_{y'=0}^{a}\left[\sqrt{a^2+y'^2+z^2}\,y'-\sqrt{y'^2+z^2}\,y'\right]dy'\end{aligned}\tag{2}$$

To complete this second integration, let $u^2 = y'^2 + z^2, 2udu = 2y'dy'$ so that

$$\begin{aligned}\int_{y'=0}^{a} y'\sqrt{y'+z^2}dy' &= \int_{z}^{\sqrt{a^2+z^2}} u^2du = \frac{u^3}{3}\Big|_{z}^{\sqrt{a^2+z^2}}\\ &= \frac{1}{3}[(a^2+z^2)^{3/2}-z^3]\end{aligned}\tag{3}$$

Similarly,

$$\int_{y'=0}^{a} y'\sqrt{y'^2+(a^2+z^2)} = \frac{1}{3}[(2a^2+z^2)^{3/2}-(a^2+z^2)^{3/2}]\tag{4}$$

so that

$$\Phi = \frac{\sigma_o}{3a^2\pi\epsilon_o}[(2a^2+z^2)^{3/2}+z^3-2(a^2+z^2)^{3/2}]\tag{5}$$

(c) At the origin,

$$\Phi = \frac{\sigma_o}{3a^2\pi\epsilon_o}[(2a^2)^{3/2}-2a^3] = \frac{2\sigma_o a(\sqrt{2}-1)}{3\pi\epsilon_o}\tag{6}$$

(d) For $z \gg a$, (5) becomes approximately

$$\begin{aligned}\Phi &\simeq \frac{\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(\frac{2a^2}{z^2}+1\right)^{3/2}-2\left(\frac{a^2}{z^2}+1\right)^{3/2}\right\}\\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{2a^2}{z^2}\right)\left(1+\frac{2a^2}{z^2}\right)^{1/2}-2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{a^2}{z^2}\right)^{1/2}\right\}\end{aligned}\tag{7}$$

For $a^2/z^2 \ll 1$, we use $(1+x)^{1/2} \simeq 1+\frac{1}{2}x$ and

$$\begin{aligned}\Phi &\simeq \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{2a^2}{z^2}\right)\left(1+\frac{a^2}{z^2}\right)-2\left(1+\frac{a^2}{z^2}\right)\left(1+\frac{a^2}{2z^2}\right)\right\}\\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{a^2}{z^2}\right)\left[1+\frac{2a^2}{z^2}-2-\frac{2a^2}{2z^2}\right]\right\}\\ &= \frac{2\sigma_o z^3}{3a^2\pi\epsilon_o}\left\{1+\left(1+\frac{a^2}{z^2}\right)\left(\frac{a^2}{z^2}-1\right)\right\}\end{aligned}\tag{8}$$

Thus,

$$\Phi = \frac{2\sigma_o a^2}{3\pi\epsilon_o z} \tag{9}$$

For a point charge Q at the origin, the potential along the z-axis is given by

$$\Phi = \frac{Q}{4\pi\epsilon_o z} \tag{10}$$

which is the same as the potential given by (9) if

$$Q = \frac{8\sigma_o a^2}{3} \tag{11}$$

(e) From (5),

$$\mathbf{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \frac{\sigma_o}{\pi a^2\epsilon_o}[z(2a^2+z^2)^{1/2} + z^2 - 2z(a^2+z^2)^{1/2}]\mathbf{i}_z \tag{12}$$

4.5.2 (a) Evaluation of (4.5.5) gives

$$\begin{aligned}\Phi &= \int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi}\frac{\sigma_o\cos\theta' R^2\sin\theta' d\phi' d\theta'}{4\pi\epsilon_o[R^2+z^2-4Rz\cos\theta']^{1/2}} \\ &= \frac{\sigma_o R^2}{4\epsilon_o}\int_{\theta'=0}^{\pi}\frac{\sin 2\theta' d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\end{aligned} \tag{1}$$

To integrate, let $u^2 = R^2 + z^2 - 2Rz\cos\theta'$ so that $2udu = 2Rz\sin\theta' d\theta'$ and note that $\cos\theta' = (R^2+z^2-u^2)/2Rz$. Thus, (1) becomes

$$\begin{aligned}\Phi &= \frac{\sigma_o}{4\epsilon_o z^2}\int_{z-R}^{(R+z)}(R^2+z^2-u^2)du \\ &= \frac{\sigma_o}{4\epsilon_o z^2}\Big[(R^2+z^2)(R+z) - \frac{(R+z)^3}{3} \\ &\quad - (R^2+z^2)(z-R) + \frac{(z-R)^3}{3}\Big] \\ &= \frac{\sigma_o R^3}{3\epsilon_o z^2}\end{aligned} \tag{2}$$

(b) Inside the shell, the lower limit of (2) becomes $(R-z)$. Then

$$\Phi = \frac{\sigma_o z}{3\epsilon_o} \tag{3}$$

(c) From (2) and (3)

$$\mathbf{E} = -\bar{\nabla}\Phi = -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \begin{cases}\frac{2\sigma_o R^3}{3\epsilon_o z^3}\mathbf{i}_z & z > R \\ -\frac{\sigma_o}{3\epsilon_o}\mathbf{i}_z & z < R\end{cases} \tag{4}$$

(d) Far away, the dipole potential on the z-axis would be $p/4\pi\epsilon_o z^2$ for the point charge dipole. By comparison of (2) to this expression the dipole moment is

$$p = \frac{4\pi\sigma_o R^3}{3} \tag{5}$$

4.5.3 (a) To find $\Phi(0,0,z)$ we use (4.5.4). For $\mathbf{r} = (0,0,z)$ and $\mathbf{r}' = a$ point on the cylinder of charge, $|\mathbf{r}-\mathbf{r}'| = \sqrt{(z-z')^2+R^2}$. This distance is valid for an entire "ring" of charge. The incremental charge element is then $\sigma 2\pi R dz$ so that (4.5.4) becomes

$$\Phi(0,0,z) = \int_0^l \frac{\sigma_o 2\pi R dz'}{4\pi\epsilon_o\sqrt{(z-z')^2+R^2}} + \int_{-l}^0 \frac{-\sigma_o 2\pi R dz'}{4\pi\epsilon_o\sqrt{(z-z')^2+R^2}} \tag{1}$$

To integrate, let $q' = z-z', dq' = -dz'$ and transform the limits

$$\begin{aligned}\Phi &= \frac{\sigma_o R}{2\epsilon_o}\Big[-\int_z^{z-l}\frac{dq'}{\sqrt{q'^2+R^2}} + \int_{z+l}^{z}\frac{dq'}{\sqrt{q'^2+R^2}}\Big]\\ &= \frac{\sigma_o R}{2\epsilon_o}\left[-\ln q' + \sqrt{R^2+q'^2}\Big|_z^{z-l} + \ln|q'+\sqrt{R^2+q'^2}\Big|_{z+l}^{z}\right]\end{aligned} \tag{2}$$

Thus,

$$\begin{aligned}\Phi &= \frac{\sigma_o R}{2\epsilon_o}\ln\left[\frac{(z+\sqrt{R^2+z^2})(z+\sqrt{R^2+z^2})}{(z-l+\sqrt{R^2+(z-l)^2})(z+l+\sqrt{R^2+(z+l)^2})}\right]\\ &= \frac{\sigma_o R}{2\epsilon_o}[2\ln(z+\sqrt{R^2+z^2}) - \ln(z-l+\sqrt{R^2+(z-l)^2})\\ &- \ln(z+l+\sqrt{R^2+(z+l)^2})]\end{aligned} \tag{3}$$

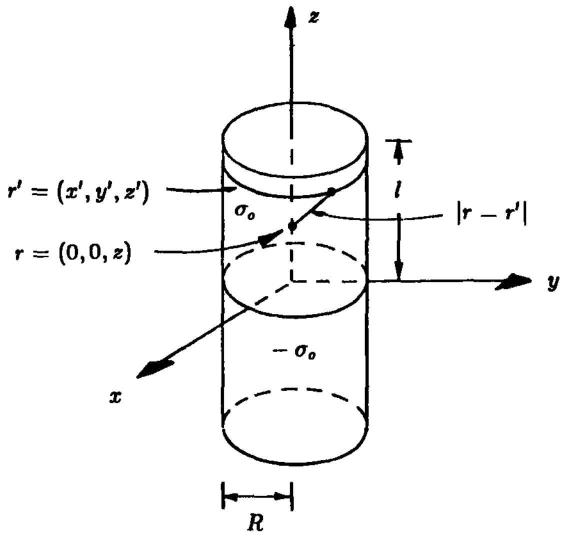


Figure S4.5.3

(b) Due to cylindrical geometry, there is no $\mathbf{i_x}$ or $\mathbf{i_y}$ field on the z axis.

$$\mathbf{E} = -\frac{\partial\Phi}{\partial z}\mathbf{i_z} = \frac{\mathbf{i_z}\sigma_o R}{2\epsilon_o}\Big[\frac{-2\left(1+\frac{z}{\sqrt{R^2+z^2}}\right)}{z+\sqrt{R^2+z^2}} + \left(\frac{1+\frac{(z-l)}{\sqrt{R^2+(z-l)^2}}}{z-l+\sqrt{R^2+(z-l)^2}}\right)$$
$$+\left(\frac{1+\frac{(z+l)}{\sqrt{R^2+(z+l)^2}}}{z+l+\sqrt{R^2+(z+l)^2}}\right)\Big] \tag{4}$$
$$= \mathbf{i_z}\frac{\sigma_o R}{2\epsilon_o}\Big[\frac{-2}{\sqrt{R^2+z^2}} + \frac{1}{\sqrt{R^2+(z-l)^2}} + \frac{1}{R^2+(z+l)^2}\Big]$$

(c) First normalize all terms in Φ to z

$$\Phi = \frac{\sigma_o R}{2\epsilon_o} ln\Big[\frac{\left(1+\sqrt{1+\frac{R^2}{z^2}}\right)\left(1+\sqrt{1+\frac{R^2}{z^2}}\right)}{\left(1-\frac{l}{z}+\sqrt{(R/z)^2+(1-\frac{l}{z})^2}\right)\left(1+\frac{l}{z}+\sqrt{(R/z)^2+(1+\frac{l}{z})^2}\right)}\Big] \tag{5}$$

Then, for $z \gg l$ and $z \gg R$,

$$\approx \frac{\sigma_o R}{2\epsilon_o} ln\Big[\frac{(1+1)(1+1)}{\left(1-\frac{l}{z}+1-\frac{l}{z}\right)\left(1+\frac{l}{z}+1+\frac{l}{z}\right)}\Big]$$
$$= \frac{\sigma_o R}{2\epsilon_o} ln\Big[\frac{4}{r(1-(l/z)^2)}\Big]$$
$$= \frac{\sigma_o R}{2\epsilon_o} ln\Big[\frac{1}{1-(l/z)^2}\Big] \approx \frac{\sigma_o R}{2\epsilon_o} ln[1+(l/z)^2] \tag{6}$$
$$\approx \frac{\sigma_o R}{2\epsilon_o}\frac{l^2}{z^2}$$

The potential of a dipole with dipole moment p is

$$\Phi_{\text{dipole}} = \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} \tag{7}$$

In our case, $\cos\theta/r^2 = 1/z^2$, so $p = 2\pi Rl^2$ (note the $p = qd, q = 2\pi Rl\sigma_o, d_{eff} = l$).

4.5.4 From (4.5.12),

$$\Phi(x,y,z) = \int_{y'=-d/2}^{d/2} \frac{\lambda dy'}{4\pi\epsilon_o\sqrt{(x-a)^2+(y-y')^2+z^2}} \tag{1}$$

To integrate, let $u = y' - y$ so that (1) becomes

$$\Phi = \frac{\lambda}{4\pi\epsilon_o}\int_{-y-d/2}^{-y+d/2}\frac{du}{\sqrt{u^2+(x-a)^2+z^2}}$$
$$= \frac{\lambda}{4\pi\epsilon_o} ln\Big[u+\sqrt{u^2+(x-a)^2+z^2}\Big]_{-y-d/2}^{-y+d/2} \tag{2}$$

which is the given expression.

4.5.5 From (4.5.12),

$$\Phi(0,0,z) = \frac{\lambda_o}{4\pi\epsilon_o l}\left\{\int_{x'=0}^{l} \frac{x'dx'}{\sqrt{x'^2+(a-z)^2}} - \frac{x'dx'}{\sqrt{x'^2+(a+z)^2}}\right\}$$
$$= \frac{\lambda_o}{4\pi\epsilon_o l}\{2z + \sqrt{l^2+(a-z)^2} - \sqrt{l^2+(a^1+z^2)}\}$$

4.5.6 From (4.5.12),

$$\begin{aligned}\Phi(0,0,z) &= \int_{z'=-a}^{a} \frac{\lambda_o z'dz'}{4\pi\epsilon_o a(z-z')} = \frac{\lambda_o}{4\pi\epsilon_o a}\int_{z'=-a}^{a}\left(-1+\frac{z}{z-z'}\right)dz' \\ &= \frac{\lambda_o}{4\pi\epsilon_o a}[-a - z ln(z-a) - z + z ln(z+a)]\end{aligned} \tag{1}$$

Thus,

$$\Phi(0,0,z) = \frac{-\lambda_o}{4\pi\epsilon_o}[2a + z ln(\frac{z-a}{z+a})] \tag{2}$$

Because of the symmetry about the z axis, the only component of $\mathbf{E}$ is in the z direction

$$\begin{aligned}\mathbf{E} &= -\frac{\partial\Phi}{\partial z}\mathbf{i}_z = \frac{\lambda_o}{4\pi\epsilon_o}\left[ln(\frac{z-a}{z+a}) + z\{\frac{1}{z-a} - \frac{1}{z+a}\}\right]\mathbf{i}_z \\ &= \frac{\lambda_o}{4\pi\epsilon_o}\left[ln(\frac{z-a}{z+a}) + \frac{2az}{z^2-a^2}\right]\mathbf{i}_z\end{aligned} \tag{3}$$

4.5.7 Using (4.5.20)

$$\begin{aligned}\Phi &= -\int_{y'=0}^{\Delta}\int_{x'=-b}^{b} \frac{\sigma_o(d-b)}{\Delta 2\pi\epsilon_o(d-x')} ln|d-x'|dx'dy' \\ &= -\frac{\sigma_o(d-b)}{2\pi\epsilon_o}\int_{x'=-b}^{b}\frac{ln(d-x')}{(d-x')}dx' \\ &= -\frac{\sigma_o(d-b)}{2\pi\epsilon_o}\{-\frac{1}{2}[ln(d-x')]^2|_{-b}^{b}\} \\ &= \frac{\sigma_o(d-b)}{4\pi\epsilon_o}\{[ln(d-b)]^2 - [ln(d+b)]^2\}\end{aligned}$$

4.5.8 From (4.5.20),

$$\Phi(d,0) = -\int_{x'=0}^{2d} \frac{\sigma_o ln|d-x'|}{2\pi\epsilon_o} dx' + \int_{x'=-2d}^{0} \frac{\sigma_o ln|d-x'|dx'}{2\pi\epsilon_o} \tag{1}$$

To integrate let $u = d - x'$ and $du = -dx'$.

$$\begin{aligned}\Phi(d,0) &= \int_{d}^{-d} \frac{\sigma_o lnudu}{2\pi\epsilon_o} - \int_{3d}^{d} \frac{\sigma_o lnudu}{2\pi\epsilon_o} \\ &= \frac{\sigma_o}{2\pi\epsilon_o}\{u(ln|u|-1)|_{d}^{-d} - u(ln|u|-1)|_{3d}^{d}\} \\ &= \frac{\sigma_o}{2\pi\epsilon_o} 3dln3\end{aligned} \tag{2}$$

Thus, setting $\Phi(d,0) = V$ gives

$$\sigma_o = \frac{2\pi\epsilon_o V}{3dln3} \tag{3}$$

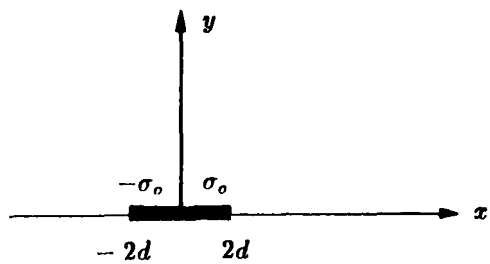


Figure S4.5.8

4.5.9 (a) (This problem might best be given while covering Sec. 8.2, where a stick model is developed for MQS systems.) At the lower end of the charge, ξ_c is the projection of $\mathbf{c}$ on $\mathbf{a}$. This is given by

$$\xi_c = \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|} \tag{1}$$

Similarly,

$$\xi_b = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} \tag{2}$$

(b) From (4.5.20),

$$\Phi = \int_{\xi'=\xi_c}^{\xi_b} \frac{\lambda d\xi'}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} = \int_{\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}}^{\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}} \frac{\lambda d\xi}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|} \tag{3}$$

where

$$|\mathbf{r}-\mathbf{r}'|^2 = \xi^2 + d^2$$

With θ defined as the angle between $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{d}| = |\mathbf{b}|\sin\theta \tag{4}$$

But in terms of $\mathbf{a}$ and $\mathbf{b}$,

$$\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \tag{5}$$

so that

$$d = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} \tag{6}$$

and

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{\xi^2 + \frac{|\mathbf{a}\times\mathbf{b}|^2}{|\mathbf{a}|^2}} \tag{7}$$

(c) Integration of (3) using (6) and (7) gives

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\xi + \sqrt{\xi^2 + \frac{|\mathbf{a}\times\mathbf{b}|^2}{|\mathbf{a}|^2}}\right|_{\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}}^{\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}} \tag{8}$$

and hence the given result.

(d) For a line charge λ_o between $(x,y,z) = (0,0,d)$ and $(x,y,z) = (d,d,d)$,

$$\begin{aligned}
\mathbf{a} &= d\mathbf{i_x} + d\mathbf{i_y} \\
\mathbf{b} &= (d-x)\mathbf{i_x} + (d-y)\mathbf{i_y} + (d-z)\mathbf{i_z} \\
\mathbf{c} &= -x\mathbf{i_x} - y\mathbf{i_y} + (d-z)\mathbf{i_z} \\
\mathbf{b}\cdot\mathbf{a} &= d(d-x) + d(d-y) \\
\mathbf{c}\cdot\mathbf{a} &= -xd - yd \\
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ d & d & 0 \\ d-x & d-y & d-z \end{vmatrix} \\
&= d(d-z)\mathbf{i_x} - d(d-z)\mathbf{i_y} + d(x-y)\mathbf{i_z} \\
|\mathbf{a}\times\mathbf{b}|^2 &= d^2[2(d-z)^2 + (x-y)^2] \\
(\mathbf{b}\cdot\mathbf{a})^2 &= d^2[(d-x)+(d-y)]^2 \\
(\mathbf{c}\cdot\mathbf{a})^2 &= d^2(x+y)^2
\end{aligned} \tag{9}$$

and evaluation of (c) of the problem statement gives (d).

4.5.10 This problem could be given in connection with covering Sec. 8.2. It illustrates the steps followed between (8.2.1) and (8.2.7), where the distinction between source and observer coordinates is also essential. Given that the potential has been found using the superposition integral, the required electric field is found by taking the gradient with respect to the observer coordinates, $\mathbf{r}$, not $\mathbf{r}'$. Thus, the gradient operator can be taken inside the integral, where it operates as though $\mathbf{r}'$ is a constant.

$$\mathbf{E} = -\nabla\Phi = -\int_V \nabla\Big[\frac{\rho(\mathbf{r}')}{4\pi\epsilon_o|\mathbf{r}-\mathbf{r}'|}\Big]dv' = -\int_{V'} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_o}\nabla\Big[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big]dv' \tag{1}$$

The arguments leading to (8.2.6) apply equally well here

$$\nabla\Big[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\Big] = -\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\mathbf{i}_{r'r} \tag{2}$$

The result given with the problem statement follows. Note that we could just as well have derived this result by superimposing the electric fields due to point charges $\rho(\mathbf{r}')dv'$. Especially if coordinates other than Cartesian are used, care must be taken to recognize how the unit vector $\mathbf{i}_{r'r}$ takes into account the vector addition.

4.5.11 (a) Substitution of the given charge density into Poisson's equation results in the given expression for the potential.

(b) If the given solution is indeed the response to a singular source at the origin, it must (i) satisfy the differential equation, (a), at every point except the origin and (ii) it must satisfy (c). With the objective of showing that (i) is true, note that in spherical coordinates with no θ or ϕ dependence, (b) becomes

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{d\Phi}{dr}\Big) - \kappa^2\Phi = s(r) \tag{1}$$

Substitution of (e) into this expression gives zero for the left hand side at every point, r, except the origin. The algebra is as follows. First,

$$\frac{d}{dr}\Big(\frac{A}{r}e^{-\kappa r}\Big) = -\frac{A\kappa}{r}e^{-\kappa r} - \frac{Ae^{-\kappa r}}{r^2} \tag{2}$$

Then,

$$-\frac{1}{r^2}\frac{d}{dr}\Big(\frac{A\kappa}{r}e^{-\kappa r} + \frac{e^{-\kappa r}}{r^2}\Big) - \kappa^2\frac{Ae^{-\kappa r}}{r} = \frac{Ak^2}{r^2}e^{-\kappa r} + \frac{Ak^2}{r}e^{-\kappa r} \tag{3}$$
$$= 0; \quad r \neq 0$$

To establish the coefficient, A, integrate Poisson's equation over a spherical volume having radius r centered on the origin. By virtue of its being singular there, what is being integrated has value only at the origin. Thus, we take the limit where the radius of the volume goes to zero.

$$\lim_{r\to 0}\Big\{\int_V \nabla\cdot\nabla\Phi dv - \kappa^2\int_V \Phi dv\Big\} = \lim_{r\to 0}\Big\{-\frac{1}{\epsilon_o}\int_V s dv\Big\} \tag{4}$$

Gauss' theorem shows that the first integral can be converted to a surface integral. Thus,

$$\lim_{r\to 0}\Big\{\oint_S \nabla\Phi\cdot d\mathbf{a} - \kappa^2\int_V \Phi dv\Big\} = \lim_{r\to 0}\Big\{-\frac{1}{\epsilon_o}\int_V s dv\Big\} \tag{5}$$

If the potential does indeed have the r dependence of (e), then it follows that

$$\lim_{r\to 0}\int_V \Phi dv = \lim_{r\to 0}\int_0^r \Phi 4\pi r^2 dr = 0 \tag{6}$$

so that in the limit, the second integral on the left in (5) makes no contribution and (5) reduces to

$$\lim_{r\to 0}\Big(-\frac{A\kappa}{r}e^{-\kappa r} - \frac{Ae^{-\kappa r}}{r^2}\Big)4\pi r^2 = -4\pi A = -\frac{Q}{\epsilon_o} \tag{7}$$

and it follows that $A = Q/4\pi\epsilon_o$.

(c) We have found that a point source, Q, at the origin gives rise to the potential

$$Q \Rightarrow \Phi = \frac{Q}{4\pi\epsilon_o}\frac{e^{-\kappa r}}{r} \tag{8}$$

Arguments similar to those given in Sec. 4.3 show that (b) is linear. Thus, given that we have shown that the response to a point source $\rho(\mathbf{r}')dv$ at $\mathbf{r} = \mathbf{r}'$ is

$$\rho(\mathbf{r}')dv \Rightarrow \Phi = \frac{\rho(\mathbf{r}')dv}{4\pi\epsilon_o}\frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{9}$$

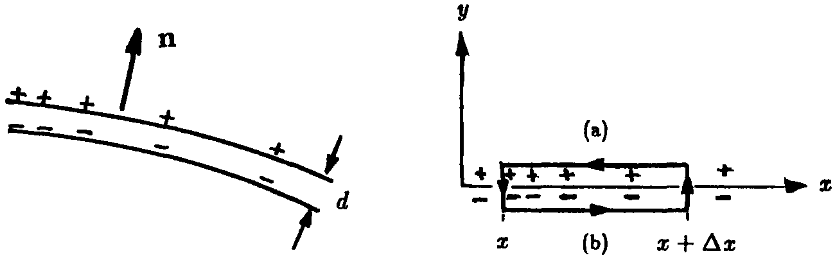
it follows by superposition that the response to an arbitrary source distribution is

$$\Phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon_o(\mathbf{r}-\mathbf{r}'|}dv \tag{10}$$

4.5.12 (a) A cross-section of the dipole layer is shown in Fig. S4.5.12a. Because the field inside the layer is much more intense than that outside and because the layer is very thin compared to distances over which the surface charge density varies with position in the plane of the layer, the fields inside are as though the surface charge density resided on the surfaces of plane parallel planes. Thus, Gauss' continuity condition applied to either of the surface charge densities

shows that the field inside has the given magnitude and the direction must be that of the normal vector.

$$\mathbf{E}_{\text{int}} = -\cancel{\epsilon_o}\sigma_s\mathbf{N}/\epsilon_o \tag{1}$$



(a) (b)

Figure S4.5.12

(b) It follows from (4.1.1) and the contour shown in Fig. S4.5.12b having incremental length Δx in the x direction that

$$-E_x^a\Delta x + E_x^b\Delta x + E_y(x+\Delta x)d - E_y(x)d = 0 \tag{2}$$

Divided by Δx, this expression becomes

$$-E_x^a + E_x^b + d\frac{\partial E_y}{\partial x} = 0 \tag{3}$$

The given expression then follows by using (1) to replace E_y with $-\cancel{\epsilon_o}\sigma_s/\epsilon_o$ and recognizing that $\pi_s \equiv \sigma_s d$.

4.6 ELECTROQUASISTATIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

4.6.1 In view of (4.5.12),

$$\Phi(0,0,a) = \int_c^b \frac{\lambda\left(\frac{a-z'}{a-c}\right)}{4\pi\epsilon_o(a-z')}dz' \tag{1}$$

The z dependence of the integrand cancels out so that the integration amounts to a multiplication.

$$\Phi(0,0,a) = \frac{\lambda_o}{4\pi\epsilon_o(a-c)}(b-c) \tag{2}$$

The net charge is

$$Q = \frac{1}{2}\left[\lambda_o\left(\frac{a-b}{a-c}\right) + \lambda_o\right](b-c) \tag{3}$$

Provied that the equipotential surface passing through $(0,0,a)$ encloses all of the segment, the capacitance of an electrode having the shape of this surface is then given by

$$C = \frac{Q}{\Phi(0,0,a)} = 2\pi\epsilon_o(2a-b-c) \tag{4}$$

4.6.2 (a) The potential is the sum of the potentials due to the charge producing the uniform field and the point charges. With $r_\pm$ defined as shown in Fig. S4.6.2a,

$$\Phi = -E_o z + \frac{q}{4\pi\epsilon_o r_+} - \frac{q}{4\pi\epsilon_o r_-} \tag{1}$$

where

$$z = r\cos\theta$$

$$r_\pm = \sqrt{r^2 + (d/2)^2 \mp 2r\frac{d}{2}\cos\theta}$$

To write (1) in terms of the normalized variables, divide by $E_o d$ and multiply and divide $r_\pm$ by d. The given expression, (b), then follows.

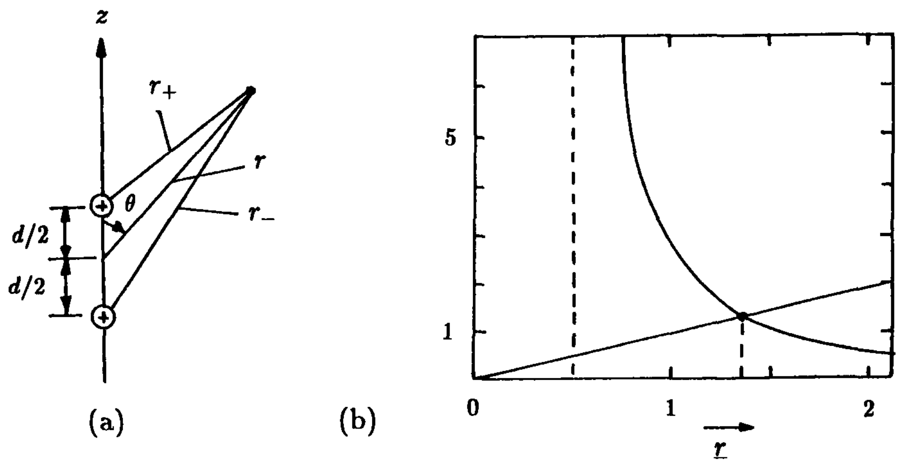


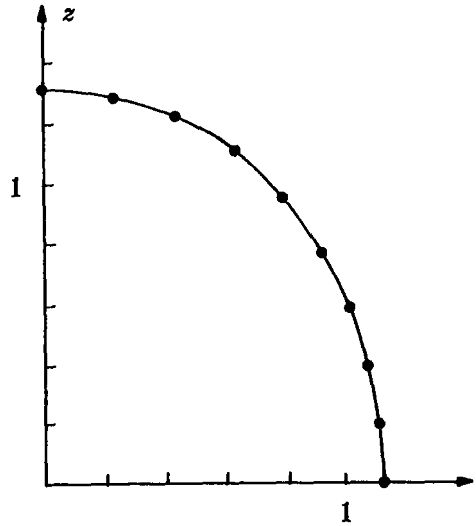
Figure S4.6.2

(b) An implicit expression for the intersection point $d/2 < r$ on the z axis is given by evaluating (b) with $\Phi = 0$ and $\theta = 0$.

$$\underline{r} = \frac{\underline{q}}{(\underline{r}-\frac{1}{2})} - \frac{\underline{q}}{(\underline{r}+\frac{1}{2})} \tag{2}$$

The graphical solution of this expression for $d/2 < r(1/2 < \underline{r})$ is shown in Fig. S4.6.2b. The required intersection point is $\underline{r} = 1.33$. Because the right hand side of (2) has an asymptote at $\underline{r} = 0.5$, there must be an intersection between the straight line representing the left side in the range $0.5 < \underline{r}$.

(c) The plot of the $\Phi = 0$ surface for $0 < \theta < \pi/2$ is shown in Fig. S4.6.2c.



(c)

Figure S4.6.2

(d) At the north pole of the object, the electric field is z-directed. It therefore follows from (b) as $(0.5 < \underline{r})$

$$\begin{aligned} E_z &= -\frac{\partial \Phi}{\partial r} = -E_o \frac{\partial \underline{\Phi}}{\partial \underline{r}} = -E_o \frac{\partial}{\partial \underline{r}}\left(-\underline{r} + \frac{\underline{q}}{\underline{r} - \frac{1}{2}} - \frac{\underline{q}}{\underline{r} + \frac{1}{2}}\right) \\ &= E_o\left[1 + \frac{\underline{q}}{\left(\underline{r} - \frac{1}{2}\right)^2} = \frac{\underline{q}}{\left(\underline{r} + \frac{1}{2}\right)^2}\right] \end{aligned} \tag{3}$$

Evauation of this expression at $\underline{r} = 1.33$ and $\underline{q} = 2$ gives $E_z = 3.33 E_o$.

(e) Gauss' integral law, applied to a surface comprised of the equipotential and the plane $z = 0$, shows that the net charge on the northern half of the object is q. For the given equipotential, $\underline{q} = 2$. It follows from the definition of $\underline{q}$ that

$$\underline{q} = 2 = \frac{q}{4\pi\epsilon_o E_o d^2} \Rightarrow Q = q = 8\pi\epsilon_o E_o d^2 \tag{4}$$

4.6.3 For the disk of charge in Fig. 4.5.3, the potential is given by (4.5.7)

$$\Phi = \frac{\sigma_o}{2\epsilon_o}\left(\sqrt{R^2 + z^2} - |z|\right) \tag{1}$$

At $(0, 0, d)$,

$$\Phi(0, 0, d) = \frac{\sigma_o}{2\epsilon_o}\left(\sqrt{R^2 + d^2} - d\right) \tag{2}$$

and

$$q = \sigma_o \pi R^2 \tag{3}$$

Thus

$$C \equiv \frac{q}{\Phi(0,0,d)} = \frac{2\epsilon_o \pi R^2}{\sqrt{R^2 + d^2} - d} \tag{4}$$

4.6.4 (a) Due to the top sphere,

$$\Phi_+ = \frac{Q}{4\pi\epsilon_o r_1} \tag{1}$$

and similarly,

$$\Phi_- = \frac{-Q}{4\pi\epsilon_o r_2} \tag{2}$$

At the bottom of the top sphere

$$\Phi_+ = \frac{Q}{4\pi\epsilon_o R} - \frac{Q}{4\pi\epsilon_o (h - R)} \tag{3}$$

while at the top of the bottom sphere

$$\Phi_-\Big|_{r=R} = \frac{-Q}{4\pi\epsilon_o R} + \frac{Q}{4\pi\epsilon_o (h - R)} \tag{4}$$

The potential difference between the two spherical conductors is therefore

$$V = \frac{2Q}{4\pi\epsilon_o R} - \frac{2Q}{4\pi\epsilon_o (h - R)} = \frac{Q}{2\pi\epsilon_o R}\Big(1 - \frac{R/h}{1 - R/h}\Big) \tag{3}$$

The maximum field occurs at $z = 0$ on the axis of symmetry where the magnitude is the sum of that due to point charges.

$$\mathbf{E}_{\max} = \frac{-2Q\mathbf{i}_z}{4\pi\epsilon_o (h/2)^2} = \frac{-2Q}{\pi\epsilon_o h^2}\mathbf{i}_z \tag{4}$$

(b) Replace point charge Q at $z = h/2$ by $Q_1 = Q\frac{R}{h}$ at $z = \frac{h}{2} - \frac{R^2}{h}$ and $Q_o = Q[1 - \frac{R}{h}]$ at $z = h/2$. The potential on the surface of the top sphere is now

$$\Phi_{\text{top}} = \frac{Q_o}{4\pi\epsilon_o R} + \frac{Q_1}{4\pi\epsilon_o \big(R - \frac{R^2}{h}\big)} - \frac{Q}{4\pi\epsilon_o (h - R)} \tag{5}$$

The potential on the surface of the bottom sphere is

$$\Phi_{\text{bottom}} = \frac{Q_o}{4\pi\epsilon_o (h - R)} + \frac{Q_1}{4\pi\epsilon_o \big(h - R - \frac{R^2}{h}\big)} - \frac{Q}{4\pi\epsilon_o R} \tag{6}$$

The potential difference is then,

$$V = \frac{Q_o}{4\pi\epsilon_o}[(\frac{1}{R} - \frac{1}{h-R})] + \frac{Q_1}{4\pi\epsilon_o}\left[\frac{1}{R - \frac{R^2}{h}} - \frac{1}{h - R - \frac{R^2}{h}}\right] \\ - \frac{Q}{4\pi\epsilon_o}(\frac{1}{h-R} - \frac{1}{R})$$

For four charges $Q_1 = QR/h$ at $z = h/2 - R^2/h$; $Q_o = Q(1 - \frac{R}{h})$ at $z = h/2$; $Q_2 = -QR/h$ at $z = -h/2 + R^2/h$; $Q_3 = -Q(1 - \frac{R}{h})$ at $z = -h/2$ and

$$\Phi_{\text{top}} = \frac{Q_o}{4\pi\epsilon_o R} + \frac{Q_1}{4\pi\epsilon_o R(1 - \frac{R}{h})} + \frac{Q_2}{4\pi\epsilon_o(h - R - \frac{R^2}{h})} \\ + \frac{Q_3}{4\pi\epsilon_o(h - R)} \tag{7}$$

which becomes

$$\Phi_{\text{top}} = \frac{Q(1 - \frac{R}{h})}{4\pi\epsilon_o R} + \frac{Q\frac{R}{h}}{4\pi\epsilon_o R(1 - \frac{R}{h})} \\ - \frac{Q(R^2/h^2)}{4\pi\epsilon_o(1 - \frac{R}{h} - \frac{R^2}{h^2})} - \frac{Q(R/h)}{4\pi\epsilon_o R} \tag{8}$$

Similarly,

$$\Phi_{\text{bottom}} = \frac{Q(R/h)}{4\pi\epsilon_o R} + \frac{Q(R^2/h^2)}{4\pi\epsilon_o R(1 - \frac{R}{h} - \frac{R^2}{h^2})} \\ - \frac{QR/h}{4\pi\epsilon_o R(1 - \frac{R}{h})} - \frac{Q(1 - R/h)}{4\pi\epsilon_o} \tag{9}$$

so that

$$V = \frac{2Q}{4\pi\epsilon_o R}\left\{1 - \frac{R}{h} + \frac{R/h}{1 - \frac{R}{h}} - \frac{R^2/h^2}{1 - \frac{R}{h} - \frac{R^2}{h^2}} - \frac{R}{h}\right\} \tag{10}$$

$$V = \frac{Q}{2\pi\epsilon_o R}\left\{1 - \frac{2R}{h} + \frac{R/h}{1 - R/h} - \frac{(R/h)^2}{1 - \frac{R}{h} - (R/h)^2}\right\} \tag{11}$$

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_o R}{1 - \frac{2R}{h} + \frac{(R/h)}{1-(R/h)} - \frac{(R/h)^2}{1-(R/h)-(R/h)^2}} \tag{12}$$

4.6.5 (a) The potential is the sum of that given by (a) in Prob. 4.5.4 and a potential due to a similarly distributed negative line charge on the line at $x = -a$ between $y = -d/2$ and $y = d/2$.

$$\begin{aligned}\Phi = \frac{\lambda_l}{4\pi\epsilon_o} ln\Bigg\{ &\left[\frac{d}{2} - y + \sqrt{(x-a)^2 + (\frac{d}{2} - y)^2 + z^2}\right] \\ &\left[-\frac{d}{2} - y + \sqrt{(x+a)^2 + (\frac{d}{2} + y)^2 + z^2}\right] / \\ &\left[-\frac{d}{2} - y + \sqrt{(x-a)^2 + (\frac{d}{2} + y)^2 + z^2}\right] \\ &\left[\frac{d}{2} - y + \sqrt{(x+a)^2 + (\frac{d}{2} - y)^2 + z^2}\right]\Bigg\}\end{aligned} \tag{1}$$

(b) The equipotential passing through $(x, y, z) = (a/2, 0, 0)$ is given by evaluating (1) at that point

$$V = \frac{\lambda_l}{4\pi\epsilon_o} ln\left\{ \frac{[\frac{d}{2} + \sqrt{\frac{a^2}{4} + \frac{d^2}{4}}][-\frac{d}{2} + \sqrt{\frac{9}{4}a^2 + \frac{d^2}{4}}]}{[-\frac{d}{2} + \sqrt{\frac{a^2}{4} + \frac{d^2}{4}}][\frac{d}{2} + \sqrt{\frac{9}{4}a^2 + \frac{d^2}{4}}]} \right\} \tag{2}$$

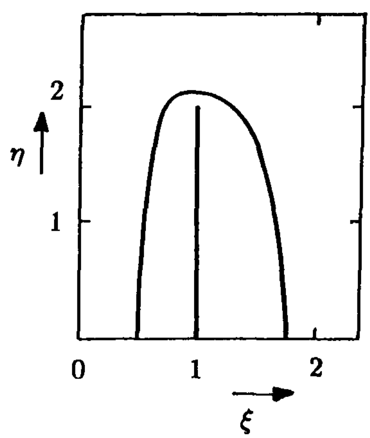


Figure S4.6.5

(c) In normalized form, (2) becomes

$$\underline{\phi} = \frac{ln\left\{ \frac{\left(2-\eta+\sqrt{(\xi-1)^2+(2-\eta)^2+z^2}\right)\left(-2-\eta+\sqrt{(\xi+1)^2+(2+\eta)^2+z^2}\right)}{\left(-2-\eta+\sqrt{(\eta-1)^2+(2+\eta)^2+z^2}\right)\left(2-\eta+\sqrt{(2-\eta)^2+(2-\eta)^2+z^2}\right)} \right\}}{ln\left\{ \frac{(4+\sqrt{1+16})(-4+\sqrt{9+16})}{(-4+\sqrt{1+16})(4+\sqrt{9+16})} \right\}} \tag{3}$$

where $\underline{\Phi} = \Phi/\Phi(\frac{a}{2},0,0), \xi = x/a, \eta = y/a$ and $d = 4a$. Thus, $\underline{\Phi} = 1$ for the equipotential passing through $(\frac{a}{2},0,0)$. This equipotential can be found by writing it in the form $f(\xi,\eta) = 0$, setting η and having a programmable calculator determine ξ. In the first quadrant, the result is as shown in Fig. S4.6.5.

(d) The lines of electric field intensity are sketched in Fig. S4.6.5.

(e) The charge on the surface of the electrode is the same as the charge enclosed by the equipotential in part (c), $Q = \lambda_l d$. Thus,

$$C = \frac{\lambda_l d}{V} = 4\pi\epsilon_o d/ln\left\{\frac{[d+\sqrt{a^2+d^2}][-d+\sqrt{9a^2+d^2}]}{[-d+\sqrt{a^2+d^2}][d+\sqrt{9a^2+d^2}]}\right\} \tag{4}$$

4.7 METHOD OF IMAGES

4.7.1 (a) The potential is due to Q and its image, $-Q$, located at $z = -d$ on the z axis.

(b) The equipotential having potential V and passing through the point $z = a < d, x = 0, y = 0$ is given by evaluating this expression and taking care in taking the square root to recognize that $d > a$.

$$V = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{d-a} - \frac{1}{d+a}\right) = \frac{Q}{4\pi\epsilon_o}\left(\frac{2a}{d^2-a^2}\right) \tag{1}$$

In general, the equipotential surface having potential V is

$$V = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{2}$$

The given expression results from equating these last two expressions.

(c) The potential is infinite at the point charge and goes to zero at infinity and in the plane $z = 0$. Thus, there must be an equipotential contour that encloses the point charge. The charge on the electrode having the shape given by (2) must be equal to Q so the capacitance follows from (1) as

$$C = \frac{Q}{V} = 2\pi\epsilon_o\frac{(d^2-a^2)}{a} \tag{3}$$

4.7.2 (a) The line charge and associated square boundaries are shown at the center of Fig. S4.7.2. In the absence of image charges, the equipotentials would be circular. However, with images that alternate in sign to infinity in each direction, as shown, a grid of square equipotentials is established and hence the boundary conditions on the central square are met. At each point on the boundary, there is an equal distance to both a positive and a negative line charge. Hence, the potential on the boundary is zero.

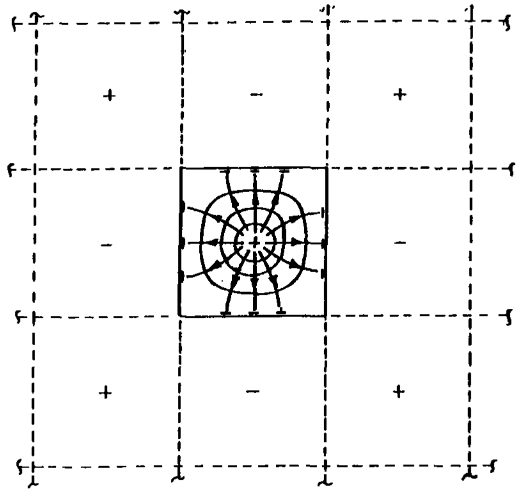


Figure S4.7.2

(b) The equipotentials close to the line charge are circular. As the other boundary is approached, they approach the square shape of the boundary. The lines of electric field intensity are as shown, terminating on negative surface charges on the surface of the boundary.

4.7.3 (a) The bird acquires the same potential as the line, hence has charges induced on it and conserves charge when it flies away.

(b) The fields are those of a charge Q at $y = h, x = Ut$ and an image at $y = -h$ and $x = Ut$.

(c) The potential is the sum of that due to Q and its image $-Q$.

$$\Phi = \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{\sqrt{(x-Ut)^2+(y-h)^2+z^2}} - \frac{1}{\sqrt{(x-Ut)^2+(y+h)^2+z^2}}\right] \quad (1)$$

(d) From this potential

$$E_y = -\frac{\partial\Phi}{\partial y} = \frac{Q}{4\pi\epsilon_o}\left\{\frac{y-h}{(x-Ut)^2+(y-h)^2+z^2]^{3/2}} - \frac{y+h}{[(x-Ut)^2+(y+h)^2+z^2]^{3/2}}\right\} \quad (2)$$

Thus, the surface charge density is

$$\sigma_s = \epsilon_o E_y\Big|_{y=0} = \frac{Q\epsilon_o}{4\pi\epsilon_o}\left[\frac{-h}{[(x-Ut)^2+h^2+z^2]^{3/2}} - \frac{h}{[(x-Ut)^2+h^2+z^2]^{3/2}}\right] = \frac{-Qh}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}} \tag{3}$$

(e) The net charge q on electrode at any given instant is

$$q = \int_{z=0}^{w}\int_{x=0}^{l} \frac{-Qh\,dx\,dz}{2\pi[(x-Ut)^2+h^2+z^2]^{3/2}} \tag{4}$$

If $w \ll h$,

$$q = \int_{x=0}^{l} \frac{-Qhw\,dx}{2\pi[(x-Ut)^2+h^2]^{3/2}} \tag{5}$$

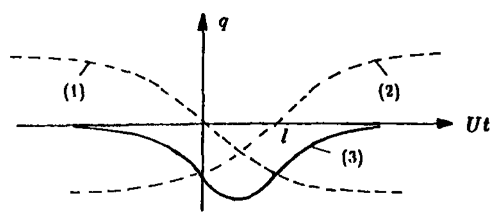
For the remaining integration, $x' = (x-Ut)$, $dx' = dx$ and

$$q = \int_{-Ut}^{l-Ut} \frac{-Qhw\,dx'}{2\pi[x'^2+h^2]^{3/2}} \tag{6}$$

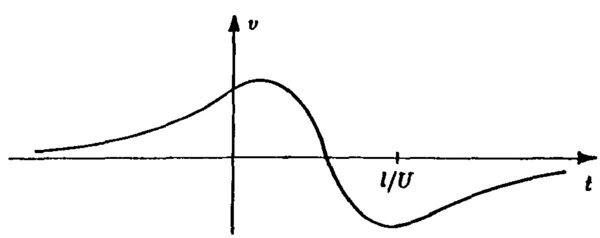
Thus

$$q = -\frac{Qw}{2\pi h}\left[\frac{l-Ut}{\sqrt{(l-Ut)^2+h^2}} + \frac{Ut}{\sqrt{(Ut)^2+h^2}}\right] \tag{7}$$

(f) The dahsed curves (1) and (2) in Fig. S4.7.3 are the first and second terms in (7), respectively. They sum to give (3)



(a)



(b)

Figure S4.7.3

(g) The current follows from (7) as

$$i = \frac{dq}{dt} = -\frac{Qw}{2\pi h}\left[\frac{-Uh^2}{[(l-Ut)^2+h^2]^{3/2}} + \frac{Uh^2}{[(Ut)^2+h^2]^{3/2}}\right] \tag{8}$$

and the voltage is then $v = -iR = -Rdq/dt$. A sketch is shown in Fig. S4.7.3b.

4.7.4 For no normal **E**, we want image charges of the same sign; $+\lambda$ at $(-a,0)$ and $-\lambda$ at $(-b,0)$. The potential in the $x=0$ plane is then,

$$\begin{aligned}\Phi &= -\frac{2\lambda}{2\pi\epsilon_o}ln(a^2+y^2)^{1/2} + \frac{2\lambda}{2\pi\epsilon_o}ln(b^2+y^2)^{1/2} \\ &= -\frac{\lambda}{2\pi\epsilon_o}ln\left(\frac{a^2+y^2}{b^2+y^2}\right)\end{aligned} \tag{1}$$

4.7.5 (a) The image to make the $x=0$ plane an equipotential is a line charge $-\lambda$ at $(x,y)=(-d,d)$. The image of these two line charges that makes the plane $y=0$ an equipotential is a pair of line charges, $+\lambda$ at $(-d,-d)$ and $-\lambda$ at $(d,-d)$. Thus

$$\begin{aligned}\Phi &= -\frac{\lambda}{4\pi\epsilon_o}ln[(x-d)^2+(y-d)^2] - \frac{\lambda}{4\pi\epsilon_o}ln[(x+d)^2+(y+d)^2] \\ &+ \frac{\lambda}{4\pi\epsilon_o}ln[(x-d)^2+(y+d)^2] + \frac{\lambda}{4\pi\epsilon_o}ln[(x+d)^2+(y-d)^2] \\ &= \frac{\lambda}{4\pi\epsilon_o}ln\left\{\frac{[(x-d)^2+(y+d)^2][(x+d)^2+(y-d)^2]}{[(x-d)^2+(y-d)^2][(x+d)^2+(y+d)^2]}\right\}\end{aligned} \tag{1}$$

(b) The surface of the electrode has the potential

$$\Phi(a,a) = \frac{\lambda}{4\pi\epsilon_o}ln\left\{\frac{[(a-d)^2+(a+d)^2][(a+d)^2+(a-d)^2]}{[(a-d)^2+(a-d)^2][(a+d)^2+(a+d)^2]}\right\} = V \tag{2}$$

Then

$$\frac{C}{\text{length}} = \frac{\lambda}{V} = \frac{4\pi\epsilon_o}{\ln\left\{\frac{[(a+d)^2+(a-d)^2]^2}{2[a-d]^2 2[a+d]^2}\right\}} = \frac{2\pi\epsilon_o}{ln\left[\frac{a^2+d^2}{a^2-d^2}\right]} \tag{3}$$

4.7.6 (a) The potential of a disk at $z=s$ is given by 4.5.7 with $z\to z-s$

$$\Phi(z>s) = \frac{\sigma_o}{2\epsilon_o}\left[\sqrt{R^2+(z-s)^2} - |z-s|\right] \tag{1}$$

The ground plane is represented by an image disk at $z=-s$; (4.5.7) with $z\to z+s$. Thus, the total potential is

$$\Phi = \frac{\sigma_o}{2\epsilon_o}\left[\sqrt{R^2+(z-s)^2} - |z-s| - \sqrt{R^2+(z+d)^2} + |z+s|\right] \tag{2}$$

(b) The potential at $z = d < s$ is

$$\begin{aligned}\Phi(z = d < s) &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} - |d-s| - \sqrt{R^2 + (d+s)^2} + |d+s|] \\ &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} - (s-d) - \sqrt{R^2 + (d+s)^2} + s + d] \\ &= \frac{\sigma_o}{2\epsilon_o}[\sqrt{R^2 + (d-s)^2} + 2d - \sqrt{R^2 + (d+s)^2}] = V\end{aligned} \tag{3}$$

Thus,

$$C = \frac{Q}{V} = \frac{2\epsilon_o \pi R^2}{\sqrt{R^2 + (d-s)^2} - \sqrt{R^2 + (d+s)^2} + 2d}$$

4.7.7 From (4.5.4),

$$\begin{aligned}\Phi(0,0,a) &= \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{\frac{\sigma_o}{R} r dr d\phi}{4\pi\epsilon_o\sqrt{r^2 + (h-a)^2}} + \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{-\frac{\sigma_o r}{R} r dr d\phi}{4\pi\epsilon_o\sqrt{r^2 + (h+a)^2}} \\ &= \frac{\sigma_o}{2\epsilon_o R}\left[\int_{r=0}^{R} \frac{r^2 dr}{\sqrt{r^2 + (h-a)^2}} - \int_{r=0}^{R} \frac{r^2 dr}{\sqrt{r^2 + (h+a)^2}}\right] \\ &= \frac{\sigma_o}{4\epsilon_o R}\left[\frac{R}{2}(\sqrt{R^2 + (h-a)^2}\right. \\ &\quad - \sqrt{R^2 + (h+a)^2}) + (h-a)^2 ln(\frac{h-a}{\sqrt{R^2 + (h-a)^2}}) \\ &\quad \left. + (h+a)^2 ln(\frac{R + \sqrt{R^2 + (h+a)^2}}{h+a})\right]\end{aligned} \tag{1}$$

The total charge in the disk is

$$Q = \int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{\sigma_o r}{R} r dr d\phi = \frac{2\pi}{3} R^2 \sigma_o$$

Thus,

$$\begin{aligned}C = \frac{Q}{V} = &\left\{2\pi R^3 \epsilon_o\right\} / \left\{\frac{R}{2}[\sqrt{R^2 + (h-a)^2}\right. \\ &- \sqrt{R^2 + (h+a)^2}] \\ &+ (h-a)^2 ln(\frac{h-a}{\sqrt{R^2 + (h-a)^2}}) \\ &\left. + (h+a)^2 ln(\frac{R^2 + \sqrt{R^2 + (h+a)^2}}{(h+a)})\right\}\end{aligned}$$

4.7.8 Because there is perfectly conducting material at $z=0$ there is the given line charge and an image from $(0,0,-d)$ to $(d,d,-d)$. Thus, for these respective line charges

$$\mathbf{a} = d\mathbf{i_x} + d\mathbf{i_y}$$
$$\mathbf{f} = (d-x)\mathbf{i_x} + (d-y)\mathbf{i_y} + (\pm d - z)\mathbf{i_z}$$
$$\mathbf{c} = -x\mathbf{i_x} - y\mathbf{i_y} + (\pm d - z)\mathbf{i_z} \tag{1}$$
$$\mathbf{b}\cdot\mathbf{a} = d[(d-x)+(d-y)]$$
$$\mathbf{c}\cdot\mathbf{a} = -xd - yd$$
$$\mathbf{a}\times\mathbf{b} = d(\pm d - z)\mathbf{i_x} - \mathbf{i_y}d(\pm d - z) + \mathbf{i_z}d[(d-y)-(d-x)]$$
$$|\mathbf{a}\times\mathbf{b}| = d^2(\pm d - z)^2 + d^2(\pm d - z)^2 + d^2[(d-y)-(d-x)]^2$$

The potential due to the line charge and its image then follows (c) of Prob. 4.5.9.

$$\Phi = \frac{\lambda}{4\pi\epsilon_o} ln\left\{\frac{2d - x - y + \sqrt{2[(d-x)^2 + (d-y)^2 + (d-z)^2]}}{-x - y + \sqrt{2[x^2 + y^2 + (d-z)^2]}} \cdot \frac{-x - y + \sqrt{2[x^2 + y^2 + (d+z)^2]}}{2d - x - y + \sqrt{2[(d-x)^2 + (d-y)^2 + (d+z)^2]}}\right\} \tag{2}$$

4.8 CHARGE SIMULATION APPROACH TO BOUNDARY VALUE PROBLEMS

4.8.1 For the six-segment system, the first two of (4.8.5) are

$$S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6 = \frac{V}{2} \tag{1}$$

$$S_{21}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3 + S_{24}\sigma_4 + S_{25}\sigma_5 + S_{26}\sigma_6 = \frac{V}{2} \tag{2}$$

Because of the symmetry,

$$\sigma_1 = \sigma_3 = -\sigma_4 = -\sigma_6, \quad \sigma_2 = -\sigma_3 \tag{3}$$

and so these two expressions reduce to two equations in two unknowns. (The other four expressions are identical to (4).)

$$\begin{bmatrix} (S_{11} + S_{13} - S_{14} - S_{16}) & (S_{12} - S_{15}) \\ (S_{21} + S_{23} - S_{24} - S_{26}) & (S_{22} - S_{25}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} V/2 \\ V/2 \end{bmatrix} \tag{4}$$

Thus,

$$\sigma_1 = \frac{V}{2D}[(S_{22} - S_{25}) - (S_{12} - S_{15})] \tag{5}$$

$$\sigma_2 = \frac{V}{2D}[(S_{11} + S_{13} - S_{14} - S_{16}) - (S_{21} + S_{23} - S_{24} - S_{26})] \tag{6}$$

where

$$D = (S_{11} + S_{13} - S_{14} - S_{16})(S_{22} - S_{25}) - (S_{21} + S_{23} - S_{24} - S_{26})(S_{12} - S_{15})$$

and from (4.8.3)

$$C = \frac{1}{V}(b/3)[2\sigma_1 + \sigma_2] \tag{7}$$

SOLUTIONS TO CHAPTER 5

5.1 PARTICULAR AND HOMOGENEOUS SOLUTIONS TO POISSON'S AND LAPLACE'S EQUATIONS

5.1.1 The particular solution must satisfy Poisson's equation in the region of interest. Thus, it is the first term in the potential, associated with the charge in the upper half plane. What remains satisfies Laplace's equation everywhere in the region of interest, so it can be called the homogeneous solution. It might also be made part of the particular solution.

5.1.2 (a) The charge density follows from Poisson's equation.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_o} \Rightarrow \rho = \rho_o \cos\beta x \tag{1}$$

(b) The first term does not satisfy Laplace's equation and indeed was responsible for the charge density, (1). Thus, it can be taken as the particular solution and the remainder as the homogeneous solution. In that case,

$$\Phi_p = \frac{\rho_o \cos\beta x}{\epsilon_o\beta^2}; \quad \Phi_h = -\frac{\rho_o \cos\beta x}{\epsilon_o\beta^2}\frac{\cosh\beta y}{\cosh\beta a} \tag{2}$$

and the homogeneous solution must satisfy the boundary conditions

$$\Phi_h(y=-a) = \Phi_h(y=a) = -\frac{\rho_o\cos\beta x}{\epsilon_o\beta^2} \tag{3}$$

(c) We could just have well taken the total solution as the particular solution.

$$\Phi_p = \Phi; \quad \Phi_h = 0 \tag{4}$$

in which case the homogeneous solution must be zero on the boundaries.

5.1.3 (a) Because the second derivatives with respect to y and z are zero, the Laplacian reduces to the term on the left. The right side is the negative of the charge density divided by the permittivity, as required by Poisson's equation.

(b) With C_1 and C_2 integration coefficients, two integrations of (b) give

$$\Phi = -\frac{4\rho_o}{d^2\epsilon_o}\frac{(x-d)^4}{12} + C_1 x + C_2 \tag{1}$$

Evaluation of this expression at each of the boundaries then serves to determine the coefficients

$$C_1 = \frac{V}{d} - \frac{\rho_o d}{3\epsilon_o}; \quad C_2 = \frac{\rho_o d^2}{2\epsilon_o} \tag{2}$$

and hence the given potential.

(c) From the derivation it is clear that the Laplacian of the first term accounts for all of the charge density while that of the remaining terms is zero.

(d) On the boundaries, the homogeneous solution, which must cancel the potential of the particular solution on the boundaries, must be (d).

5.1.4 (a) The derivatives with respect to y and z are by definition zero, so Poisson's equation reduces to

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho_o}{\epsilon_o}\sin\left(\frac{\pi x}{d}\right) \tag{1}$$

(b) Two integrations of (1) give

$$\Phi = \frac{\rho_o d^2}{\epsilon_o \pi^2}\sin\left(\frac{\pi x}{d}\right) + C_1 x + C_2 \tag{2}$$

and evaluation at the boundaries determines the integration coefficients.

$$C_2 = 0; \quad C_1 = v/d \tag{3}$$

It follows that the required potential is

$$\Phi = \frac{\rho_o d^2}{\epsilon_o \pi^2}\sin\left(\frac{\pi x}{d}\right) + \frac{Vx}{d} \tag{4}$$

(c) From the derivation, the first term in (4) accounts for the charge density while the remaining terms have no second derivative and hence no Laplacian. Thus, the first term must be included in the particular solution while the remaining term can be defined as the homogeneous solution.

$$\Phi_p = \frac{\rho_o d^2}{\epsilon_o \pi^2}\sin\left(\frac{\pi x}{d}\right); \quad \Phi_h = \frac{Vx}{d} \tag{5}$$

(d) In the case of (c), it follows that the boundary conditions satisfied by the homogeneous solution are

$$\Phi_h(0) = -\Phi_p(0) = 0; \quad \Phi_h(d) = V - \Phi_p(d) = V \tag{6}$$

5.1.5 (a) There is no charge density, so the potential must satisfy Laplace's equation. $\mathbf{E} = (-v/d)\mathbf{i}_z = -\partial\Phi/\partial z$

$$\nabla^2\Phi = \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right) = 0 \tag{1}$$

(b) The surface charge density on the lower surface of the upper electrode follows from applying Gauss' continuity condition to the interface between the highly conducting metal and the free space just below. Because the field is zero in the metal,

$$\sigma_s = \epsilon_o[0 - E_z^b] = \frac{\epsilon_o v}{d} \tag{2}$$

(c) The capacitance follows from the integration of the surface charge density over the surface of the electrode having the potential v. That amounts to multiplying (2) by the area A of the electrode.

$$q = A\sigma_s = \frac{\epsilon_o A}{d} v = CV \tag{3}$$

(d) Enclose the upper electrode by the surace S having the volume V and the integral form of the charge conservation law is

$$\oint_S \mathbf{J} \cdot \mathbf{n} da + \frac{d}{dt}\int_V \rho dV = 0 \tag{4}$$

Contributions to the first term are confined to where the wire carrying the total current i into the volume passes through S. By definition, the second term is the total charge, q, on the electrode. Thus, (4) becomes

$$-i + \frac{dq}{dt} = 0 \tag{5}$$

Introduction of (3) into this expression then gives the current

$$i = C\frac{dv}{dt} \tag{6}$$

5.1.6 (a) Well away from the edges, the fields between the plates are the potential difference divided by the spacings. Thus, they are as given.

(b) The surface charge densities on the lower surface of the upper electrode and on the upper plus lower surfaces of the middle electrode are, respectively

$$\sigma_1 = \epsilon_o(0 + E_1) = \frac{\epsilon_o v_1}{2d}; \quad \sigma_m = \epsilon_o(0 + E_m) = \epsilon_o(v_1 - v_2)/d \tag{1}$$

$$\sigma_{mm} = -\epsilon_o(v_1 - v_2)/d + \epsilon_o v_2/d \tag{2}$$

Thus, the total charge on these electrodes is these quantities multiplied by the respective plate areas

$$q_1 = w[\sigma_1(L - l) + \sigma_m l] \tag{3}$$

$$q_2 = \epsilon_o l w \sigma_m \tag{4}$$

These are the expressions summarized in matrix notation by (a).

5.2 UNIQUENESS OF SOLUTIONS OF POISSON'S EQUATION

5.3 CONTINUITY CONDITIONS

5.3.1 (a) In the plane $y = 0$, the respective potentials are

$$\Phi^a(0) = V\cos\beta x = \Phi^b(0) \tag{1}$$

and are therefore equal.

(b) The tangential fields follow from the given potentials.

$$E_x^a = \beta V \sin\beta x e^{-\beta y}; \quad E_x^b = \beta V \sin\beta x e^{\beta y} \tag{2}$$

Evaluated at $y = 0$, these are also equal. That is, if the potential is continuous in a given plan, then so also is its slope in any direction within that plane.

(c) From Gauss' continuity condition applied to the plane $y = 0$,

$$\sigma_s = -\epsilon_o\left[\frac{\partial\Phi^a}{\partial y} - \frac{\partial\Phi^b}{\partial y}\right]_{y=0} = 2\beta V\cos\beta x \tag{3}$$

and this is the given surface charge density.

5.3.2 (a) The y dependence is not given. Thus, given that $\mathbf{E} = -\nabla\Phi$, only the x and z derivatives and hence x and z components of $\mathbf{E}$ can be found. These are the components of $\mathbf{E}$ tangential to the surface $y = 0$. If these components are to be continuous, then to within a constant so must be the potential in the plane $y = 0$.

(b) For this particular potential,

$$E_x = -\beta V\cos\beta x\sin\beta z; \quad E_z = -\beta V\sin\beta x\cos\beta z \tag{1}$$

If these are to be the tangential components of $\mathbf{E}$ on both sides of the interface, then the $x-z$ dependence of the potential from which they were derived must also be continuous (within a constant that must be zero if the electric field normal to the interface is to remain finite).

5.4 SOLUTIONS TO LAPLACE'S EQUATION IN CARTESIAN COORDINATES

5.4.1 (a) The given potential satisfies Laplace's equation. Evaluated at either $x = 0$ or $y = 0$ it is zero, as required by the boundary conditions on these boundaries. At $x = a$, it has the required potential, as it does at $y = a$ as well. Thus, it is the required potential.

(b) The plot of equipotentials and lines of electric field intensity is obtained from Fig. 4.1.3 by cutting away that part of the plot that is outside the boundaries at $x = a, y = a, x = 0$ and $y = 0$. Note that the distance between the equipotentials along the line $y = a$ is constant, as it must be if the potential is to have a linear distribution along this surface. Also, note that except for the special point at the origin (where the field intensity is zero anyway), the lines of electric field intensity are perpendicular to the zero potential surfaces. This is as it must be because there is no component of the field tangential to an equipotential.

5.4.2 (a) The potentials on the four boundaries are

$$\Phi(a, y) = V(y + a)/2a; \quad \Phi(-a, y) = V(y - a)/2a$$

$$\Phi(x, a) = V(x + a)/2a; \quad \Phi(x, -a) = V(x - a)/2a \tag{1}$$

(b) Evaluation of the given potential on each of the four boundaries gives the conditions on the coefficients

$$\Phi(\pm a, y) = \frac{V}{2a}y \pm \frac{V}{2} = \pm Aa + By + C + Dxy$$

$$\Phi(x, \pm a) = \frac{V}{2a}x \pm \frac{V}{2} = Ax \pm Ba + C + Dxy \tag{2}$$

Thus, $A = B = V/2a, C = 0$ and $D = 0$ and the equipotentials are straight lines having slope -1.

$$\Phi = \frac{V}{2a}(x + y) \tag{3}$$

(c) The electric field intensity follows as being uniform and having x and y components of equal magnitude.

$$\mathbf{E} = -\nabla\Phi = -\frac{V}{2a}(\mathbf{i_x} + \mathbf{i_y}) \tag{4}$$

(d) The sketches of the potential, (3), and field intensity, (4), are as shown in Fig. S5.4.2.

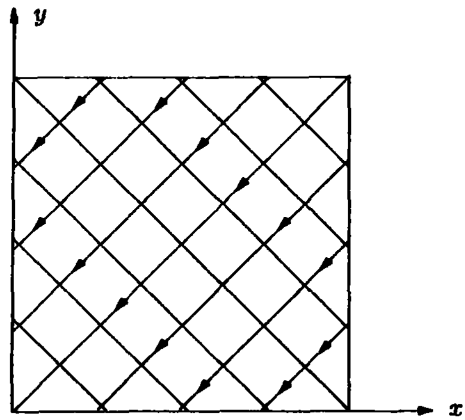


Figure S5.4.2

(e) To make the potential zero at the origin, $C = 0$. Evaluation at $(x, y) = (0, a)$ where the potential must also be zero shows that $B = 0$. Similarly, evaluation at $(x, y) = (a, 0)$ shows that $A = 0$. Evaluation at $(x, y) = (a, a)$ gives $D = V/2a^2$ and hence the potential

$$\Phi = \frac{V}{2a^2} xy \tag{5}$$

Of course, we are not guaranteed that the postulated combination of solutions to Laplace's equation will satisfy the boundary conditions everywhere. However, evaluation of (5) on each of the boundaries shows that it does. The associated electric field intensity is

$$\mathbf{E} = -\nabla\Phi = -\frac{V}{2a^2}(y\mathbf{i_x} + x\mathbf{i_y}) \tag{6}$$

The equipotentials and lines of field intensity are as shown by Fig. 4.1.3 inside the boundaries $x = \pm a$ and $y = \pm a$.

5.4.3 (a) The given potential, which has the form of the first term in the second column of Table 5.4.1, satisfies Laplace's equation. It also meets the given boundary conditions on the boundaries enclosing the region of interest. Therefore, it is the required potential.

(b) In identifying the equipotential and field lines of Fig. 5.4.1 with this configuration, note that $k = \pi/a$ and that the extent of the plot that is within the region of interest is between the zero potentials at $x = -\pi/2k$ and $x = \pi/2k$. The plot is then adapted to representing our potential distribution by multiplying each of the equipotentials by V_o divided by the potential given on the plot at $(x, y) = (0, b)$. Note that the field lines are perpendicular to the walls at $x = \pm a/2$.

5.4.4 (a) Write the solution as the sum of two, each meeting zero potential conditions on three of the boundaries and the required sinusoidal distribution on the fourth.

$$\Phi = V_o \sin\left(\frac{\pi x}{a}\right)\frac{\sinh(\pi y/a)}{\sinh(\pi)} + V_o \sin\frac{\pi y}{a}\frac{\sinh[\frac{\pi}{a}(a-x)]}{\sinh(\pi)} \tag{1}$$

(b) The associated electric field is

$$\mathbf{E} = -\frac{V_o\pi}{a\sinh(\pi)}\Big\{\Big[\cos(\pi x/a)\sinh(\pi y/a) - \sin(\pi y/a)\cosh\big[\frac{\pi}{a}(a-x)\big]\Big]\mathbf{i_x} + \Big[\sin(\pi x/a)\cosh(\pi y/a) + \cos(\pi y/a)\sinh\big[\frac{\pi}{a}(a-x)\big]\Big]\mathbf{i_y}\Big\} \tag{2}$$

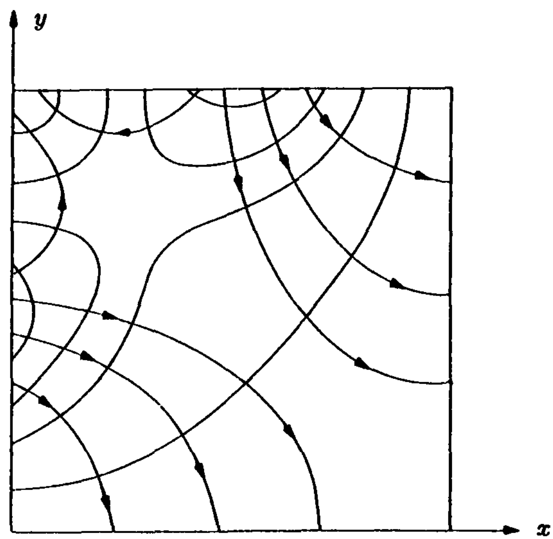


Figure S5.4.4

(c) A sketch of the equipotentials and field lines is shown in Fig. S5.4.4.

5.4.5 (a) The given potential, which has the form of the second term in the second column of Table 5.4.1, satisfies Laplace's equation. The electrodes have been shaped and constrained in potential to match the potential. For example, between $y = -b$ and $y = b$, we obtain the y coordinate of the boundary $\eta(x)$ as given by (a) by setting (b) equal to the potential v of the electrode, $y = \eta$ and solving for η.

(b) The electric field follows from (b) as $\mathbf{E} = -\nabla\Phi$.

(c) The potential given by (b) and field given by (c) have the same (x, y) dependence as that represented by Fig. 5.4.2. To adjust the numbers given on the plot for the potentials, note that the potential at the location $(x, y) = (0, a)$ on the upper electrode is v. Thus, to make the plot fit this situation, multiply each of the given potentials by v divided by the potential given on the plot at the location $(x, y) = (0, a)$.

(d) The charge on the electrode is found by enclosing it by a surface S and using Gauss' integral law. To make the integration over the surface enclosing the electrode convenient, the surface is selected as enclosing the electrode in an arbitrary way in the field free region above the electrode, passing through the slits in the planes $x = \pm l$ to the y equal zero plane and closing in the $y = 0$ plane. Thus, with y_1 defined as the height of the electrode at its left and right extremities, the net charge is

$$\begin{aligned} q &= d\epsilon_o \int_{y=0}^{y_1} -E_x(x=-l)dy + d\epsilon_o \int_{y=0}^{y_1} E_x(x=l)dy \\ &+ d\epsilon_o \int_{x=-l}^{l} -E_y(y=0)dx \\ &= -\frac{vd\pi\epsilon_o}{2b\sinh\left(\frac{\pi a}{2b}\right)}\left[\int_0^{y_1} -\sin\frac{\pi l}{2b}\sinh\frac{\pi y}{2b}dy \right. \\ &+ \int_0^{y_1} -\sin\frac{\pi l}{2b}\sinh\frac{\pi y}{2b}dy \\ &+ \left. \int_{-l}^{l} -\cos\frac{\pi x}{2b}dx\right] \end{aligned} \tag{2}$$

Note that

$$\sinh ky_1 = \frac{\sinh ka}{\cos kl}; \quad -\sinh^2 ky + \cosh^2 ky = 1 \tag{3}$$

and (2) becomes the given result.

(e) Conservation of charge for a surface enclosing the electrode through which the wire carrying the current i passes requires that $i = dq/dt$. Thus, given the result of (d) and the voltage dependence, (e) follows.

5.4.6 (a) Reversing the potentials on the lower electrodes turns the potential from an even to an odd function of y. Thus, the potential takes the form of the first term in the second column of Table 5.4.1.

$$\Phi = A\cosh\left(\frac{\pi y}{2b}\right)\cos\frac{\pi x}{2b} \tag{1}$$

To make the potential be v at $(x, y) = (0, a)$, the coefficient is adjusted so that

$$\Phi = v\cos kx\frac{\cosh ky}{\cosh ka}; \quad k \equiv \frac{\pi}{2b} \tag{2}$$

The shape of the upper electrode in the range between $x = -b$ and $x = b$ is then obtained by solving (2) with $\Phi = v$ and $y = \eta$ for η.

$$\eta = \frac{1}{k}\cosh^{-1}\left[\frac{\cosh ka}{\cos kx}\right] \tag{3}$$

(b) The electric field intensity follows from (2) as

$$\mathbf{E} = -\frac{vk}{\cosh ka}[-\sin(kx)\cosh(ky)\mathbf{i}_x + \cos kx \sinh ky \mathbf{i}_y] \tag{4}$$

(c) The equipotentials and field lines are as shown by Fig. 5.4.2. To adjust the given potentials, multiply each by v divided by the potential given from the plot at the location $(x, y) = (0, a)$.

(d) The charge on the electrode segment is obtained by using Gauss' integral law with a surface that encloses the electrode. This surface is arbitrary in the field free region above the electrode. For convenience, it passes through the slits to the $y = 0$ plane in the planes $x = \pm l$ and closes in the $y = 0$ plane. Note that there is no electric field perpendicular to this latter surface, so the only contributions to the surface integration come from the surfaces at $x = \pm l$.

$$\begin{aligned} q &= 2d\epsilon_o \int_0^y \left[\frac{vk}{\cosh ka}\sin(kl)\cosh(ky)\right]dy \\ &= \frac{2d\epsilon_o v}{\cosh ka}\sin kl \sinh ky_1 \end{aligned} \tag{5}$$

With the use of the identities

$$\cosh(ky_1) = \frac{\cosh ka}{\cos kl}; \qquad \cosh^2 ky_1 - \sinh^2 ky_1 = 1 \tag{6}$$

(5) becomes

$$q = Cv = \frac{2d\epsilon_o v}{\cosh ka}\sin kl \sqrt{\left[\frac{\cosh(ka)}{\cos kl}\right]^2 - 1} \tag{7}$$

(e) From conservation of charge,

$$i = C\frac{dv}{dt} = -CV_o\omega \sin \omega t$$

5.5 MODAL EXPANSIONS TO SATISFY BOUNDARY CONDITIONS

5.5.1 (a) The solutions superimposed by the infinite series of (a) are chosen to be zero in the planes $x = 0$ and $x = b$ and to be the linear combination of exponentials in the y direction that are zero at $y = b$. To evaluate the coefficients, multiply both sides by $\sin(m\pi x/a)$ and integrate from $x = 0$ to $x = a$

$$\int_0^a \Phi(x, 0)\sin\left(\frac{m\pi x}{a}\right)dx = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\frac{m\pi x}{a}dx \tag{1}$$

The integral on the right is zero except for $m = n$, in which case the integral of $\sin^2(n\pi x/a)$ over the interval $x = 0$ to $x = a$ gives the average value of 1/2 multiplied by the length a, $a/2$. Thus, (1) can be solved for the coefficient A_m, to obtain (b) as given (if $m \to n$).

(b) In the specific case where the distribution is as given, the integration of (b) gives

$$\begin{aligned} A_n &= \frac{2}{a \sinh\left(-\frac{n\pi b}{a}\right)} \int_{a/4}^{3a/4} V_1 \sin\left(\frac{n\pi x}{a}\right) dx \\ &= \frac{2V_1}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \left[\cos\left(\frac{n\pi x}{a}\right)\right]_{a/4}^{3a/4} \end{aligned} \tag{2}$$

which becomes (c) as given.

5.5.2 (a) This problem illustrates how the modal approach can be applied to finding the solutions in a rectangular region for arbitrary boundary conditions on all four of the boundaries. In general, four infinite series would be used, each with zero potential on three of the walls and with coefficients to match the potential boundary condition on the fourth wall. Here, the potential is zero on two of the walls, so only two infinite series are used. The first is zero in the planes $y = 0, y = b$ and $x = a$ and, because the potential is constant in the plane $x = 0$, has coefficients that are as given by (5.5.8). (The roles of a and b are reversed relative to those in the section for this first term and the minus sign results because the potential is being matched at $x = 0$. Note that the argument of the sinh function is negative within the region of interest.) The coefficients of the second series are similarly determined. (This time, the roles of x and y and of a and b are as in the section discussion, but the surface where the uniform potential is imposed is at $y = 0$ rather than $y = b$.)

(b) The surface charged density on the wall at $x = a$ is

$$\sigma_s = \epsilon_o[-E_x(x = a)] = -\epsilon_o \frac{\partial \Phi}{\partial x}(x = a) \tag{1}$$

Evaluation using (a) results in (b).

5.5.3 (a) For arbitrary distributions of potential in the plane $y = 0$ and $x = 0$, the potential is taken as the superposition of series that are zero on all but these planes, respectively.

$$\begin{aligned} \Phi &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left[\frac{n\pi}{a}(y - b)\right] \\ &+ \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left[\frac{n\pi}{b}(x - a)\right] \end{aligned} \tag{1}$$

The first of these series must satisfy the boundary condition in the plane $y = 0$,

$$\Phi(x = 0) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi}{a} x\right) \tag{2}$$

where

$$\Phi(x,0) = \begin{cases} 2V_a x/a; & 0 < x < a/2 \\ 2V_a(1 - x/a); & a/2 < x < a \end{cases} \tag{3}$$

Multiplication of both sides of (2) by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ gives

$$\begin{aligned} \frac{2V_a}{a}\int_0^{a/2} x \sin\left(\frac{m\pi x}{a}\right)dx + 2V_a \int_{a/2}^{a} \sin\left(\frac{m\pi x}{a}\right)dx \\ - \frac{2V_a}{a}\int_{a/2}^{a} x\sin\left(\frac{m\pi x}{a}\right)dx \\ = A_m \frac{a}{2}\sinh\left(-\frac{m\pi b}{a}\right) \end{aligned} \tag{4}$$

Integration, solution for $A_m \to A_n$ then gives $A_n = 0$, n even and for n odd

$$A_n = -\frac{8V_a \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh\left(\frac{n\pi b}{a}\right)} \tag{5}$$

Evalution on the boundary at $x = 0$ leads to a similar term with the roles of V_a and a replaced by those of V_b and b, respectively. Thus, $B_n = 0$ for n even and for n odd

$$B_n = -\frac{8V_b \sin\left(\frac{n\pi}{a}\right)}{n^2\pi^2 \sinh\left(\frac{n\pi a}{b}\right)} \tag{5}$$

(b) The surface charge density in the plane $y = b$ is

$$\begin{aligned} \sigma_s &= \epsilon_o[-E_y(y = b)] = \epsilon_o \frac{\partial \Phi}{\partial y}(y = b) \\ &= \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \left[A_n\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right) - B_n\left(\frac{n\pi}{b}\right)\sinh\left[\left(\frac{n\pi}{b}\right)(x - a)\right]\right] \end{aligned} \tag{6}$$

where A_n and B_n are given by (5) and (6).

5.5.4 (a) Far to the left, the system appears as a parallel plate capacitor. A uniform field satisfies both Laplace's equation and the boundary conditions.

$$\mathbf{E} = -\frac{V}{d}\mathbf{i}_y \Rightarrow \Phi_a = \frac{Vy}{d} \tag{1}$$

(b) Because the uniform field part of this solution, Φ_a, satisfies the conditions far to the left, the aditional part must go to zero there. However, the first term produces a field tangential to the right boundary which must be cancelled by the second term. Thus, conditions on the second term are that it also satisfy Laplace's equation and the boundary conditions as given

(c) Because of the homogeneous boundary conditions in the $y = 0$ and $y = d$ planes, the solution is selected as being sinusoidal in the y direction. Because the region extends to infinity in the $-x$ direction, exponential solutions are used in that direction, with the sign of the exponent arranged to assure decay in the $-x$ direction.

$$\Phi_b = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right) e^{n\pi x/d} \tag{2}$$

The coefficients are determined by the requirement on this part of the potential at $x = 0$.

$$-\frac{Vy}{d} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{d}\right) \tag{3}$$

Multiplication by $\sin(m\pi y/d)$, integration from $y = 0$ to $y = d$, solution for A_m and replacement of A_m by A_n gives

$$A_n = \frac{2V}{n\pi}\cos n\pi = \frac{2V}{n\pi}(-1)^n \tag{4}$$

The sum of the potentials of (1) and (2) with the coefficient given by (4) is (e).

(d) The equipotential lines must be those of a plane parallel capacitor, (1), far to the left where the associated field lines are y directed and uniform. Because the boundaries are either at the potential V or at zero potential to the right, these equipotential lines can only terminate in the gap at $(x, y) = (0, d)$, where the potential makes an abrupt excursion from the zero potential of the right electrode to the potential V of the top electrode. In this local, the potential lines converge and become radially symmetric. The boundaries are themselves equipotentials. The electric field, which is perpendicular to the equipotentials and directed from the upper electrode toward the bottom and right electrodes, can then be pictured as shown by Fig. 6.6.9c turned upside down.

5.5.5 (a) The potential far to the left is that of a plane parallel plate capacitor. It takes the form $Ax + B$, with the coefficients adjusted to meet the boundary conditions at $x = 0$ and $x = a$.

$$\Phi(y \to -\infty) \to \Phi_a = \frac{V_o}{2}\left(1 - \frac{2x}{a}\right) \tag{1}$$

(b) With the total potential written as

$$\Phi = \Phi_a + \Phi_b \tag{2}$$

the potential Φ_b can be used to make the total potential satisfy the boundary condition at $y = 0$. Because the first part of (2) satisfies Laplace's equation and the boundary conditions far to the left, the second part must go to zero there. Thus, it is taken as a superposition of solutions to Laplace's equation that are zero in the planes $y = 0$ and $y = a$ (so that the potential there as given by the first term is not disturbed) and that decay exponentially in the $-y$ direction.

$$\Phi_b = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{n\pi y/a} \tag{3}$$

At $y = 0, \Phi(x,0) = \Phi_d(x)$. Thus, $\Phi_b(x,0) = \Phi_d(x) - \Phi_a(x)$ and evaluation of (3) at $y = 0$, multiplication by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ gives

$$\int_0^a \left[\Phi_d(x) - \frac{V_o}{2}\left(1 - \frac{2x}{a}\right)\right] \sin\frac{m\pi x}{a} dx = A_m \frac{a}{2} \tag{4}$$

from which it follows that

$$A_n = \frac{2}{a}\int_0^a \Phi_d(x) \sin\left(\frac{n\pi x}{a}\right) dx - \begin{cases} \frac{2V_o}{n\pi}; & n \text{ even} \\ 0; & n \text{ odd} \end{cases} \tag{5}$$

Thus, the potential between the plates is

$$\Phi = \frac{V_o}{2}\left(1 - \frac{2x}{a}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{n\pi y/a} \tag{6}$$

where A_n is given by (5).

5.5.6 The potential is taken as the sum of two, the first being zero on all but the boundary at $x = a$ where it is $V_o y/a$ and the second being zero on all but the boundary at $y = a$, where it is $V_o x/a$. The second solution is obtained from the first by interchanging the roles of x and y. For the first solution, we take

$$\Phi_1 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh n\pi} \tag{1}$$

The coefficients follow by evaluating this expression at $x = a$, multiplying by $\sin(m\pi y/a)$ and integrating from $y = 0$ to $y = a$.

$$\int_0^a \frac{V_o x}{a} \sin\left(\frac{n\pi x}{a}\right) dx = A_n(a/2) \tag{2}$$

Thus,

$$A_n = -\frac{2V_o}{n\pi}(-1)^n \tag{3}$$

The first part of the solution is given by substituting (3) into (1). It follows that the total solution is

$$\Phi = \sum_{n=1}^{\infty} -\frac{2V_o}{n\pi}\frac{(-1)^n}{\sinh(n\pi)}\left[\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) + \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)\right] \tag{4}$$

5.5.7 (a) The total potential is zero at $y = 0$ and so also is the first term. Thus, Φ_1 must be zero as well at $y = 0$. The first term satisfies the boundary condition at $y = b$, so Φ_1 must be zero there as well. However, in the planes $x = 0$ and $x = a$, the first term has a potential Vy/b that must be cancelled by the second term so that the sum of the two terms is zero. Thus, Φ_1 must satisfy the conditions summarized in the problem statement.

(b) To satisfy the conditions at $x = 0$ and $x = a$, the y dependence is taken as $\sin(n\pi y/b)$. The product form x dependence is a linear combination of exponentials having arguments $(n\pi y/b)$. Because the boundary conditions in the $x = 0$ and $x = a$ planes are even about the plane $x = a/2$, this linear combination is taken as being the cosh function displaced so that its origin is at $x = a/2$.

$$\Phi = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \cosh\left[\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right] \tag{1}$$

Thus, if the boundary condition is satisfied at $x = a$, it is at $x = 0$ as well. Evaluation of (1) at $x = a$, multiplication by $\sin(m\pi y/b)$ and integration from $y = 0$ to $y = b$ then gives an expression that can be solved for A_m and hence A_n

$$A_n = \frac{2V(-1)^n}{n\pi \cosh(n\pi a/2b)} \tag{2}$$

In terms of these coefficients, the desired solution is then

$$\Phi = \frac{Vy}{b} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \cosh\left[\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right] \tag{3}$$

5.6 SOLUTIONS TO POISSON'S EQUATION WITH BOUNDARY CONDITIONS

5.6.1 The potential is the sum of two homogenous solutions that satisfy Laplace's equation and a third inhomogeneous solution that makes the potential satisfy Poisson's equation for each point in the volume. This latter solution, which follows from assuming $\Phi_p = \Phi_p(y)$ and integration of Poisson's equation, is arranged to give zero potential on each of the boundaries, so it is up to the first two to satisfy the boundary conditions. The first solution is zero at $y = 0$, has the same x dependence as the wall at $y = d$ and has a coefficient that has been adjusted so that the magnitude of the potential matches that at $y = d$. The second solution is zero at $y = d$ (the displaced sinh function is a linear combination of the sinh and cosh functions in column 2 of Table 5.4.1) and so does not disturb the potential already satisfied by the first term at that boundary. At $y = 0$, where the first term has been arranged to make no contribution, it has the same y dependence as the potential in the $y = 0$ plane and has its coefficient adjusted so that it has the correct magnitude on that boundary as well.

5.6.2 The particular solution is found by assuming that the particular potential is only a function of y and integration of Poisson's equation twice. With the two integration coefficients adjusted to make the potential of this particular solution zero on each of the boundaries, it is the same as the last term in (a) of Prob. 5.6.1. Thus, the homogeneous solution must be zero at $y = 0$, suggesting that it has a sinh function y dependence. The x dependence of the potential at $y = d$ then suggests the x dependence of the potential be made $\sin(kx)$. With the coefficient of this homogeneous solution adjusted so that the condition at $y = d$ is satisfied, the desired potential is

$$\Phi = \Phi_o \sinh kx \frac{\sinh ky}{\sinh kd} - \frac{\rho_o}{2\epsilon_o} y(y-d) \tag{1}$$

5.6.3 (a) In the volume, Poisson's equation is satisfied by a potential that is independent of y and z,

$$\nabla^2 \Phi_p = \frac{\partial^2 \Phi_p}{\partial x^2} = -\frac{\rho_o}{\epsilon_o} \cos k(x-\delta) \tag{1}$$

Two integrations give the particular solution

$$\Phi_p = \frac{\rho_o}{\epsilon_o k^2} \cos k(x-\delta) \tag{2}$$

$$\mathbf{E}_p = \frac{\rho_o}{\epsilon_o k} \sin k(x-\delta)\mathbf{i_x} \tag{3}$$

(b) The boundary conditions at $y = \pm d/2$ are

$$E_x = E_{px} + E_{hx} = E_o \cos kx \tag{4}$$

Because the configuration is symmetric with respect to the $x-z$ plane, use $\cosh(ky)$ as the y dependence. Thus, in view of the two x dependencies, the homogeneous potential is assumed to take the form

$$\Phi_h = [A \sin kx + B \cos k(x-\delta)] \cosh ky \tag{5}$$

The condition of (4) then requires that

$$E_{xh} = -[A \cos kx - B \sin k(x-\delta)] k \cosh ky \tag{6}$$

and it follows from the fact that at $y = d/2$ that (3) + (6) = (4)

$$A = -E_o/k \cosh(kd/2); \quad B = -\rho_o/\epsilon_o k^2 \cosh(kd/2) \tag{7}$$

so that the total potential is as given by (d) of the problem statement.

(c) First note that because of the symmetry with respect to the z plane, there is no net force in the y direction. In integrating ρE_x over the volume, note that E_x is

$$E_x = \frac{\rho_o}{\epsilon_o k}\sin k(x-\delta) + \frac{\cosh kh}{\cosh\left(\frac{kd}{2}\right)}\left[E_o\cos kx - \frac{\rho_o}{\epsilon_o k}\sin k(x-\delta)\right] \tag{8}$$

In view of the x dependence of the charge density, only the second term in this expression makes a contribution to the integral. Also, $\rho = \rho_o\cos k(x-\delta) = \rho_o[\cos k\delta\cos kx - \sin k\delta\sin kx]$ and only the first of these two terms makes a contribution also.

$$\begin{aligned} f_x &= \int_0^{2\pi/k}\int_{-d/2}^{d/2}\rho_o\cos k\delta\cos kx\frac{\cosh ky}{\cosh\left(\frac{kd}{2}\right)}E_o\cos kx\,dy\,dx \\ &= [2\pi\rho_o E_o\cos k\delta\tanh(kd/2)]/k^2 \end{aligned} \tag{9}$$

5.6.4 (a) For a particular solution, guess that

$$\Phi = A\cos k(x-\delta) \tag{1}$$

Substitution into Poisson's equation then shows that $A = \rho_o/\epsilon_o k^2$ so that the particular solution is

$$\Phi_p = \frac{\rho_o}{\epsilon_o k^2}\cos k(x-\delta) \tag{2}$$

(b) At $y = 0$

$$\Phi_h = -\Phi_p = -\frac{\rho_o}{\epsilon_o k^2}\cos k(x-\delta) \tag{3}$$

while at $y = d$,

$$\Phi_h = V_o\cos kx - \frac{\rho_o}{\epsilon_o k^2}\cos k(x-\delta) \tag{4}$$

(c) The homogeneous solution is itself the sum of a part that satisfies the conditions

$$\Phi_1(y=d) = V_o\cos kx, \quad \Phi_1(y=0) = 0 \tag{5}$$

and is therefore

$$\Phi_1 = V_o\cos kx\frac{\sinh ky}{\sinh kd} \tag{6}$$

and a part satisfying the conditions

$$\Phi_2(y=d) = -\frac{\rho_o}{k^2\epsilon_o}\cos k(x-\delta); \quad \Phi_2(0) = -\frac{\rho_o}{\epsilon_o k^2}\cos k(x-\delta) \tag{7}$$

which is therefore

$$\Phi_2 = -\frac{\rho_o}{k^2\epsilon_o}\cos k(x-\delta)\frac{\cosh k(y-\frac{d}{2})}{\cosh(kd/2)} \tag{8}$$

Thus, the total potential is the sum of (2), (6) and (8).

$$\Phi = \frac{\rho_o}{\epsilon_o k^2}\cos k(x-\delta)\left[1 - \frac{\cosh k(y-\frac{d}{2})}{\cosh\left(\frac{kd}{2}\right)}\right] + V_o \cos kx \frac{\sinh ky}{\sinh kd} \tag{9}$$

(d) In view of the given charge density and (9), the force density in the x direction is

$$\begin{aligned} F_x = &\frac{\rho_o}{k\epsilon_o}\sin k(x-\delta)\cos k(x-\delta)\left[1 - \frac{\cosh k(y-\frac{d}{2})}{\cosh\left(\frac{kd}{2}\right)}\right] \\ &+ \rho_o k V_o \sin kx \cos k(x-\delta)\frac{\sinh ky}{\sinh kd} \end{aligned} \tag{10}$$

The first term in this expression integrates to zero while the second gives a total force of

$$f_x = \frac{\rho_o k V_o}{\sinh kd}\int_0^{2\pi/k}\int_0^d \sin kx \cos k(x-\delta)\sinh ky\,dy\,dx \tag{11}$$

With the use of $\cos k(x-\delta) = \cos kx \cos k\delta + \sin kx \sin k\delta$, this integration gives

$$f_x = \rho_o \pi V_o \frac{(\cosh kd - 1)\sin k\delta}{k \sinh kd} \tag{12}$$

5.6.5 By inspection, we know that if we look for a particular solution having only a y dependence, it will have the same y dependence as the charge distribution (the second derivative of the sin function is once again a sin function). Thus, we substitute $A\sin(\pi y/b)$ into Poisson's equation and evaluate A.

$$\Phi_p = \frac{\rho_o b^2}{\epsilon_o \pi^2}\sin\left(\frac{\pi y}{b}\right) \tag{1}$$

The homogeneous solution must therefore be zero on the boundaries at $y = b$ and $y = 0$ and must be $-\rho_o b^2 \sin(\pi y/b)/\epsilon_o \pi^2$ at $x = \pm a$. This latter condition is even in x and can be matched by the solution to Laplace's equation

$$\Phi_h = A\sin\left(\frac{\pi y}{b}\right)\frac{\cosh(\pi x/b)}{\cosh(\pi a/b)} \tag{2}$$

if the coefficient, A, is made

$$A = -\rho_o b^2/\epsilon_o \pi^2 \tag{3}$$

Thus, the solution is the sum of (1) and (2) with A given by (3).

5.6.6 (a) The charge distribution follows from Poisson's equation.

$$-\frac{\rho}{\epsilon_o} = \nabla^2 \Phi_p \Rightarrow \rho = \epsilon_o V \sin \beta x \sin \frac{\pi y}{b}(\beta^2 + \frac{\pi^2}{b^2}) \tag{1}$$

(b) To make the total solution satisfy the zero potential conditions, the homogeneous solution must also be zero at $y = 0$ and $y = b$. At $x = 0$ it must also be zero but at $x = a$ the homogeneous solution must be $\Phi_h = -V \sin(\pi y/b) \sin \beta a$. Thus, we select the homogeneous solution

$$\Phi_h = A \sin \frac{\pi y}{b} \frac{\sinh(\pi x/b)}{\sinh(\pi a/b)} \tag{2}$$

make $A = -V \sin \beta a$ and obtain the potential distribution

$$\Phi = V \sin\left(\frac{\pi y}{b}\right)\left[\sin \beta x - \sin \beta a \frac{\sinh(\pi x/b)}{\sinh(\pi a/b)}\right] \tag{3}$$

5.6.7 A particular solution is found by assuming that it only depends on x and integrating Poisson's equation twice to obtain

$$\Phi_p = -\frac{\rho_o l^2}{6\epsilon_o}\left(\frac{x}{l} - \frac{x^3}{l^3}\right) \tag{1}$$

The two integration constants have been assigned so that the potential is zero at $x = 0$ and $x = l$. The homogeneous solution must therefore satisfy the boundary conditions

$$\Phi_h(x = 0) = \Phi_h(x = l) = 0$$

$$\Phi_h(y = \pm d) = -\frac{\rho_o l^2}{6\epsilon_o}\left(\frac{x}{l} - \frac{x^3}{l^3}\right) \tag{2}$$

The first two of these are satisfied by the following solutions to Laplace's equation.

$$\Phi_h = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \frac{\cosh\left(\frac{n\pi y}{d}\right)}{\cosh\left(\frac{n\pi d}{l}\right)} \tag{3}$$

This potential has an even y dependence, reflecting the fact that the boundary conditions are even in y. To determine the coefficients in (3), note that the second pair of boundary conditions require that

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = -\frac{\rho_o l^2}{6\epsilon_o}\left(\frac{x}{l} - \frac{x^3}{l^3}\right) \tag{4}$$

Multiplication of both sides of this expression by $\sin(m\pi x/l)$, and integration gives

$$A_m \frac{l}{2} = -\frac{\rho_o l}{6\epsilon_o} \int_0^l x \sin\left(\frac{m\pi x}{l}\right) dx + \frac{\rho_o}{6\epsilon_o l} \int_0^l x^3 \sin \frac{m\pi x}{l} dx \tag{5}$$

or

$$A_n = \frac{2l^3 \rho_o}{\epsilon_o (n\pi)^3}(-1)^n$$

Thus, the required potential is

$$\Phi = \frac{\rho_o l^2}{6\epsilon_o}\Big(\frac{x}{l} - \frac{x^3}{l^3}\Big) + \sum_{n=1}^{\infty} \frac{2}{l}\Big(\frac{l}{n\pi}\Big)^3 \frac{\rho_o}{\epsilon_o}(-1)^n \sin\frac{n\pi x}{l}\frac{\cosh\left(\frac{n\pi y}{l}\right)}{\cosh\left(\frac{n\pi d}{l}\right)} \tag{6}$$

5.6.8 (a) The charge density can be found using Poisson's equation to confirm that the charge density is that given. Thus, the particular solution is indeed as given.

(b) Continuity conditions at the interface where $y = 0$ are

$$\Phi^a = \Phi^b \tag{1}$$

$$\frac{\partial \Phi^a}{\partial y} = \frac{\partial \Phi^b}{\partial y} \tag{2}$$

To satisfy these conditions, add to the particular solution a solution to Laplace's equation in the respective regions having the same x dependence and decaying to zero far from the interface.

$$\Phi^a = A\cos\beta x e^{-\beta y} \tag{3}$$

$$\Phi^b = \frac{\rho_o}{\epsilon_o(\beta^2 - \alpha^2)}\cos\beta x e^{\alpha y} + B\cos\beta x e^{\beta y} \tag{4}$$

Substitution of these relations into (1) and (2) shows that

$$A = \frac{\rho_o}{\epsilon_o(\beta^2 - \alpha^2)2}\Big(1 - \frac{\alpha}{\beta}\Big) \tag{5}$$

$$B = \frac{-\rho_o}{\epsilon_o(\beta^2 - \alpha^2)2}\Big(1 + \frac{\alpha}{\beta}\Big) \tag{6}$$

and substitution of these coefficients into (3) and (4) results in the given potential distribution.

5.6.9 (a) The potential in each region is the sum of a part due to the wall potentials without the surface charge in the plane $y = 0$ and a part due to the surface charge and having zero potential on the walls. Each of these is continuous in the $y = 0$ plane and even in y. The x dependence of each is determined by the respective x dependencies of the wall potential and surface charge density distribution. The latter is the same as that part of its associated potential so that Gauss' continuity condition can be satisfied. Thus, with A a yet to be determined coefficient, the potential takes the form

$$\Phi = \begin{cases} V\frac{\cosh\beta y}{\cosh\beta a}\cos\beta x - A\sinh\beta(y-a)\sin\beta(x-x_o); & 0 < y < a \\ V\frac{\cosh\beta y}{\cosh\beta a}\cos\beta x - A\sinh\beta(y+a)\sin\beta(x-x_o); & -a < y < 0 \end{cases} \tag{1}$$

The coefficient is determined from Gauss' condition to be

$$-\epsilon_o\left[\frac{\partial\Phi^a}{\partial y}-\frac{\partial\Phi^b}{\partial y}\right]_{y=0}=\sigma_o\sin\beta(x-x_o)\Rightarrow A=\frac{-\sigma_o}{2\epsilon_o\beta\cosh\beta a} \tag{2}$$

(b) The force is

$$f_x=d\int_x^{x+2\pi/\beta}\sigma_o\sin\beta(x-x_o)E_x(y=0)dx \tag{3}$$

From (1),

$$E_x(y=0)=V\beta\frac{\sin\beta x}{\cosh\beta a}-\frac{\sigma_o\sinh\beta a}{2\epsilon_o\cosh\beta a}\cos\beta(x-x_o) \tag{4}$$

The integration of the second term in this expression in (3) will give no contribution. Substitution of the first term gives

$$f_x=\frac{d\sigma_o V\beta}{\cosh\beta a}\int_0^{x+2\pi/\beta}\sin\beta(x-x_o)\sin\beta x dx=\sigma_o V\beta(\frac{d\pi}{\beta})\frac{\cos\beta x_o}{\cosh\beta a} \tag{5}$$

(d) Because the charge and wall potential are synchronous, that is $U=\omega/\beta$, the new potential distribution is just that found with x replaced by $x-Ut$. Thus, the force is that already found. The force acts on the external mechanical system (acts to accelerate the charged particles). Thus, Uf_x is the mechanical power output and $-Uf_x$ is the mechanical power input. Because the system is loss free and the system is in the steady state so that there is no energy storage, $-Uf_x$ is therefore the electrical power output.

$$\text{Electrical Power Out}=-Uf_x=-Ud\sigma_o V\beta\frac{\pi}{\beta}\frac{\cos\beta x_o}{\cosh\beta a} \tag{6}$$

(e) For (6) to be positive so that the system is a generator, $\frac{\pi}{2}<\beta x_o<\frac{3\pi}{2}$.

5.7 SOLUTIONS TO LAPLACE'S EQUATION IN POLAR COORDINATES

5.7.1 The given potentials have the correct values at $r=a$. With $m=5$, they are solutions to Laplace's equation. Of the two possible solutions in each region having $m=5$ and the given distribution, the one that is singular at the origin is eliminated from the inner region while the one that goes to infinity far from the origin is eliminated from the outer solution. Hence, the given solution.

5.7.2 (a) Of the two potentials have the same ϕ dependence as the potential at $r = R$, the one that is not singular at the origin is

$$\Phi = \frac{V}{R} r \sin\phi = \frac{V}{R} y \tag{1}$$

Note that this potential is also zero on the $y = 0$ plane, so it satisfies the potential conditions on the enclosing surface.

(b) The surface charge density on the equipotential at $y = 0$ is

$$\sigma_s = \epsilon_o E_y = -\epsilon_o \frac{\partial \Phi}{\partial y} = -\epsilon_o \frac{V}{R} \tag{2}$$

and hence is uniform.

5.7.3 The solution is written as the sum of two solutions, Φ^a and Φ^b. The first of these is the linear combination of solutions matching the potential on the outside and being zero on the inside. Thus, when added to the second solution, which is zero on the outside but assumes the given potential on the inside, it does not disturb the potential on the inside boundary. Nor does the second potential disturb the potential of the first solution on the outside boundary. Note also that the correct combination of solutions, $(r/b)^3$ and $(b/r)^3$ in the first solution and (r/a) and (a/r) in the second solution can be determined by inspection by introducing r normalized to the radius at which the potential must be zero. By using the appropriate powers of r, this approach can be used for any ϕ dependence of the given potential.

5.7.4 From Table 5.7.1, column two, the potentials that are zero at $\phi = 0$ and $\phi = \alpha$ are

$$r^{\pm m} \sin m\phi \tag{1}$$

with $m = n\pi/\alpha, n = 1, 2, \ldots$ In taking a linear combination of these that is zero at $r = a$, it is convenient to normalize the r dependence to a and write the linear combination as

$$\Phi = A(r/a)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) + B(a/r)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{2}$$

where A and B are to be determined. It can be seen from (2) that to make $\Phi = 0$ at $r = a, A = -B$ and the solution becomes

$$\Phi = A\big[(r/a)^{n\pi/\alpha} - (a/r)^{n\pi/\alpha}\big] \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

Finally, the last coefficient and n are adjusted so that the potential meets the condition at $r = b$. Thus,

$$\Phi = V_b \frac{[(r/a)^{n\pi/\alpha} - (a/r)^{n\pi/\alpha}]}{[(b/a)^{n\pi/\alpha} - (a/b)^{n\pi/\alpha}]} \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{4}$$

5.7.5 To make the potential zero at $\phi = 0$, use the second and fourth solutions in the third column of Table 5.7.1.

$$\cos[pln(r)]\sinh p\phi, \quad \sin[pln(r)]\sinh p\phi \tag{1}$$

The linear combination of these solutions that is zero at $r = a$ is obtained by simply normalizing r to a in the second solution. This can be seen by using the double-angle formula to write that solution as

$$\begin{aligned} A\sin[pln(r/a)]\sinh p\phi &= A\sin[pln(r) - pln(a)]\sinh p\phi \\ &= A\{\sin[pln(r)]\cos[pln(a)] \\ &- \cos[pln(r)]\sin[pln(a)]\}\sinh p\phi \end{aligned} \tag{2}$$

This solution is made to be zero at $r = b$ by making $p = n\pi/ln(b/a)$, where n is any integer. Finally, the last boundary condition at $\phi = 0$ is met by adjusting the coefficient A and selecting $n = 3$.

$$A = V/\sinh[3\pi\alpha/ln(b/a)] \tag{3}$$

5.7.6 The potential is a linear combination of the first two in column one of Table 5.7.1.

$$\Phi = A\phi + B = -\frac{V}{(3\pi/2)}\left(\phi - \frac{3\pi}{2}\right) = V\left(1 - \frac{2\phi}{3\pi}\right) \tag{1}$$

This potential and the associated electric field are sketched in Fig. S5.7.6.

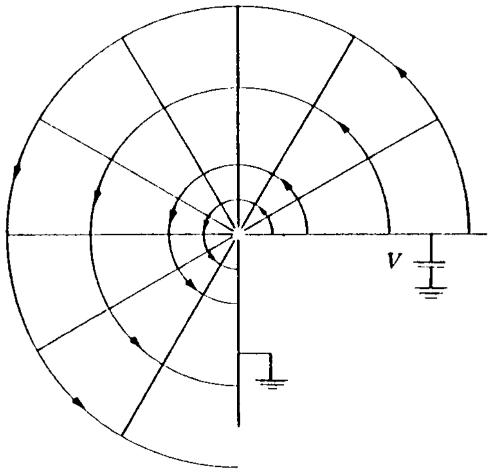


Figure S5.7.6

5.8 EXAMPLES IN POLAR COORDINATES

5.8.1 Either from (5.8.4) or from Fig. 5.8.2, it is clear that outside of the cylinder, the $z = 0$ plane is one having the same zero potential as the surface of the cylinder. Therefore, the potential and field as respectively given by (5.8.4) and (5.8.5) also describe the given situation.

Intuitively, we would expect the maximum electric field to be at the top of the cylinder, at $r = R, \phi = \pi/2$. From (5.8.5), the field at this point is

$$E_{\max} = 2E_o \tag{1}$$

and this maximum field is indeed independent of the cylinder radius. To be more rigorous, from (5.8.5), the magnitude of $\mathbf{E}$ is

$$|\mathbf{E}| = E_o\mathcal{E} \tag{2}$$

where

$$\mathcal{E} \equiv \sqrt{[1 + (R/r)^2]^2 \cos^2\theta + [1 - (R/r)^2]^2 \sin^2\theta}$$

If this function is pictured as the vertical coordinate in a three dimensional plot where the floor coordinates are r and ϕ, its extremes are located at (r, ϕ) where the derivatives in the r and ϕ directions are zero. These are the locations where the surface represented by (2) is level and where the surface is either a maximum, a minimum or a saddle point. Thus, to locate the coordinates which are candidates for giving the maximum, note that

$$\frac{\partial \mathcal{E}}{\partial \theta} = \frac{E_o}{\mathcal{E}}\{-[1 + (R/r)^2]^2 + [1 - (R/r)^2]^2\} \cos\phi \sin\phi = 0 \tag{3}$$

and

$$\frac{\partial \mathcal{E}}{\partial r} = \frac{E_o}{\mathcal{E}}\frac{2R^2}{r^3}\{[1 + (R/r)^2]^2 \cos^2\theta + [1 - (R/r)^2] \sin^2\theta\} = 0 \tag{4}$$

Locations where (3) is satisfied are either at

$$\phi = 0 \tag{5}$$

or at

$$\phi = \pi/2 \tag{6}$$

with r not equal to R or at

$$r = R \tag{7}$$

with ϕ not given by (5) or (6). Putting (5) into (4) shows that there is no solution for r while putting (6) into (4) shows that the associated value of r is $r = R$. Finally, putting (7) into (4) gives the same location, $r = R$ and $\phi = \pi/2$. Inspection of (5) shows that this is the location of a maximum, not a minimum.

5.8.2 Because there is no ϕ dependence of the potential on the boundaries, we use the second $m = 0$ potential from Table 5.7.1.

$$\Phi = A ln r + B \tag{1}$$

Here, a constant potential has been added to the ln function. The two coefficients, A and B, are determined by requiring that

$$V_b = A ln b + B \tag{2}$$

$$V_a = A ln a + B \tag{3}$$

Thus,

$$\begin{aligned} A &= (V_a - V_b)/ln(a/b) \\ B &= \{V_b ln\, a - V_a ln\, b\}/ln(a/b) \end{aligned} \tag{4}$$

and the required potential is

$$\begin{aligned} \Phi &= V_a \frac{ln(r/b)}{ln(a/b)} - V_b \frac{ln(r/b)}{ln(a/b)} + V_b \\ &= [V_a ln(r/b) - V_b ln(r/a)]/ln(a/b) \end{aligned} \tag{5}$$

The electric field follows as being

$$\mathbf{E} = -\frac{\mathbf{i}_r (V_a - V_b)}{ln(a/b)} \frac{1}{r} \tag{6}$$

and evaluation of this expression at $r = b$ shows that the field is positive on the inner cylinder, and everywhere else for that matter, if $V_a < V_b$.

5.8.3 (a) The given surface charge distribution can be represented by a Fourier series that, like the given function, is odd about $\phi = \phi_o$

$$\sigma_s = \sum_{n=1}^{\infty} \sigma_n \sin n\pi(\phi - \theta_o) \tag{1}$$

where the coefficients σ_n are determined by multiplying both sides of (1) by $\sin m\pi(\phi - \phi_o)$ and integrating over a half-wavelength.

$$\int_{\phi_o}^{\phi_o+\pi} \sigma_s(\phi) \sin m(\phi - \theta_o d\phi = \int_{\phi_o}^{\phi_o+\pi} \sum_{n=1}^{\infty} \sigma_n \sin n(\phi - \phi_o) \sin m(\phi - \theta_o) d\phi \tag{2}$$

Thus,

$$\sigma_n = \frac{4\sigma_o}{n\pi}; \quad n \text{ odd} \tag{3}$$

and $\sigma_n = 0, n$ even. The potential response to this surface charge density is written in terms of solutions to Laplace's equation that i) have the same ϕ dependence as (1), ii) go to zero far from the rotating cylinder (region a) and at the inner cylinder where $r = R$ and are continuous at $r = a$.

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \Phi_n \left\{ \begin{matrix} [(a/R)^n - (R/a)^n](R/r)^n \\ (R/a)^n[(r/R)^n - (R/r)^n] \end{matrix} \right\} \sin n(\phi - \theta_o) \begin{matrix} a < r \\ R < r < a \end{matrix} \tag{4}$$

The coefficients Φ_n are determined by the "last" boundary condition, requiring that

$$\sigma_s(r = a) = -\epsilon_o \left[\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r} \right]_{r=a} \tag{5}$$

Substitution of (1), (3) and (4) into (5) gives

$$\Phi_n = \frac{2\sigma_o a}{\epsilon_o \pi n^2} \tag{6}$$

(b) The surface charge density on the inner cylinder follows from using (4) to evaluate

$$\sigma_s(r = R) = -\epsilon_o \frac{\partial \Phi^b}{\partial r} \Big|_{r=R} = -\frac{\epsilon_o 2}{R} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \Phi_n n (R/a)^n \sin n(\phi - \theta_o) \tag{7}$$

Thus, the total charge on the electrode segment in the wall of the inner cylinder is

$$q = w \int_0^{\alpha} \sigma_s(R) R d\phi = -\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} Q_n [\cos n\theta_o - \cos n(\alpha - \theta_o)] \tag{8}$$

where

$$Q_n \equiv \frac{4\sigma_o w a}{\pi} (R/a)^n \frac{1}{n^2}$$

(c) The output voltage is then evaluated by substituting $\theta_o \to \Omega t$ into (8) and taking the temporal derivative.

$$v_o = -R_o \frac{dq}{dt} = -\Omega R_o \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} n Q_n [\sin n\Omega t + \sin n(\alpha - \Omega t)] \tag{9}$$

5.8.4 The Fourier representation of the square-wave of surface charge density is carried out as in Prob. 5.8.3, (1) through (3), resulting in

$$\sigma_s = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \sigma_n \sin n\pi(\phi - \theta_o) \tag{1}$$

where

$$\sigma_n = \frac{4\sigma_o}{n\pi}; \quad n \text{ odd}$$

The potential between the moving sheet at $r = R$ and the outer cylindrical wall at $r = a$, and inside the moving sheet, are respectively

$$\Phi = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \Phi_n = \left\{ \begin{matrix} (a/R)^n[(r/R)^n - (R/r)^n] \\ (r/R)^n[(a/R)^n - (R/a)^n] \end{matrix} \right\} \sin n(\phi - \theta_o) \begin{matrix} a < r < R \\ r < a \end{matrix} \tag{2}$$

where the coefficient has been adjusted so that the potential is zero at $r = R$ and continuous at the surface of the moving sheet, where $r = a$. The coefficients are determined by using Gauss' continuity condition with the surface charge density written as (1) and the potential given by (2);

$$\begin{aligned} -\epsilon_o\left(\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r}\right)_{r=a} &= \sigma_s \Rightarrow -\epsilon_o \Phi_n (a/R)^n \left[\frac{n}{a}(a/R)^n + \frac{n}{a}(R/a)^n\right] \\ &+ \frac{n}{a}(a/R)^n\left[(a/R)^n - (R/a)^n\right] = \frac{4\sigma_o}{n\pi} \end{aligned} \tag{3}$$

which implies that

$$\Phi_n = -\frac{2\sigma_o a}{n^2 \pi \epsilon_o} \tag{4}$$

The surface charge on the detection segment is

$$\sigma_s = \epsilon_o \frac{\partial \Phi^a}{\partial r}\bigg|_{r=R} = -\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{4\sigma_o}{\pi n}(a/R)^{n+1} \sin n(\phi - \theta_o) \tag{5}$$

and so the total charge on that segment is

$$q = w\int_0^{\alpha} \sigma_s(r = R) R d\phi = -\sum_{\substack{n=1\\ \text{odd}}}^{\infty} Q_n[\cos n\theta_o - \cos n(\alpha - \theta_o)] \tag{6}$$

where

$$Q_n = \frac{4\sigma_o w R}{\pi}(a/R)^{n+1}\frac{1}{n^2}$$

Finally, with $\theta_o = \Omega t$, the detected voltage is therefore

$$v_o = -R_o \frac{dq}{dt} = -\Omega R_o \sum_{\substack{n=1\\ \text{odd}}}^{\infty} n Q_n[\sin n\Omega t + \sin n(\alpha - \Omega t)] \tag{7}$$

5.8.5 Of the potentials in the second column of Table 5.7.1, the requirement that the potential be zero where $\phi = 0$ selects the two that vary as $\sin(m\phi)$ while the fact that the space of interest extends to the origin precludes those with negative exponents, for $m > 0$, the last two. The potential will be zero at $\phi = \alpha$ if $m = n\pi/d, n = 1, 2, \ldots$ Thus, candidate potentials are

$$\Phi = \sum_{m=1}^{\infty} A_n (r/R)^{n\pi/\alpha} \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{1}$$

Evaluated at $r = R$, this potential takes the form of a Fourier series, used here to represent the uniform potential.

$$V = \sum_{m=1}^{\infty} A_n \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{2}$$

Multiplication by $\sin(q\pi\phi/\alpha)$ and integration from $\phi = 0$ to $\phi = \alpha$ gives an expression which can be solved for the coefficients in (2).

$$-\frac{V\alpha}{q\pi} \cos\left(\frac{q\pi\phi}{\alpha}\right)\Big]_0^\alpha = A_q \frac{\alpha}{2} \Rightarrow A_n = \frac{4V}{\pi} \begin{cases} 1/n; & n \text{ odd} \\ 0; & n \text{ even} \end{cases} \tag{3}$$

Thus, (1) and (3) are the given answer.

5.8.6 Far from $r = R$, the field becomes that of a pair of electrodes extending from the origin to infinity in the planes $\phi = 0$ (with zero potential) and $\phi = \alpha$ (with potential V). The associated electric field is ϕ directed and simply the voltage V divided by the distance αr between the electrodes, following lines of constant r.

$$\Phi(r \to \infty) = V\frac{\phi}{\alpha} \Rightarrow \mathbf{E}(r \to \infty) = \frac{V}{\alpha r}\mathbf{i}_\phi \tag{1}$$

Although this potential satisfies the boundary conditions on the "wedge" electrodes, it does not satisfy the boundary conditions over the surface at $r = R$. On that surface, the potential should be the constant V. To satisfy this boundary condition, we add to (1) a potential that is zero on the surfaces $\phi = 0$ and $\phi = \alpha$ where (1) already satisfies the boundary conditions and that goes to zero at $r \to \infty$, where (1) is also the correct potential.

$$\Phi = V\frac{\phi}{\alpha} + \sum_{n=1}^{\infty} A_n (r/R)^{-(n\pi/\alpha)} \sin\left(\frac{n\pi}{\alpha}\phi\right) \tag{2}$$

The coefficients A_n are determined from evaluating (2) on the electrode at $r = R$, where

$$V = \frac{V\phi}{\alpha} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

The first term on the right in (3) is transferred to the left, both sides of the expression multiplied by $\sin(m\pi\phi/\alpha)$ and both sides integrated from $\phi = 0$ to $\phi = \alpha$ to obtain

$$V\left\{-\frac{\alpha}{m\pi}\cos\left(\frac{m\pi\phi}{\alpha}\right) - \frac{\alpha}{(m\pi)^2}\left[\sin\frac{m\pi\phi}{\alpha} - \frac{m\pi\phi}{\alpha}\cos\left(\frac{m\pi\phi}{\alpha}\right)\right]\right\}_0^\alpha = \frac{A_m\alpha}{2} \tag{4}$$

This expression can be solved for the coefficient, which (with $m \to n$) is

$$A_n = \frac{2V}{n\pi} \tag{5}$$

Evaluated using this coefficient, (2) is the desired potential.

5.8.7 (a) From the four equations in the second column of Table 5.7.1, the sin functions satisfy the boundary conditions that $\Phi = 0$ at $\phi = 0$ and $\phi = 2\pi$ if $m = n/2, n = 1, 2, \ldots$ With the understanding that n is positive, the solutions with exponents $-m$ are excluded so that the potential is finite as $r \to 0$. Thus, the remaining potential is the superposition of the modes

$$\Phi = \sum_{n=1}^{\infty} A_n (r/R)^{n/2} \sin\left(\frac{n}{2}\phi\right) \tag{1}$$

(b) The boundary condition at $r = R$ requires that

$$V_o = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n}{2}\phi\right) \tag{2}$$

Multiplication of both sides of this expression by $\sin(p\phi/2)$ and integration gives

$$\int_0^{2\pi} V_o \sin\left(\frac{m}{2}\phi\right) d\phi = \sum_{n=1}^{\infty} \int_0^{2\pi} A_n \sin\left(\frac{n}{2}\phi\right) \sin\left(\frac{m}{2}\phi\right) d\phi \tag{3}$$

or

$$-\frac{2}{m} V_o[\cos(m\pi) - 1] = \pi A_m \tag{4}$$

so that it follows that $A_n = 0, n$ even and for n odd

$$A_n = \frac{4V_o}{n\pi} \tag{5}$$

Substitution of this coefficient into (1) then gives the desired potential.

$$\Phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4V_o}{n\pi} (r/R)^{n/2} \sin\left(\frac{n}{2}\phi\right) \tag{6}$$

(c) The associated electric field follows from this expression as

$$\mathbf{E} = -\frac{4V_o}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n}\Big[\mathbf{i}_r\frac{n}{2}\frac{r^{\frac{n}{2}-1}}{R^{n/2}}\sin\big(\frac{n}{2}\phi\big)+\mathbf{i}_\phi\frac{n}{2}\frac{r^{\frac{n}{2}-1}}{R^{n/2}}\cos\big(\frac{n}{2}\phi\big)\Big] \tag{7}$$

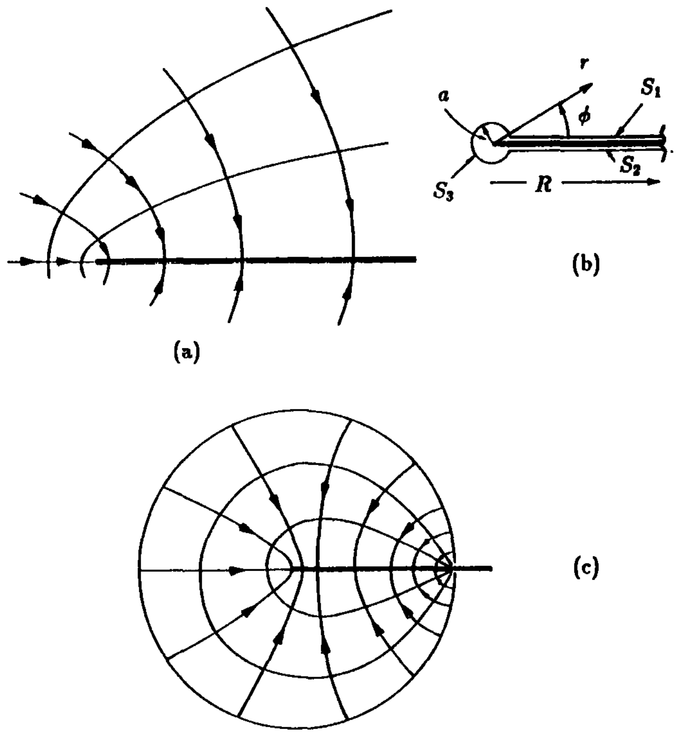


Figure S5.8.7

A sketch of the lead term in (6) and (7) is shown in Fig. S5.8.7a. The potential is finite at the tip of the fin but the electric field intensity varies as $1/\sqrt{r}$ at the tip. On the surface S_1 shown in Fig. S5.8.7b, the surface charge density follows from (7) as

$$\epsilon_o E_\phi(r,\phi=0) = -\frac{4\epsilon_o V_o}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{2}\frac{r^{\frac{n}{2}-1}}{R^{n/2}} \tag{8}$$

On the circular cylindrical surface S_2 at radius a, also shown in Fig. S5.8.7b,

$$\epsilon_o E_r(r=a,\phi) = -\frac{4\epsilon_o V_o}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{2}\frac{a^{\frac{n}{2}-1}}{R^{n/2}}\sin\big(\frac{n}{2}\phi\big) \tag{9}$$

while on surface S_3,

$$-\epsilon_o E_\phi = -\frac{4\epsilon_o V_o}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{2}\frac{r^{\frac{n}{2}-1}}{R^{n/2}} \tag{10}$$

The total charge represented by the first mode in the series is therefore

$$\frac{2\epsilon_o V_o}{\pi\sqrt{R}}\Big[-\int_a^R r^{-1/2}dr - \int_0^{2\pi} a^{-1/2}\sin(\phi/2)ad\phi - \int_a^R r^{-1/2}dr\Big] = \frac{8\epsilon_o V_o}{\pi} \tag{11}$$

(d) The potential and field distribution is sketched in Fig. S5.8.7b.

5.8.8 The potential takes the form of (5.8.15) with azimuthal coordinate displaced so that $\phi \rightarrow \phi_o - \phi$.

$$\Phi = \sum_{n=1}^{\infty} A_n \sin\left[n\pi\frac{ln(r/b)}{ln(a/b)}\right]\sinh\left[\frac{n\pi}{ln(a/b)}(\phi_o - \phi)\right] \tag{1}$$

Evaluated at $\phi = 0$, this expression is then the same as (5.8.15) evaluated at $\phi = \phi_o$. Thus, the coefficients are the same as given by (5.8.17). For n even, $A_n = 0$ and for n odd

$$A_n = 4v ln(a/b)/n\pi \sinh\big[\frac{n\pi}{ln(a/b)}\phi_o\big] \tag{2}$$

5.8.9 The radial distribution $R_n(r)$ is governed by (5.7.5).

$$r\frac{d}{dr}\Big(r\frac{dR_n}{dr}\Big) + p_n^2 R_n = 0 \tag{1}$$

Multiplication of this expression by another of the eigenfucntions and the weighting factor $1/r$ and integration results in the expression

$$\int_a^b \left[\frac{R_m}{r}r\frac{d}{dr}\Big(r\frac{dR_n}{dr}\Big) + p_n^2\frac{1}{r}R_nR_m\right]dr = 0 \tag{2}$$

With the identification $udv = d(uv) - vdu$ where

$$du = d\Big(r\frac{dR_n}{dr}\Big), \quad v = R_m \tag{3}$$

Eq. (2) can be integrated by parts

$$r\frac{dR_n}{dr}R_m\Big]_b^a - \int_b^a \Big(r\frac{dR_n}{dr}\frac{dR_m}{dr}\Big)dr + p_n^2\int_b^a \frac{1}{r}R_nR_m dr = 0 \tag{4}$$

This same procedure can be repeated with the roles of n and m reversed. Substraction of the resulting expression from (4) gives

$$r\left[\frac{dR_n}{dr}R_m - R_n\frac{dR_m}{dr}\right]_b^a + (p_n^2 - p_m^2)\int_b^a \frac{1}{r}R_nR_m dr = 0 \tag{5}$$

If boundary conditions require that the first term is zero, or in particular that $R_n(a) = 0$ and $R_m(b) = 0$, then the orthogonality condition follows.

$$(p_n^2 - p_m^2)\int_b^a \frac{1}{r}R_nR_m dr = 0 \tag{6}$$

5.9 THREE SOLUTIONS TO LAPLACE'S EQUATION IN SPHERICAL COORDINATES

5.9.1 (a) The given surface potential has the same θ dependence as for the uniform field potential of (5.9.4) and the dipole field potential of (5.9.3). With the coefficients of these potentials adjusted to match the given potential at $r = a$,

$$\Phi = \begin{cases} V(r/a)\cos\theta; & r < a \\ V(a/r)^2\cos\theta; & a < r \end{cases} \tag{1}$$

(b) A sketch of Φ and $\mathbf{E}$ is shown in Fig. 6.3.1.

5.9.2 (a) The surface charge density has the same θ dependence at $r = a$ as the discontinuity in the normal derivative of the potential. This suggests representing the potentials inside and outside the sphere with the same θ dependence as the given surface charge distribution. In addition, these potentials must be finite at the origin and at infinity. The natural choices are the uniform field potential given by (5.9.4) inside the sphere and the dipole potential of (5.9.3) outside the sphere.

$$\Phi = \begin{cases} A(a/r)^2\cos\theta; & a < r \\ A(r/a)\cos\theta; & r < a \end{cases} \tag{1}$$

The coefficients have already been adjusted so that the potential is continuous at $r = a$. Gauss' continuity condition then requires that

$$-\epsilon_o\left(\frac{\partial\Phi^a}{\partial r} - \frac{\partial\Phi^b}{\partial r}\right)_{r=a} = \sigma_o\cos\theta \Rightarrow -\epsilon_o\left[\frac{1}{a} + \frac{2}{a}\right]A = \sigma_o \tag{2}$$

so that $A = \sigma_o a/3\epsilon_o$ and the potential is as given with the problem.

(b) In Example 6.3.1, the potentials inside and outside the sphere take the same form as in (1) [(6.3.9) and (6.3.8)] and satisfy boundary conditions which take the same form as used here [(6.3.6) and (6.3.7)]. Indeed, we will see in Sec. 6.3 that with the polarization density given the polarization charge density is specified and the determination of the associated potential and field is much the same as in this chapter when the charge is specified. Hence, Fig. 6.3.1 portrays the potential and field.

5.9.3 Because the given charge density does not depend on ϕ, the potential is also independent of ϕ. In that case, Poisson's equation in spherical coordinates reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = -\frac{\rho_o\cos\theta}{\epsilon_o} \tag{1}$$

First, given the dependence of the charge density on θ, look for a particular solution having the form $\Phi_p = Ar^p\cos\theta$. Substitution into (1) then shows that $p = 2$ and $A = -\rho_o/4\epsilon_o$ so that a particular solution is

$$\Phi_p = -\frac{\rho_o}{4\epsilon_o}r^2\cos\theta \tag{2}$$

The sum of this potential and a solution to Laplace's equation must satisfy the condition that the potential be zero at $r = a$. Again, for the θ dependence of the particular solution, it is natural to take a uniform field as the homogeneous solution. Thus, with B an adjustable coefficient,

$$\Phi = -\frac{\rho_o}{4\epsilon_o} r^2 \cos\theta + Br\cos\theta \tag{3}$$

and by requiring that the total potential be zero at $r = a$, it follows that $B = \rho_o a/4\epsilon_o$ so that the potential is as given with the problem statement.

5.9.4 Because the given charge density does not depend on ϕ, the potential is also independent of ϕ. In that case, Poisson's equation in spherical coordinates reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) = -\frac{\rho_o}{\epsilon_o}(r/a)^m\cos\theta \tag{1}$$

First, given the dependence of the charge density on θ, look for a particular solution having the form $(r/a)\cos\theta$. Substitution into (1) then shows that $p = m + 2$ and $A = -\rho_o a^2/\epsilon_o(m+1)(m+4)$ so that a particular solution is

$$\Phi_p = -\frac{\rho_o a^2}{\epsilon_o(m+1)(m+4)}(r/a)^{m+2}\cos\theta \tag{2}$$

The sum of this potential and a solution to Laplace's equation must satisfy the condition that the potential be zero at $r = a$. Again, for the θ dependence of the particular solution, it is natural to take a uniform field as the homogeneous solution. Thus, with B an adjustable coefficient,

$$\Phi = \Phi_p + B(r/a)\cos\theta \tag{3}$$

and by requiring that the total potential be zero at $r = a$, it follows that the required potential is

$$\Phi = \frac{-\rho_o a^2}{\epsilon_o(m+1)(m+4)}(r/a)[(r/a)^{m+1} - 1]\cos\theta \tag{4}$$

5.10 THREE-DIMENSIONAL SOLUTIONS TO LAPLACE'S EQUATION

5.10.1 Given the zero potential surfaces at $y = 0$ and $y = b$ and at $z = 0$ and $z = w$, it is natural to construct the solution from product solutions having the form

$$\Phi = X(x)\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{w} \tag{1}$$

where, to satisfy Laplace's equation

$$X(x) = \begin{cases} \sinh k_{mn} x \\ \cosh k_{mn} x \end{cases}$$

and

$$k_{mn} = \sqrt{(m\pi/b)^2 + (n\pi/w)^2}$$

The boundary conditions on the surfaces at $x = 0$ and $x = a$ are the same. Thus, if $X(x)$ is chosen to be even about an origin at $x = a/2$, the potential that satisfies the condition of being v at $x = 0$ will also be v at $x = a$. Thus, $X(x)$ is made a linear combination of the solutions given with (1) which is the cosh function displaced so that its argument is zero where $x = a/2$.

$$X(x) = A_{mn} \cosh k_{mn}\Big(x - \frac{a}{2}\Big) \tag{2}$$

The solution therefore takes the form of (a) given with the problem. At $x = 0$, the condition at $x = 0$ requires that

$$v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cosh\Big(\frac{k_{mn} a}{2}\Big) \sin\Big(\frac{m\pi y}{b}\Big) \sin\Big(\frac{n\pi z}{w}\Big) \tag{3}$$

Note that this expression is the same as (11) if the $\sinh(k_{mn} b)$ is replaced by $\cosh(k_{mn} a/2)$ and $x/a \rightarrow y/b$. The evaluation of the coefficient using the orthogonality of the product solutions is therefore essentially the same as given by (5.10.11)-(5.10.15), resulting in (b) as given with the problem.

5.10.2 Given the x and z dependence of the surface charge density, which is the same as that of the components of $\mathbf{E}$ in the z direction on either side of the surface $y = a/2$, look for solutions of the form

$$\Phi = Y(y) \sin\Big(\frac{\pi x}{a}\Big) \sin\Big(\frac{\pi z}{w}\Big) \tag{1}$$

where

$$Y(y) = \begin{cases} \sinh k_{11} y \\ \cosh k_{11} y \end{cases}$$

and

$$k_{11} = \sqrt{(\pi/a)^2 + (\pi/b)^2}$$

To satisfy the continuity conditions at $y = b/2$, the potential function is given a piece-wise representation. The function in the upper region must be zero at $y = b$, so $Y(y)$ is chosen as a sinh with its argument displaced to $y = b$. In the lower region, the sinh function with its origin at $y = 0$ does the job. Thus,

$$\Phi = \begin{Bmatrix} A \sinh k_{11}(y - b) \\ B \sinh k_{11} y \end{Bmatrix} \sin\Big(\frac{\pi x}{a}\Big) \sin\Big(\frac{\pi z}{w}\Big) \tag{2}$$

At $y = b/2$, the potential must be continuous and Gauss' continuity condition must be satisfied.

$$-A \sinh(k_{11} b/2) = B \sinh(k_{11} b/2) \tag{3}$$

$$-\epsilon_o k_{11}(A - B) \cosh(k_{11} b/2) = \sigma_o \tag{4}$$

It follows that the coefficients in (2) are

$$A = -B = -\sigma_o / 2\epsilon_o k_{11} \cosh(k_{11} b/2) \tag{5}$$

5.10.3 In each case, the solution can be regarded as the superposition of a particular solution to Poisson's equation and a homogeneous solution to satisfy the boundary conditions. The determination of representation begins with the selection of the former.

As a first solution, select a particular solution that is only x dependent. Then, Poisson's equation reduces to

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho_o}{\epsilon_o} \tag{1}$$

and the particular solution that (for convenience) is also zero at $x = 0$ and $x = a$ is

$$\Phi_p = -\frac{\rho_o}{2\epsilon_o}x^2 + Ax + B = -\frac{\rho_o}{2\epsilon_o}x(x-a) \tag{2}$$

With this potential satisfying the boundary conditions on two of the surfaces, the homogeneous solution must assure satisfying the conditions on the remaining four surfaces. This is done by adding to (2) solutions designed to satisfy the conditions at $y = 0$ and $y = b$ while being zero at all the other surfaces and therefore neither disturbing the already satisfied conditions at $x = 0$ and $x = a$ nor those to be satisfied by the next homogeneous solution. To satisfy both the conditions at $y = 0$ and $y = b$, the y dependence is taken as even about $y = b/2$. A second homogeneous solution is then added to this one to assure satisfaction of the conditions at $z = 0$ and $z = w/2$ while not disturbing the potential at the other four surfaces. Thus, the potential takes the form

$$\begin{aligned}\Phi = \frac{-\rho_o}{2\epsilon_o}x(x-a) &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cosh k_{mn}\Big(y-\frac{b}{2}\Big)\sin\Big(\frac{m\pi}{a}x\Big)\sin\Big(\frac{n\pi}{w}z\Big)\\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cosh k_{mn}\Big(z-\frac{w}{2}\Big)\sin\Big(\frac{m\pi}{a}x\Big)\sin\Big(\frac{n\pi}{b}y\Big)\end{aligned} \tag{3}$$

The coefficients B_{mn} and C_{mn} are determined by requiring that the potential indeed be zero on the surfaces $y = 0$ and $z = 0$ (and hence also at $y = b$ and $z = w$).

$$\frac{\rho_o}{2\epsilon_o}x(x-a) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cosh\Big(\frac{k_{mn}b}{2}\Big)\sin\Big(\frac{m\pi}{a}x\Big)\sin\Big(\frac{n\pi}{w}z\Big) \tag{4}$$

$$\frac{\rho_o}{2\epsilon_o}x(x-a) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\cosh\Big(\frac{k_{mn}w}{2}\Big)\sin\Big(\frac{m\pi}{a}x\Big)\sin\Big(\frac{n\pi}{b}y\Big) \tag{5}$$

The coefficients therefore follow from the same procedure as illustrated by (5.10.11) through (5.10.15). For m or n even the coefficients are zero. For m and n odd,

$$\begin{aligned}B_{mn} &= \frac{\rho_o}{2\epsilon_o\cosh\big(\frac{k_{mn}b}{2}\big)}(4/n\pi)\int_0^a x(x-a)\sin\Big(\frac{m\pi}{a}x\Big)dx\\ &= \frac{-\rho_o}{2\epsilon_o\cosh\big(\frac{k_{mn}b}{2}\big)}(4/n\pi)\frac{8a^2}{(m\pi)^3}\end{aligned} \tag{6}$$

$$\begin{aligned} C_{mn} &= \frac{\rho_o}{2\epsilon_o \cosh\left(\frac{k_{mn}w}{2}\right)}(4/n\pi)\int_0^a x(x-a)\sin\left(\frac{m\pi}{a}x\right)dx \\ &= \frac{-\rho_o}{2\epsilon_o \cosh\left(\frac{k_{mn}w}{2}\right)}(4/n\pi)\frac{8a^2}{(m\pi)^3} \end{aligned} \tag{7}$$

Two more solutions are obtained by replacing the role of x with that of y and of z. As a fourth solution, expand the charge distribution in a three dimensional Fourier series

$$\rho_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} R_{mnq}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{w}\right) \tag{8}$$

The coefficients R_{mnq} follow by multiplying by

$$\sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi y}{b}\right)\sin\left(\frac{u\pi z}{w}\right)$$

integrating over the volume and solving for R_{rsu}. Then, with $rsu \rightarrow mnq$,

$$R_{mnq} = \frac{16\rho_o}{mnq\pi^3} \tag{9}$$

for m and n and q odd and zero for m or n or q even. Given this (x, y, z) dependence and given that the second derivative of each of the sinusoids results in the same sinusoidal function, we are motivated to look for a particular solution having the same form.

$$\Phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \Phi_{mnq}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{w}\right) \tag{10}$$

Substitution of this expression into Poisson's equation shows that term by term it is not only a solution to Poisson's equation (and therefore a particular solution) if

$$\Phi_{mnq} = \frac{R_{mnq}}{\epsilon_o\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{q\pi}{w}\right)^2\right]} \tag{11}$$

but satisfies the boundary conditions as well.

SOLUTIONS TO CHAPTER 6

6.1 POLARIZATION DENSITY

6.1.1 (a) From (6.1.6), the polarization charge density is

$$\rho_p = -\nabla\cdot\mathbf{P} = P_o\beta\sin\beta x \tag{1}$$

(b) The polarization surface charge density at the respective surfaces follows from (6.1.7) evaluated at the respective interfaces.

$$\begin{aligned}\sigma_{sp} &= -\mathbf{n}\cdot(\mathbf{P}^a - \mathbf{P}^b)\\ &= \begin{cases} -(0 - P_o\cos\beta x) = P_o\cos\beta x; & y = d\\ -(P_o\cos\beta x - 0) = -P_o\cos\beta x; & y = 0\end{cases}\end{aligned} \tag{2}$$

6.2 LAWS AND CONTINUITY CONDITIONS WITH POLARIZATION

6.2.1 (a) Given the polarization density, the polarization current density follows from (6.2.9).

$$\mathbf{J}_p = \frac{\partial\mathbf{P}}{\partial t} = \frac{dP_o}{dt}\cos\beta x(\mathbf{i_x} + \mathbf{i_y}) \tag{1}$$

The polarization charge density is as found in Prob. 6.1.1.

(b) Substitution of these quantities into (6.2.10) gives

$$\frac{\partial\rho_p}{\partial t} + \nabla\cdot\mathbf{J}_p = \frac{\partial P_o}{\partial t}\beta\sin\beta x - \frac{dP_o}{dt}\beta\sin\beta x = 0 \tag{2}$$

6.3 PERMANENT POLARIZATION

6.3.1 (a) The polarization charge density between the electrodes is

$$\rho_p = -\nabla\cdot\mathbf{P} = P_o\beta\sin\beta x \tag{1}$$

Thus, at each point between the electrodes,

$$\nabla^2\Phi = -\frac{\rho_p}{\epsilon_o} = -\frac{P_o\beta}{\epsilon_o}\sin\beta x \tag{2}$$

and a particular solution is gotten from

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{P_o\beta}{\epsilon_o}\sin\beta x \Rightarrow \Phi_p = \frac{P_o}{\beta\epsilon_o}\sin\beta x \tag{3}$$

To satisfy the boundary conditions, use the homogeneous solutions Ax

$$\Phi = Ax + \frac{P_o}{\beta\epsilon_o}\sin\beta x \tag{4}$$

which satisfies $\Phi(x=0)=0$. To make $\Phi(x=a)=-V$,

$$A = -\frac{V}{a} - \frac{P_o}{\beta\epsilon_o a}\sin\beta a \tag{5}$$

so that (6.3.4) becomes

$$\Phi = \frac{P_o}{\beta\epsilon_o}(\sin\beta x - \frac{x}{a}\sin\beta a) - V\frac{x}{a} \tag{6}$$

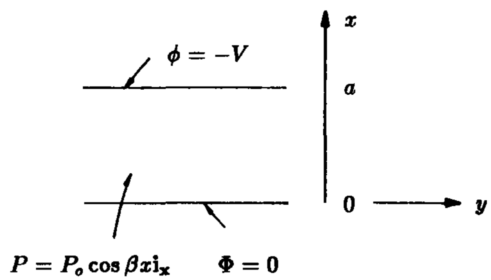


Figure S6.3.1

(b) In this case,

$$\rho_p = -\nabla\cdot\mathbf{P} = \beta P_o\sin\beta y \tag{7}$$

and

$$\frac{\partial^2\Phi_p}{\partial y^2} = -\frac{\beta P_o}{\epsilon_o}\sin\beta y \Rightarrow \Phi_p = \frac{P_o}{\epsilon_o\beta}\sin\beta y \tag{8}$$

Boundary conditions at $x=0$ and $x=a$, are satisfied by $\Phi = -Vx/a$. Thus, we let

$$\Phi = \frac{P_o}{\epsilon_o\beta}\sin\beta y - V\frac{x}{a} + \Phi_1 \tag{9}$$

Then Φ_1 must be $-(P_o/\epsilon_o\beta)\sin\beta y$ at $x=0$ and $x=a$. Such a solution to Laplace's equation is symmetric about $x=a/2$;

$$\Phi = \frac{P_o}{\epsilon_o\beta}\sin\beta y - \frac{Vx}{a} + A\cosh\beta\left(x-\frac{a}{2}\right)\sin\beta y \tag{10}$$

To satisfy boundary conditions

$$A = -\frac{P_o}{\epsilon_o\beta}\frac{1}{\cosh\left(\frac{\beta a}{2}\right)} \tag{11}$$

Thus, from (6.3.4) and (6.3.5),

$$\Phi = \frac{P_o}{\epsilon_o\beta}\sin\beta y\left[1 - \frac{\cosh\beta\left(x-\frac{a}{2}\right)}{\cosh\left(\frac{\beta a}{2}\right)}\right] - \frac{Vx}{a}$$

6.3.2 The polarization charge density inside the rectangular region is

$$\rho_p = -\nabla \cdot \mathbf{P} = P_o \frac{\pi}{a} \sin \frac{\pi}{a} x \tag{1}$$

The potential is therefore a solution to

$$\nabla^2 \Phi = -P_o \frac{\pi}{a\epsilon_o} \sin \frac{\pi}{a} x \tag{2}$$

that satisfies the zero potential boundary conditions. Two of these conditions are satisfied by the particular solution to (6.3.2) that follows from assuming that it, like the charge distribution, only depends on x.

$$\frac{d^2\Phi_p}{dx^2} = -P_o \frac{\pi}{a\epsilon_o} \sin \frac{\pi}{a} x \tag{3}$$

Two integrations, with the integration constants adjusted to make the potential at $x = a$ and $x = 0$ zero then give the particular solution

$$\Phi_p = P_o \frac{a}{\pi\epsilon_o} \sin \frac{\pi}{a} x \tag{4}$$

The homogeneous solution must also be zero on these boundaries and cancel this particular solution when evaluated at $y = \pm b$.

$$\Phi_h(x, \pm b) = -P_o \frac{a}{\pi\epsilon_o} \sin \frac{\pi}{a} x \tag{5}$$

Because these conditions are even in y, and because of the former boundary conditions, the potential is therefore taken as having a cosh dependence on x and the potential distribution suggested by the conditions of (6.3.5) at $y = \pm b$.

$$\Phi_h = -P_o \frac{a}{\pi\epsilon_o} \sin \frac{\pi}{a} x \frac{\cosh \frac{\pi}{a} y}{\cosh \frac{\pi}{a} b} \tag{6}$$

The required potential is then the sum of the particular and homogeneous solutions, (6.3.4) and (6.3.6).

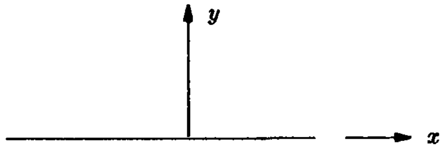
$$\Phi = P_o \frac{a}{\pi\epsilon_o} \sin \frac{\pi}{a} x \left[1 - \frac{\cosh \frac{\pi}{a} y}{\cosh \frac{\pi}{a} b}\right] \tag{7}$$

6.3.3 (a) First,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{\partial P_y}{\partial y} = 0 \tag{1}$$

and

$$\sigma_{sp} = -\mathbf{n} \cdot [\mathbf{P}^a - \mathbf{P}^b] = P_o \cos\left[\left(\frac{2\pi}{\Lambda}\right)x\right] \tag{2}$$



$\mathbf{P} = P_o \cos[(2\pi/\Lambda)x]\mathbf{i}_y$

Figure S6.3.3

(b) The Φ above and below must satisfy Laplace's equation and the boundary conditions that at $y = 0$

$$\Phi^a = \Phi^b \tag{3}$$

$$\epsilon_o E_x^a - \epsilon_o E_x^b = -\epsilon_o \frac{\partial \Phi^a}{\partial y}\bigg|_{x=0} + \epsilon_o \frac{\partial \Phi^b}{\partial y}\bigg|_{x=0} = P_o \cos\left[\left(\frac{2\pi}{\Lambda}\right)x\right] \tag{4}$$

To these ends, and to make $\Phi \to 0$ at $y \to \pm\infty$, make

$$\Phi = \frac{P_o \Lambda}{4\epsilon_o \pi} \cos\left(\frac{2\pi}{\Lambda}x\right) e^{\mp(2\pi y/\Lambda)} \tag{5}$$

6.3.4 In the region $-a < y < 0$, the divergence of the polarization density is zero and so the polarization charge density is zero as well. Thus, in both regions (a) and (b), the potential must satisfy Laplace's equation. Boundary conditions on the potential are that it be the given values at $y = \pm a$, that it be continuous at $y = 0$ and that it satisfy Gauss' continuity condition at $y = 0$. This condition requires that

$$-\mathbf{n} \cdot \left[\frac{\partial \Phi^a}{\partial y} - \frac{\partial \Phi^b}{\partial y}\right]_{y=0} = \sigma_{sp} \tag{1}$$

where the polarization surface charge density follows from

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = -[0 - P_o \sin\beta(x - x_o)] = P_o \sin\beta(x - x_o) \tag{2}$$

Thus, the fields are the same as if there were an unpaired surface charge density $\sigma_{su} = P_o \sin\beta(x - x_o)$ in the plane $y = 0$. With the identification of $\sigma_o \to P_o$, the physical situation is the same as considered in Prob. 5.6.12 and the solution as outlined there.

6.3.5 The given polarization density is uniform, so there is no volume polarization charge density. The polarization surface charge density at the cavity interface is

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = -\mathbf{i_r} \cdot \mathbf{i_z} P_o = -P_o \cos\theta \tag{1}$$

Thus, the boundary conditions at $r = R$ are

$$\Phi^a = \Phi^b \tag{2}$$

$$\mathbf{n} \cdot (\epsilon_o \mathbf{E}^a - \epsilon_o \mathbf{E}^b) = \epsilon_o E_r^a - \epsilon_o E_r^b = \sigma_o - P_o \cos\theta \tag{3}$$

On the right in this last expression is the sum of the polarization and unpaired surface charge densities. Superposition can be used to find the potentials due to the respective terms on the right and then their sum can be taken. Symmetry and Gauss' integral law give the electric field due to the uniform unpaired surface charge density.

$$4\pi\epsilon_o r^2 E_r^a = 4\pi R^2 \sigma_o \Rightarrow E_r^a = \frac{1}{\epsilon_o}(R/r)^2 \sigma_o \tag{4}$$

There is no electric field intensity inside, so the potential there is what it is on the surface. Thus,

$$\Phi = \begin{cases} \frac{R\sigma_o}{\epsilon_o}; & r \le R \\ \frac{R^2\sigma_o}{r}; & r \ge R \end{cases} \tag{5}$$

To find the potential from the second term in (6.3.3), assume that

$$\Phi = \begin{cases} A\frac{r}{R}\cos\theta \\ A\frac{R^2}{r^2}\cos\theta \end{cases} \tag{6}$$

where the coefficient has been adjusted to satisfy (6.3.2). Substitution of these expressions into (6.3.3) then gives

$$A = -\frac{P_o R}{3\epsilon_o} \tag{7}$$

The sum of (6.3.5) and (6.3.6) with the latter evaluated using (6.3.7) is the given potential.

6.3.6 In polar coordinates, the uniform y directed polarization density is

$$\mathbf{P} = P_o \mathbf{i_y} = P_o(\cos\phi \mathbf{i_r} - \sin\phi \mathbf{i_\phi}) \tag{1}$$

Because the divergence of $\mathbf{P}$ is zero in the volume, the only polarization charge is a surface charge density at $r = R$. This is

$$\sigma_{sp} = -\mathbf{n} \cdot \cos(\mathbf{P}^a - \mathbf{P}^b) = -P_o \cos\phi \tag{2}$$

The equipotential boundary condition in the plane $y = 0$ is met by assuming solutions

$$\Phi^a = \frac{A}{r}\cos\phi; \quad \Phi^b = Br\cos\phi \tag{3}$$

Boundary conditions at $r = R$ are

$$-\left[\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r}\right]_{r=R} = \frac{\sigma_{sp}}{\epsilon_o} \Rightarrow \left(\frac{A}{R^2} + B\right)\cos\phi = -P_o\cos\phi \tag{4}$$

$$\frac{A}{R} = BR \tag{5}$$

Simultaneous solution for the coefficients gives

$$A = -\frac{P_o R^2}{2}; \quad B = -\frac{P_o}{2} \tag{6}$$

and hence,

$$\Phi^a = -\frac{P_o R}{2}\frac{R}{r}\cos\phi; \quad \Phi^b = -\frac{P_o R}{2}\frac{r}{R}\cos\phi \tag{7}$$

6.3.7 The fields in regions (a) and (b), respectively above and below the interface, are taken as uniform. Because the line integral of $\mathbf{E}$ between the electrodes is zero,

$$aE_x^a + bE_x^b = 0 \tag{1}$$

At the interface, there is a polarization surface charge density

$$\sigma_{sp} = -(P_x^a - P_x^b) = P_o \tag{2}$$

Thus, Gauss' continuity condition requires that

$$\epsilon_o(E_x^a - E_x^b) = P_o \tag{3}$$

Solution of (6.3.1) and (6.3.3) then gives

$$E_x^a = \frac{P_o}{\epsilon_o}\frac{1}{\left(1 + \frac{a}{b}\right)}; \quad E_x^b = -\frac{P_o}{\epsilon_o}\frac{1}{\left(\frac{b}{a} + 1\right)} \tag{4}$$

6.3.8 (a) The polarization charge density is

$$\rho_p = -\nabla \cdot \mathbf{P} = -\nabla \cdot \nabla\psi = -\nabla^2\psi \tag{1}$$

where $\psi = P_o r\cos(\phi - \alpha)$ is a solution to Laplace's equation. Thus, $\rho_p = 0$.

(b) The surface polarization charge density at $r = b$ is

$$\sigma_{sp} = -\mathbf{n}\cdot(\mathbf{P}^I - \mathbf{P}^{II}) = \mathbf{i_r}\cdot\nabla P_o r\cos(\phi-\alpha) = P_o\cos(\phi-\alpha) \tag{2}$$

It is assumed that there is no unpaired surface charge density on this interface, so the boundary conditions are

$$\Phi^I(r = a) = 0 \tag{3}$$

$$\Phi^I(r = b) = \Phi^{II}(r = b) \tag{4}$$

$$\epsilon_o E_r^I - \epsilon_o E_r^{II} = P_o\cos(\phi - \alpha) \tag{5}$$

Solutions to Laplace's equation that have the same dependence as the right hand side of (6.3.5) take the form

$$\Phi = \begin{cases} A[(r/a) - (a/r)]\cos(\phi-\alpha) \\ B(r/b)\cos(\phi-\alpha) \end{cases} \tag{6}$$

Here, the solution that is infinite at the origin has been omitted and the two contributions to the outer potential adjusted to satisfy (6.3.3). Substitution of (6.3.6) into (6.3.4) and (6.3.5) then gives

$$B = A(\frac{b}{a} - \frac{a}{b}) = -\frac{P_o b^2}{2\epsilon_o a}(\frac{b}{a} - \frac{a}{b}) \tag{7}$$

Thus, (6.3.6) and (6.3.7) are the given potentials.

6.3.9 (a) Note that the scalar function inside the gradient operator is a solution to Laplace's equation. Thus,

$$\rho_p = -\nabla\cdot\mathbf{P} = -\nabla^2[P_o(r^m/r^{m-1})\cos m\phi] = 0 \tag{1}$$

and there is no polarization charge density in the volume of the rotor. However, at the interface there is a surface polarization charge density given by

$$\sigma_{sp} = -\mathbf{n}\cdot(\mathbf{P}^a - \mathbf{P}^b) = P_r^b(r = b) = P_o m\cos m\phi \tag{2}$$

(b) Boundary and continuity conditions are

$$\Phi^a(r = a) = 0; \quad \Phi^a(r = b) = \Phi^b(r = b) \tag{2}$$

$$-\epsilon_o\left[\frac{\partial\Phi^a}{\partial r}(r=b)-\frac{\partial\Phi^b}{\partial r}(r=b)\right]=P_o m\cos m\phi \tag{3}$$

Given the ϕ dependence of σ_{sp}, solutions to Laplace's equation are assumed to take the form

$$\Phi^a = A[(r/a)^m-(a/r)^m]\cos m\phi \tag{4a}$$

$$\Phi^b = A(r/b)^m[(b/a)^m-(a/b)^m]\cos m\phi \tag{4b}$$

where a linear combination of solutions has been selected in the annular region, (a), that satisfies the zero potential condition at $r=a$ and the coefficients have been arranged so that the potential is continuous at $r=b$. The last of the boundary conditions then determines A.

$$P_o m\cos m\phi = -\epsilon_o A\frac{2m}{b}(a/b)^m\cos m\phi \Rightarrow A=-\frac{bP_o}{2\epsilon_o}(b/a)^m \tag{5}$$

Thus, the potential is

$$\Phi=-\frac{bP_o}{2\epsilon_o}(b/a)^m\cos m\phi\begin{cases}[(r/a)^m-(a/r)^m]; & b<r<a\\ (r/b)^m[(b/a)^m-(a/b)^m]; & r<b\end{cases} \tag{6}$$

(c) With the substitution $\phi\rightarrow\phi-\Omega t$, at a given instant in time there is only a shift in the origin of ϕ. Because the field laws do not involve a time rate of change, they are satisfied by the new solution. To stay at a point of constant $\phi-\Omega t$ and hence constant $\mathbf{P}$ requires being at the angular position $\phi=\Omega t+$ constant. Thus, the new solution is one that represents the fields associated with a rotor having the angular velocity Ω.

(d) From (6.3.6), the surface charge density on the wall at $r=a$ is

$$\sigma_{su}=\epsilon_o\frac{\partial\Phi^a}{\partial r}(r=a)=-\frac{bP_o}{2}(b/a)^m\frac{2m}{a}\cos m(\phi-\Omega t) \tag{7}$$

The net charge on the segment is then

$$\begin{aligned}q&=l\int_{-\pi/m}^{0}\sigma_{su}a\,d\phi=-lbP_o(b/a)^m\left[\sin(-m\Omega t)-\sin m\left(-\frac{\pi}{m}-\Omega t\right)\right]\\&=lbP_o(b/a)^m[\sin(m\Omega t)-\sin(\pi+m\Omega t)]\\&=2lbP_o(b/a)^m\sin(m\Omega t)\end{aligned} \tag{8}$$

and hence the output voltage is

$$v_o=-R\frac{dq}{dt}=-2lbP_oR(b/a)^m m\Omega\cos(m\Omega t) \tag{9}$$

6.3.10 (a) The potential in regions (a) and (b), respectively $0 < x$ and $x < 0$, take the form of (6.6.26) and (6.6.27) with $V = 0$ in the latter because both of the electrodes are grounded over their full length in the x direction and $a \to d$.

$$\Phi^a = \sum_{n=1}^{\infty} V_n e^{-\frac{n\pi}{d}x} \sin \frac{n\pi}{d} y \tag{1}$$

$$\Phi^b = \sum_{n=1}^{\infty} V_n e^{\frac{n\pi}{d}x} \sin \frac{n\pi}{d} y \tag{2}$$

The coefficient in these expressions, V_n, has been adjusted so that the potential is continuous at the interface. Given $V(y)$, these coefficients follow by evaluating either of these expressions at $y = 0$

$$\sum_{n=1}^{\infty} V_n \sin \frac{n\pi}{d} y = V(y) \tag{3}$$

multiplying by $\sin(m\pi y/d)$ and integrating from $y = 0$ to $y = d$.

$$V_m = \frac{2}{d} \int_0^d V(y) \sin \frac{m\pi}{d} y dy \tag{4}$$

Given $V(y)$, this integral can be evaluated and the coefficients needed to complete (1) and and (2) determined.

(b) In addition to the continuity of potential which is already satisfied by (1) and (2), the continuity condition at $x = 0$ is

$$-\epsilon_o \frac{\partial \Phi^a}{\partial x} + \epsilon_o \frac{\partial \Phi^b}{\partial x} = \sigma_{sp} = P_o \tag{5}$$

(c) With P_o now the given quantity, substitution of (1) and (2) into (5) gives

$$\sum_{n=1}^{\infty} V_n \left(2\epsilon_o \frac{n\pi}{d}\right) \sin \frac{n\pi}{d} y = P_o \tag{6}$$

The coefficients are evaluated in this case by the same procedure as leading to (4).

$$V_m = \frac{2}{d[2\epsilon_o(n\pi/d)]} \int_0^d P_o \sin \frac{n\pi}{d} y dy = \frac{dP_o}{\epsilon_o} \frac{[1 - (-1)^m]}{(m\pi)^2} \tag{7}$$

Evaluated using this coefficient, (1) and (2) become the given potential.

6.3.11 In region (b), the potential must satisfy Poisson's equation with the charge density found in Prob. 6.1.1.

$$\nabla^2\Phi^b = -\frac{P_o\beta}{\epsilon_o}\sin\beta x \tag{1}$$

while in region (a) it satisfies Laplace's equation. At the interface, the potential must be continuous and satisfy Poisson's continuity condition for the polarization surface charge density found in Prob. 6.1.1.

$$-\epsilon_o\left[\frac{\partial\Phi^a}{\partial y}(y=d) - \frac{\partial\Phi^b}{\partial y}(y=d)\right] = P_o\cos\beta x \tag{2}$$

Finally, the potential must go to zero as $y\to\infty$ and be zero in the plane $y=0$. The particular solution to (1) is taken as depending only on x. Thus, two integrations give

$$\Phi_p^b = \frac{P_o}{\epsilon_o\beta}\sin\beta x \tag{3}$$

The x dependence of the potential due to the surface charge density is $\cos(\beta x)$ while that due to the volume charge denisty is $\sin(\beta x)$. The potential is taken as the sum of potentials due to these two sources, $\Phi = \Phi_s + \Phi_v$. The potential due to the surface charge satisfies Laplace's equation in each region and takes the form

$$\Phi_s = \begin{cases} A_s e^{-\beta(y-d)}\cos\beta x \\ A_s\frac{\sinh(\beta y)}{\sinh(\beta d)}\cos\beta x \end{cases} \tag{4}$$

Here, the coefficients have been adjusted to make the potential continuous at $y=d$, while the sinh function satisfies the zero potential boundary condition at $y=0$. The coefficient is determined by requiring that Gauss' continuity condition be satisfied with the surface charge density given by (2).

$$-\epsilon_o A_s[-\beta - \beta\coth(\beta d)]\cos(\beta x) = P_o\cos(\beta x) \Rightarrow A_s = \frac{P_o}{\epsilon_o\beta[1+\coth(\beta d)]} \tag{5}$$

The part of the potential due to the bulk charge takes the form

$$\Phi_v = \begin{cases} A_v e^{-\beta(y-d)}\sin(\beta x) \\ \frac{P_o}{\epsilon_o\beta}[1 + B_v\sinh(\beta y) + C_v\cosh(\beta y)]\sin(\beta x) \end{cases} \tag{6}$$

where the solution in the lower region has been taken as the sum of the particular solution, (3), and two solutions to Laplace's equation. This part of the potential must also be zero at $y=0$, so $C=-1$. In addition both the potential and its normal derivative must be continuous at $y=d$.

$$A_v = \frac{P_o}{\epsilon_o\beta}[1 + B_v\sinh(\beta d) - \cosh(\beta d)] \tag{7}$$

$$-\beta A_v = \frac{P_o}{\epsilon_o}[B_v \cosh(\beta d) - \sinh(\beta d) \tag{8}$$

Simultaneous solution of these expressions gives ($\sinh^2 x - \cosh^2 x = -1$).

$$A_v = \frac{P_o}{\epsilon_o \beta}\frac{[\cosh(\beta d) - 1]}{[\cosh(\beta d) + \sinh(\beta d)]} \tag{9}$$

$$B_v = \frac{\cosh(\beta d) + \sinh(\beta d) - 1}{\cosh(\beta d) + \sinh(\beta d)} \tag{10}$$

Finally, the total solution is the sum of (4) and (6) with the coefficients given by (5), (9), and (10).

6.4 POLARIZATION CONSTITUTIVE LAWS

6.4.1 In terms of the number density N, the polarization density is given by

$$\mathbf{P} = Nq\mathbf{d} = (\epsilon - \epsilon_o)\mathbf{E} \tag{1}$$

It follows that the separation d of single electronic charges needed to account for the given polarization is

$$d = \frac{(\epsilon - \epsilon_o)E}{Ng} = \frac{(1.5)(8.85 \times 10^{-12})(10^7)}{(6 \times 10^{26}/8)10^3(1.6 \times 10^{-19})} = 1.1 \times 10^{-14} \tag{2}$$

This is less than 1/1000 of a dimension typical of an atom.

6.5 FIELDS IN THE PRESENCE OF ELECTRICALLY LINEAR DIELECTRICS

6.5.1 (a) The divergence of $\epsilon\mathbf{E}$ is zero

$$\nabla \cdot \epsilon\mathbf{E} = \frac{\partial}{\partial y}\left[\epsilon(x)\frac{v}{d}\right] = 0 \tag{1}$$

and the curl of $\mathbf{E}$, a uniform field, is as well. Given that the field is normal to the perfectly conducting boundaries, which extend to infinity, it follows that the solution, which does indeed satisfy the relevant field laws, is uniquely specified.

(b) On the upper surface of the lower electrode in the regions to right and left,

$$\sigma_s = \epsilon_a \frac{v}{d}; \quad \epsilon_b \frac{v}{d} \tag{2}$$

It follows that the net charge on the lower electrode is

$$q = ac\epsilon_a \frac{v}{d} + bc\epsilon_b \frac{v}{d} = \frac{c}{d}(a\epsilon_a + b\epsilon_b)v \tag{3}$$

and hence the capacitance is as given.

(c) In this case, the surface charge density on the lower electrode is

$$\sigma_s = \epsilon(x)\frac{v}{d} \tag{4}$$

and so the net charge on the lower electrode is

$$q = c\int_0^l \sigma_s dx = \frac{cv}{d}\int_0^l \epsilon_o\left(1 + \frac{x}{l}\right)dx = \left(\frac{\epsilon_o c l}{d}\frac{3}{2}\right)v \tag{5}$$

so that C is as given.

6.5.2 A uniform electric field, $\mathbf{E} = (v/d)\mathbf{i}_y$ is irrotational and satisifies Gauss' law with the permittivity varying with x, a direction perpendicular to the proposed electric field.

$$\nabla \cdot \epsilon\mathbf{E} = \frac{\partial}{\partial y}\left[\epsilon_a(1 + \alpha\cos\beta x)\frac{v}{d}\right] = 0 \tag{1}$$

Thus, $\mathbf{E}$ is indeed uniform and

$$D_y = \epsilon_a(1 + \alpha\cos\beta x)\frac{v}{d} \tag{2}$$

This is also the density of unpaired surface charge on the lower electrode, so the total charge on that electrode is

$$\begin{aligned} q &= c\int_0^l \epsilon_a(1 + \alpha\cos\beta x)\frac{v}{d}dx \\ &= \frac{c\epsilon_a}{d}\left[x + \frac{\alpha}{\beta}\sin\beta x\right]_0^l v = Cv \end{aligned} \tag{3a}$$

$$C \equiv \frac{c\epsilon_a}{d}\left[l + \frac{\alpha}{\beta}\sin\beta l\right] \tag{3b}$$

6.5.3 (a) Because the field is independent of x and z,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_y}{\partial y} = 0 \tag{1}$$

and from this it follows that $D_y = D_y(t)$.

(b) In terms of the given distribution of permittivity,

$$D_y = \epsilon_o\Big[1 + \chi_a\big(1 + \frac{y}{l}\big)\Big]E_y \tag{2}$$

This expression can be solved for E_y and hence for the y dependence of E_y. To determine the unknown D_y, that expression is integrated from the lower to the upper electrode and the result equated to the voltage.

$$\int_0^l E_y dy = v = \int_0^l \frac{D_y dy}{\epsilon_o\big[1 + \chi_a\big(1 + \frac{y}{l}\big)\big]} = \frac{D_y l}{\epsilon_o \chi_a} ln \frac{(1 + 2\chi_a)}{(1 + \chi_a)} \tag{3}$$

The total charge on the lower electrode, and hence C, follows from this result

$$q = AD_y = \Big[\frac{\epsilon_o \chi_a A}{l} / ln \frac{(1 + 2\chi_a)}{(1 + \chi_a)}\Big] v \tag{4}$$

6.5.4 Because the field is independent of x and z,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_y}{\partial y} = 0 \tag{1}$$

and from this it follows that $D_y = D_y(t)$. This means that

$$D_y = \epsilon_p e^{-y/d} E_y(y, t) = D_y(t) \tag{2}$$

is independent of y and can be solved for E_y. The voltage is then

$$v = \int_0^l E_y dy = \frac{D_y}{\epsilon_p} \int_0^l e^{-y/d} dy = \frac{D_y d}{\epsilon_p}(1 - e^{-l/d}) \tag{3}$$

and this expression can be solved for D_y, which is the surface charge density on the lower electrode.

$$q = AD_y = Cv; \quad C \equiv \frac{A\epsilon_p}{d(1 - e^{-l/d})} \tag{4}$$

6.5.5 (a) For each, the electric field intensity in each region takes the form $\mathbf{E} = \mathbf{i_r} A/r$ [the potential takes the form $\Phi = Aln(r)$]. In the first case, the integral of this field between the electrodes must be the same whether it is taken in the dielectric or in the free space region. Thus, in the first case,

$$v = \int_b^a \frac{A}{r} dr = Aln(a/b) \Rightarrow \mathbf{E} = \mathbf{i_r} v/rln(a/b) \tag{1}$$

Note that this solution satisfies the conditions that the tangential field be continuous at the dielectric-free space interfaces and that the normal $\mathbf{D}$ be continuous (there is no normal $\mathbf{D}$). The field is normal to the circular cylinder electrodes and so these are equipotentials, as required. In the second case, the coefficient A has a different value in each of the regions. The two coefficients are found by requiring that

$$v = \int_b^R \frac{A_1}{r} dr + \int_R^a \frac{A_2}{r} dr = A_1 ln(R/b) + A_2 ln(a/r) \tag{2}$$

and that at the interface

$$\epsilon \frac{A_1}{R} = \epsilon_o \frac{A_2}{R} \tag{3}$$

Thus, in the second case,

$$\mathbf{E} = \frac{\mathbf{i}_r v}{[ln(R/b) + \frac{\epsilon}{\epsilon_o} ln(a/R)]r} \begin{cases} 1; & b < r < R \\ \epsilon/\epsilon_o; & R < r < a \end{cases} \tag{4}$$

(b) The capacitance follows from integrating the surface charge density over the inner electrode. In the first case,

$$q = l[\alpha b \epsilon E_r(r = b) + (2\pi - \alpha) b \epsilon_o E_r(r = b)] = Cv; \tag{5a}$$

$$C \equiv l[\alpha\epsilon + (2\pi - \alpha)\epsilon_o]/ln(a/b) \tag{5b}$$

while in the second case

$$q = l 2\pi b D_r(r = b) \equiv Cv; \quad C \equiv 2\pi l \epsilon / [ln(R/b) + \frac{\epsilon}{\epsilon_o} ln(a/R)] \tag{6}$$

6.5.6 Based on experience in the special case where the wedge is of uniform permittivity (so that the spatial variations in permittivity are bumps at the interfaces) postulate that the electric field is no different than if the dielectric wedge were not present.

$$\mathbf{E} = [v/r ln(a/b)]\mathbf{i}_r \tag{1}$$

Because the electric field is perpendicular to the gradient in permittivity, there is no induced polarization charge, (6.5.9), and hence no distortion of this field by the dielectric. The field of (1) has no divergence (and of course no curl) and hence does satisfy the bulk conditions throughout the volume. It also has no tangential value on the boundaries, as required. The given capacitance follows from integrating the unpaired surface charge over the surface of the inner electrode.

$$\begin{aligned} q &= \int_0^l \int_0^{2\pi} (\epsilon_a + \epsilon_b \cos^2\phi) \frac{v}{b ln(a/b)} b d\phi dz \\ &= v \left[\frac{2\pi\epsilon_a + \epsilon_b \pi}{ln(a/b)} \right] l \end{aligned}$$

6.6 PIECE-WISE UNIFORM ELECTRICALLY LINEAR DIELECTRICS

6.6.1 Given that the imposed potential takes the form

$$\Phi(r \to \infty) = -E_o r \cos\theta \tag{1}$$

assume postentials of the form

$$\Phi = -E_o r \cos\theta + \frac{A}{r^2}\cos\theta; \quad r > R \tag{2}$$

$$\Phi = B\frac{r}{R}\cos\theta = \left(E_o R + \frac{A}{R^2}\right)\frac{r}{R}\cos\theta; \quad r < R \tag{3}$$

Here, the coefficients have already been adjusted to make the potential continuous at $r = R$. The remaining condition is that

$$-\epsilon_o \frac{\partial \Phi}{\partial r} + \epsilon_s \frac{\partial \Phi}{\partial r} = 0 \tag{4}$$

from which it follows that

$$A = \frac{E_o R^3(\epsilon_s - \epsilon_o)}{(2\epsilon_o + \epsilon_s)} \tag{5}$$

The given potentials follow from substitution of (5) into (2) and (3).

6.6.2 (a) Assume a potential within the cavity that is consistent with the dipole being at the origin with the addition term satisfying Laplace's equation while having the same θ dependence as the dipole and being finite at the origin. Outside the cavity, the potential again has the θ dependence of the dipole and goes to zero at infinity.

$$\Phi = \begin{cases} \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} + Br\cos\theta; & r < a \\ A\frac{\cos\theta}{r^2}; & a < r \end{cases} \tag{1}$$

Potential continuity and continuity of normal $\mathbf{D}$ at $r = a$ requires that the coefficients A and B satisfy

$$\begin{bmatrix} \frac{1}{a^2} & -a \\ \frac{2\epsilon}{a^3} & \epsilon_o \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{p}{4\pi\epsilon_o a^2} \\ \frac{2p}{4\pi a^3} \end{bmatrix} \tag{2}$$

Thus,

$$A = \frac{3p}{4\pi(\epsilon_o + 2\epsilon)}; \quad B = \frac{-2p}{4\pi a^3}\frac{\left(\frac{\epsilon}{\epsilon_o} - 1\right)}{(\epsilon_o + 2\epsilon)} \tag{3}$$

so that the required potential is

$$\Phi = \frac{p\cos\theta}{4\pi\epsilon_o}\begin{cases} \frac{1}{r^2} - \frac{2}{a^3}\frac{\left(\frac{\epsilon}{\epsilon_o} - 1\right)}{\left(1 + \frac{2\epsilon}{\epsilon_o}\right)}r; & r < a \\ \frac{3}{1 + 2\frac{\epsilon}{\epsilon_o}}\frac{1}{r^2}; & a < r \end{cases} \tag{4}$$

The electric field follows as

$$\mathbf{E} = \frac{p}{4\pi\epsilon_o}\begin{cases} \left[\frac{2}{r^3} + \frac{2}{a^3}\frac{\left(\frac{\epsilon}{\epsilon_o}-1\right)}{\left(1+\frac{2\epsilon}{\epsilon_o}\right)}\right]\cos\theta\mathbf{i}_r + \left[\frac{1}{r^3} - \frac{2\left(\frac{\epsilon}{\epsilon_o}-1\right)}{a^3\left(1+\frac{2\epsilon}{\epsilon_o}\right)}\right]\sin\theta\mathbf{i}_\theta & r < a \\ \frac{6}{1+\frac{2\epsilon}{\epsilon_o}}\frac{\cos\theta}{r^3}\mathbf{i}_r + \frac{3}{1+\frac{2\epsilon}{\epsilon_o}}\frac{\sin\theta}{r^3}\mathbf{i}_\theta; & a < r \end{cases} \tag{5}$$

(b) In the limit $\epsilon \to \infty$, the tangential electric field at $r = a$ becomes

$$\lim_{\epsilon\to\infty} \frac{1}{a^3} - \frac{2\left(\frac{\epsilon}{\epsilon_o} - 1\right)}{a^3\left(1 + \frac{2\epsilon}{\epsilon_o}\right)} = \frac{1}{a^3} - \frac{1}{a^3} = 0 \tag{6}$$

and the potential inside the cavity, (1a), becomes

$$\lim_{\epsilon\to\infty} \Phi^b = \frac{p\cos\theta}{4\pi\epsilon_o}\left(\frac{1}{r^2} - \frac{r}{a^3}\right) \tag{7}$$

(c) If the cavity is regarded as an equipotential at the outset, it follows from (1a) that

$$\frac{p}{4\pi\epsilon_o}\frac{1}{a^2} + Ba = 0 \Rightarrow B = -\frac{p}{4\pi\epsilon_o a^3} \tag{8}$$

in agreement with what was obtained by taking the limit, (7).

6.6.3 From (5.8.4), the potential around a perfectly conducting rod of radius R in a uniform electric field is

$$\Phi = -E_a R\left(\frac{r}{R} - \frac{R}{r}\right)\cos\phi \tag{1}$$

The potential for a two-dimensional electric dipole is given by (a) of Prob. 4.4.1.

$$\Phi = \frac{\lambda_l d}{2\pi\epsilon_o}\frac{\cos\phi}{r} \tag{2}$$

Comparison of these expressions shows that the induced two-dimensional dipole moment is

$$\lambda_l d = (2\pi\epsilon_o R^2)E_a \tag{3}$$

The density of the rods is $(1/s^2)$ per unit area and therefore the polarization density is

$$P = (\lambda_l d/s^2) = \frac{2\pi\epsilon_o R^2}{s^2}E_a = \epsilon_o\chi_c E_a \tag{4}$$

6.6.4 The dielectric spheres have induced dipole moments that follow from (a) of Prob. 6.6.1.

$$p = 4\pi\epsilon_o E_o R^3\frac{(\epsilon_s - \epsilon_o)}{(\epsilon_s + 2\epsilon_o)} \tag{1}$$

Using the arguments of (6.6.6)-(6.6.9), it follows that the equivalent permittivity is

$$\epsilon = 1 + 4\pi(R/s)^3\frac{(\epsilon_s - \epsilon_o)}{(\epsilon_s + 2\epsilon_o)} \tag{2}$$

6.6.5 Writing the potential in the upper region as that of the point charge q at $y = h$ and its yet to be determined image at $y = -h$, both on the y axis, we have (Sec. 4.4)

$$\Phi = \frac{1}{4\pi\epsilon_o}\begin{cases} \frac{q}{r_+} - \frac{q_b}{r_-}; & 0 < y \\ \frac{q_a}{r_+}; & y < 0 \end{cases} \tag{1}$$

where

$$r_+ = \sqrt{x^2 + (y-h)^2 + z^2}; \quad r_- = \sqrt{x^2 + (y+h)^2 + z^2}$$

In their respective regions, these have been chosen to satisfy Poisson's equation (in the upper region) and Laplace's equation. At the interface, where $y = 0$ in (1), the potential must be continuous for all x and z

$$q - q_b = q_a \tag{2}$$

and the normal electric flux density must be continuous (there is no unpaired surface charge density)

$$\epsilon_a(q + q_b) - \epsilon_b q_a = 0 \tag{3}$$

Simultaneous solution of these expressions gives the relations for q_a and q_b summarized by (b) in the problem. To determine the force on the charge caused by the surface polarization charge it induces at the interface of the dielectric, compute the electric field at $y = h, x = 0, z = 0$ using (1) and ignoring the self field (it can produce no net force on itself) and multiply by q.

$$\mathbf{f} = \mathbf{i}_y qE_y(x = 0, y = h, z = 0) \tag{4}$$

Thus, the force is one of attraction, as given.

6.6.6 In the upper half space, the particular solution is that of a line charge. Because it has the same x dependence of its potential in the $y = 0$ plane, a homogeneous solution is added to this which is the potential of an image line charge at $(x, y) = (0, -h)$.

$$\Phi^a = -\frac{\lambda}{2\pi\epsilon_a} ln\sqrt{x^2 + (y-h)^2} - \frac{\lambda_a}{2\pi\epsilon_a} ln\sqrt{x^2 + (y+h)^2} \tag{1}$$

In the lower half space, the potential is taken as that due to a line charge located at $(x, y) = (0, h)$.

$$\Phi^b = -\frac{\lambda_b}{2\pi\epsilon_b} ln\sqrt{x^2 + (y-h)^2} \tag{2}$$

The coefficients, λ_a and λ_b are now adjusted to satisfy the continuity conditions on the potential and the normal dielectric flux density in the $y = 0$ plane.

$$\frac{\lambda}{\epsilon_a} ln\sqrt{x^2 + h^2} + \frac{\lambda_a}{\epsilon_a} ln\sqrt{x^2 + h^2} = \frac{\lambda_b}{\epsilon_b}\sqrt{x^2 + h^2} \tag{3}$$

$$-\lambda\frac{h^2}{\sqrt{x^2 + h^2}} + \lambda_a\frac{h^2}{\sqrt{x^2 + h^2}} = -\lambda_b\frac{h^2}{\sqrt{x^2 + h^2}} \tag{4}$$

Because each of the terms in one or the other of these expressions has (by design) the same x dependence, the boundary conditions can be satisfied by adjusting the coefficients. Simultaneous solution gives

$$\lambda_a = \lambda(\epsilon_a - \epsilon_b)/(\epsilon_a + \epsilon_b) \tag{5}$$

$$\lambda_b = 2\epsilon_b\lambda/(\epsilon_a + \epsilon_b) \tag{5}$$

In the limit where $\epsilon_b \to \infty$ the field in the upper region, (5), becomes that of a line charge over a ground plane, where the image line charge is equal in magnitude and opposite in sign to that of the line charge and the field lines are perpendicular to the surface. In the opposite extreme where the upper region has a very large ϵ, the field lines in the upper region tend to have no normal component. One way to see this is to observe that in the limit $\epsilon_a \to \infty$ the image line charge becomes equal to the line charge.

6.6.7 (a) The uniform electric field that would exist if the permittivities were equal is written in polar coordinates as

$$\Phi = -E_o r\cos\phi \Rightarrow \mathbf{E} = E_o(\cos\phi\mathbf{i_r} - \sin\phi\mathbf{i}_\phi) \tag{1}$$

(b) The surface polarization charge density induced by this imposed field is

$$\begin{aligned}\sigma_{sp} &= -P_r^a + P_r^b = -(\epsilon_a - \epsilon_b)E_r\\ &= \epsilon_b\big(1 - \frac{\epsilon_a}{\epsilon_b}\big)E_r = \kappa\epsilon_b E_o\cos\phi\end{aligned} \tag{2}$$

(c) The potential induced by this surface charge density is of the form

$$\Phi = \begin{cases} A\frac{\cos\phi}{r/R}; & r > R\\ A(r/R)\cos\phi; & r < R\end{cases} \tag{3}$$

where the outer solution leaves the field as that imposed at infinity and the coefficients have been adjusted to insure continuity of the potential at the surface. The continuity condition from Gauss' law then gives

$$-\left(\epsilon_a\frac{\partial\Phi^a}{\partial r} - \epsilon_b\frac{\partial\Phi^b}{\partial r}\right) = \kappa\epsilon_b E_o\cos\phi \tag{4}$$

hence

$$A = \frac{\kappa E_o R}{2} \tag{5}$$

Substitution of this coefficient into (3) confirms the given potential.

(d) The exact solution given by (6.6.21) and (6.6.22) is first written in terms of κ.

$$\Phi^a = -RE_o\cos\phi\Big[\frac{r}{R} - \frac{R}{r}\frac{\kappa}{2-\kappa}\Big] \tag{6}$$

$$\Phi^b = -RE_o\cos\phi\Big[\frac{r}{R}\frac{2(1-\kappa)}{2-\kappa}\Big] \tag{7}$$

To linear terms in κ, note that

$$\frac{\kappa}{2-\kappa} \to \frac{\kappa}{2}; \quad \frac{2(1-\kappa)}{2-\kappa} \to 1 - \frac{\kappa}{2} \tag{8}$$

Using these expressions in (6) and (7) gives the same approximate expressions for the potential as given with the problem.

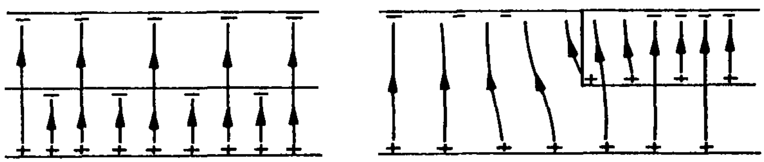
6.6.8 (a) If the dielectric is uniform, then so is the electric field.

$$\mathbf{E} \approx \mathbf{i_y}\frac{v}{d} \equiv \mathbf{i}_y E_o \tag{1}$$

(b) From (6.6.25), if (1) approximates the electric field then the approximate polarization surface charge density is

$$\sigma_{sp} \simeq \epsilon_o E_o\big(1 - \frac{\epsilon_a}{\epsilon_b}\big) = \epsilon_o \kappa E_o \tag{2}$$

(c) For $\epsilon_b < \epsilon_a$, $\sigma_{sp} < 0$. Thus, the distribution of surface charge density and hence electric field is as shown in Fig. S6.6.8.



(a) (b)

Figure S6.6.8

(d) For the second case, the polarization surface charge density is as sketched in Fig. S6.6.8b.

6.6.9 (a) The potential is represented as a piece-wise continuous function. In regions (a) and (b) where the permittivity is uniform, it is expanded in solutions to Laplace's equation that have zero potential on the boundaries. To satisfy the potential boundary condition to the left, V is added to the potential in region (b).

$$\Phi = \begin{cases} \displaystyle\sum_{n=1}^{\infty} -A_n \frac{\sinh\left[\frac{n\pi}{d}(x-a)\right]}{\sinh(n\pi a/d)} \sin\big(\frac{n\pi}{d}y\big); & 0 < x < a \\ \displaystyle\sum_{n=1}^{\infty} B_n \frac{\sinh\left[\frac{n\pi}{d}(x+a)\right]}{\sinh(n\pi a/d)} \sin\big(\frac{n\pi}{d}y\big) + V; & -a < x < 0 \end{cases} \tag{1}$$

Continuity of D_x at the interface requires that

$$\epsilon_a A_n\big(\frac{n\pi}{d}\big)\coth\big(\frac{n\pi a}{d}\big) = -B_n\epsilon_b\big(\frac{n\pi}{d}\big)\coth\big(\frac{n\pi a}{d}\big) \tag{2}$$

which gives

$$B_n = -\frac{\epsilon_a}{\epsilon_b}A_n \tag{3}$$

To make the potential continuous at $x = 0$, the constant V is expanded in the same Fourier series as representing the y dependence in the other terms in (1).

$$V = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{d}\right) \tag{4}$$

Multiplication by $\sin(m\pi y/d)$ and integration on y from 0 to d then gives an expression that can be solved for the coefficients.

$$C_n = \begin{cases} \frac{4V}{n\pi}; & n \text{ odd} \\ 0; & n \text{ even} \end{cases} \tag{5}$$

The potential continuity condition is then satisfied by each term in the series.

$$A_n = B_n + \frac{4V}{n\pi}; \quad n \text{ odd} \tag{6}$$

It follows from (3) and (6) that the coefficients in (1) are $A_n = B_n = 0$ for n even and for n odd.

$$A_n = \frac{4V}{n\pi}\frac{1}{1+\epsilon_a/\epsilon_b}; \quad B_n = -\frac{4V}{n\pi}\frac{1}{1+\epsilon_b/\epsilon_a} \tag{7}$$

(b) In sketching Φ and $\mathbf{E}$, as shown in Fig. S6.6.9a for the case where the permittivities are equal, note that the potential varies from V to 0 across the gaps. Every other point on the boundaries is either at potential V or potential 0. Thus, equipotentials all terminate and originate in the gaps. The equipotential $\Phi = V/2$ is in the $x = 0$ plane. Thus, the potential and field lines in each region are as shown in Fig. 5.5.3.

(c) The surface charge density is given by using (6.6.25) with $\mathbf{E}$ approximated by what it would be if the permittivities were equal.

$$\sigma_{sp} = \epsilon_o E_x\left(1 - \frac{\epsilon_a}{\epsilon_b}\right) \tag{8}$$

In the case where $\epsilon_a/\epsilon_b > 1$, $\sigma_{sp} < 0$, as illustrated by Fig. S6.6.9b. Some of the field lines originating to the left terminate in the negative σ_{sp} on the interface. Thus, the dielectric to the right tends to shield out the field. With $\epsilon_a/\epsilon_b < 1$, the surface charge density is positive, and the field tends to be shielded out of the material to the left.

(d) With $\epsilon_a \gg \epsilon_b$, the surface becomes an equipotential and the field is concentrated in the region to the left, as shown in Fig. S6.6.9c.

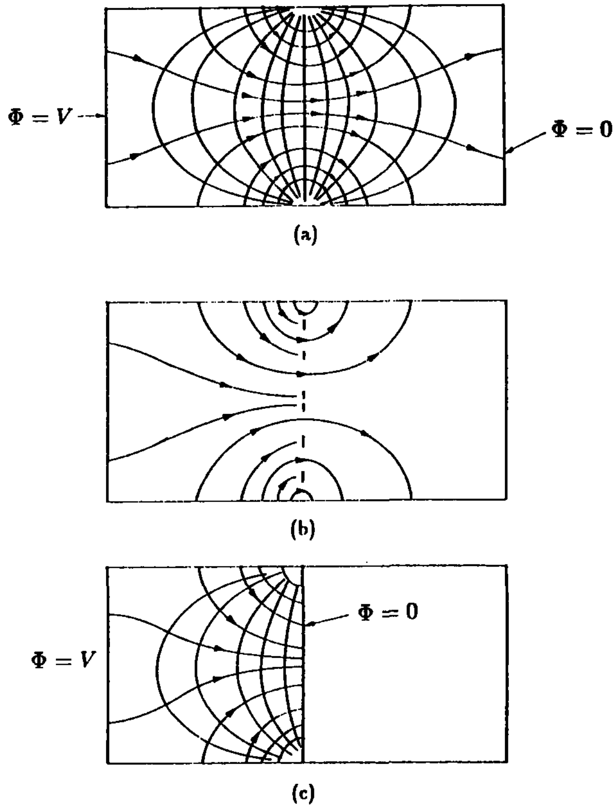


Figure S6.6.9

(e) With $\epsilon_b \gg \epsilon_a$, the field is shielded out of the region to the left. The field looks much as in Fig. S6.6.9c except that the fields are on the right rather than the left. The equipotential $\Phi = V/2$ is in the $x = 0$ plane. Thus, the potential and field lines in each region are as shown in Fig. 5.5.3.

6.7 SMOOTHLY INHOMOGENEOUS ELECTRICALLY LINEAR DIELECTRICS

6.7.1 Far from the lower end, the system becomes a pair of parallel plates having the potential difference V separated by a dielectric having its permittivity gradient in the y direction. Thus, as $y \to \infty$, the potential becomes simply

$$\Phi(y \to \infty) \to V\frac{x}{a} \tag{1}$$

The product solutions throughout the region between the plates are as developed in Example 6.7.1. From those, we add to (1) those that are zero at $x = 0$ and

$x = a$ (so as not to disturb the fact that (1) already satisfies the conditions on the potential there) and that go to zero as $y \to \infty$ [again, so that the potential there becomes (1)].

$$\Phi = \sum_{n=1}^{\infty} V_n e^{[\frac{\beta}{2} - \sqrt{(\beta/2)^2 + (n\pi/a)^2}]y} \sin\frac{n\pi}{a}x + V\frac{x}{a} \tag{2}$$

To determine the coefficients, V_n, (2) is evaluated at $y = 0$ and set equal to the potential there.

$$V = \sum_{n=1}^{\infty} V_n \sin\frac{n\pi}{a}x + V\frac{x}{a} \tag{3}$$

Multiplication by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ then gives the coefficients and hence the potential.

$$V_n = \frac{2}{a}\int_0^a V\Big(1 - \frac{x}{a}\Big)\sin\frac{n\pi}{a}x dx = (2/n\pi)V \tag{4}$$

6.7.2 The solutions to (6.7.2) for the given distributions of permittivity are as found in Example 6.7.1 with the roles of x and y interchanged. In the region to the left, $\beta \to -\beta$. Because the system extends to infinity in the $\pm x$ directions, exponential solutions are selected in each of the regions that decay to zero at infinity.

$$\Phi = \begin{cases} \displaystyle\sum_{n=1}^{\infty} A_n e^{[\frac{\beta}{2} - \sqrt{(\beta/2)^2 + (n\pi/a)^2}]x} \sin\frac{n\pi}{a}y; & 0 < x \\ \displaystyle\frac{V}{a}(a - y) + \sum_{n=1}^{\infty} B_n e^{[-\frac{\beta}{2} + \sqrt{(\beta/2)^2 + (n\pi/a)^2}]x} \sin\frac{n\pi}{a}y; & x < 0 \end{cases} \tag{1}$$

Continuity of D_x gives one condition on the coefficients.

$$[\epsilon_p e^{\beta x} E_x^b]_{x=0} = [\epsilon_p e^{-\beta x} E_x^a]_{x=0} \Rightarrow B_n = -A_n \tag{2}$$

To match the potential in the $x = 0$ plane, the first term in the solution to the left, in (1b), is expanded in the same series as the other terms.

$$\frac{V}{a}(a - y) = \sum_{n=1}^{\infty} C_n \sin\Big(\frac{n\pi}{a}y\Big) \tag{3}$$

The coefficient is found by multiplying this expression by $\sin(m\pi y/a)$ and integrating from $y = 0$ to $y = a$.

$$C_n = \frac{2V}{n\pi} \tag{4}$$

It follows from (2) and (4) that

$$A_n = -B_n = \frac{V}{n\pi} \tag{5}$$

6.7.3 (a) The polarization charge density is approximately

$$\begin{aligned}\rho_p &= -\nabla \cdot [(\epsilon - \epsilon_o)\mathbf{E}] \simeq -\nabla \cdot [(\epsilon - \epsilon_o)E_o\mathbf{i}_y] = -E_o\frac{\partial \epsilon}{\partial y} \\ &= E_o\epsilon_a\chi_p\frac{2y}{a^2}\exp[-(x^2+y^2+z^2)/a^2]\end{aligned} \tag{1}$$

(b) For $E_o\chi_p > 0$, the field induces positive and negative regions of charge density in the upper and lower regions respectively. These are centered on the y axis where the function $y\exp(-y^2/a^2)$ peaks, at $y = a/\sqrt{2}$. Thus, some of the field lines entering from below in Fig. S6.7.3 terminate on the negative charge while some leaving at the top originate on the positive charge. The field is that of a diffuse dipole.

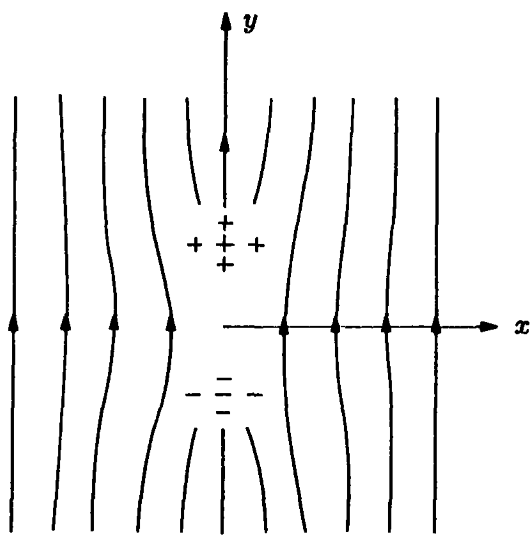


Figure S6.7.3

SOLUTIONS TO CHAPTER 7

7.1 CONDUCTION CONSTITUTIVE LAWS

7.1.1 If there are as many conduction electrons as there are atoms, then their number density is

$$N_- = \frac{A}{M_o}\rho = \frac{(6.023 \times 10^{26}(8.9 \times 10^3)}{63.5} = 8.4 \times 10^{28} \frac{\text{electrons}}{m^3} \tag{1}$$

The mobility is then

$$\mu_- = \frac{\sigma}{N_- q_-} = \frac{5.8 \times 10^7}{(8.4 \times 10^{38})(1.6 \times 10^{-19})} = 4.3 \times 10^{-3} \tag{2}$$

The electric field required to produce a current density of $1A/\text{cm}^2$ is

$$E = \frac{J}{\sigma} = \frac{10^4}{5.9 \times 10^7} = 1.7 \times 10^{-4} v/m \tag{3}$$

Thus, in copper, the velocity of the electrons giving rise to this current density is only

$$v_- = \mu_- E = (4.3 \times 10^{-3})(1.7 \times 10^{-4}) = 7.4 \times 10^{-7} m/s \tag{4}$$

7.2 STEADY OHMIC CONDUCTION

7.2.1 Boundary conditions on the conducting region are that $\Phi = 0, \Phi = v$ on the perfectly conducting surfaces at $r = a$ and $r = b$ respectively and that there is no normal current density on the insulating surfaces where $z = 0, z = d$. The latter are satisfied by a potential that is independent of the axial coordinate, so an appropriate solution to Laplace's equation, arranged to be zero on the outer electrode, is

$$\Phi = A ln(r/a) \tag{1}$$

The coefficient is adjusted to make the potential v on the inner electrode so that $A = v/ln(b/a)$ and (1) becomes

$$\Phi = v ln(r/a)/ln(b/a) \tag{2}$$

The current density is

$$J_r = \sigma E_r = -\frac{v\sigma}{ln(b/a)}\frac{1}{r} \tag{3}$$

and so the total current is

$$i = 2\pi b d J_r = -\frac{2\pi b d\sigma}{ln(b/a)}\frac{1}{b} = \frac{2\pi\sigma d}{ln(a/b)}v = \frac{v}{R} \tag{4}$$

Thus, R is as given.

7.2.2 The net current passing through the wire connected to the inner spherical electrode, i, must be equal to the net current at any radius r.

$$i = \int_S \mathbf{J} \cdot d\mathbf{a} = 4\pi r^2 \sigma E_r \Rightarrow E_r = \frac{i}{4\pi\sigma r^2} \tag{1}$$

Thus,

$$v = \int_b^a E_r dr = \frac{i}{4\pi\sigma}\int_b^a \frac{dr}{r^2} = \frac{i}{4\pi\sigma}\Big[\frac{1}{b} - \frac{1}{a}\Big] \tag{2}$$

By definition $v = iR$ so $R = \left(\frac{1}{b} - \frac{1}{a}\right)/4\pi\sigma$.

7.2.3 (a) Associated with the uniform field is the potential

$$\Phi = -\frac{v}{d}(y-d) \tag{1}$$

If the surrounding region is insulating relative to that between the electrodes, the normal component of the current density on the conductor surfaces bounded by the insulating surroundings is zero. The potential is constrained on the remainder of the surface enclosing the conductors, so the solution is uniquely specified. Provided the laws are satisfied everywhere inside the conducting region, the solution is exact. The given solution does indeed satisfy the boundary conditions on the surfaces of the conducting region. In the case of (a), the potential and normal component of current density must be continuous across the interior interface. Further, in the uniformly conducting regions of (a), Laplace's equation must be satisfied, as it is by a uniform field. In the case of (b), (7.2.4) is satisfied by the given potential.

(b) The total current is related to v by integrating the current density over the surface of the lower conductor.

$$i = J^a ac + J^b bc = \sigma_a ac\frac{v}{d} + \sigma_b bc\frac{v}{d} = \frac{c(a\sigma_a + b\sigma_b)}{d}v = Gv \tag{2}$$

(c) A similar calculation gives the resistance in the second case.

$$i = c\int_0^l \frac{\sigma v}{d} x dx = \frac{c}{d}\sigma_a v\int_0^l \Big(1 + \frac{x}{l}\Big)dx = \frac{2\sigma_a lc}{2d}v = Gv \tag{3}$$

7.2.4 The potential in each of the uniformly conducting regions takes the form

$$\Phi^a = -A\phi + C; \quad \Phi^b = -B\phi + F \tag{1}$$

where the four coefficients are adjusted to make the potentials zero and v on the respective electrodes, and make both the potential and the normal current density continuous at the interface between the conductors. On the surfaces at $r = a$ and

$r = b$, the current density must be zero, as it is for the potential of (1) because the electric field

$$\mathbf{E} = -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = \begin{cases} (A/r)\mathbf{i}_\phi \\ (B/r)\mathbf{i}_\phi \end{cases} \tag{2}$$

has no radial component. Rather than proceeding to determine the four coefficients in (1), we work directly with the electric field. The integration of $\mathbf{E}$ from one electrode to the other must be equal to the applied voltage.

$$\frac{\pi}{2}r\frac{A}{r} + \frac{\pi}{2}r\frac{B}{r} = v \tag{3}$$

Further, the current density must be continuous at the interface.

$$\sigma_a\frac{A}{r} = \frac{\sigma_b B}{r} \tag{4}$$

It follows from these relations that

$$A = 2v/\pi(1 + \sigma_a/\sigma_b) \tag{5}$$

The current through any cross-section of the material [say region (a)] must be equal to that through the wire. Thus,

$$i = d\int_b^a \sigma_a E_\phi dr = \left[\frac{2d\sigma_a}{\pi(1+\sigma_a/\sigma_b)}\int_b^a \frac{dr}{r}\right]v \equiv Gv \tag{6}$$

and the resistance is

$$G = \frac{2d\sigma_a}{\pi(1+\frac{\sigma_a}{\sigma_b})}ln(a/b) \tag{7}$$

7.2.5 (a) From (7.2.23)

$$G = A\sigma_o/\int_0^d (1+\frac{y}{a})dy = \frac{A\sigma_o}{d[1+\frac{1}{2}\frac{d}{a}]} \tag{1}$$

(b) We need the electric field, which follows from (7.2.19) by using the result of (1) to evaluate $J_o = i/A = Gv/A$

$$E_y = \frac{J_o}{\sigma} = \frac{G}{A\sigma_o}(1+\frac{y}{a})v \tag{2}$$

Thus, the unpaired charge density is evaluated using (7.2.8).

$$\rho_u = -\frac{\epsilon(1+\frac{y}{a})}{\sigma_o}\frac{G(1+\frac{y}{a})}{A\sigma_o}\frac{d}{dy}\left[\frac{\sigma_o}{(1+\frac{y}{a})}\right] = \frac{\epsilon G}{A\sigma_o a} \tag{3}$$

7.2.6 (a) The inhomogeneity in permittivity has no effect on the resistance. It is therefore given by (7.2.25).

(b) With the steady conduction laws stipulating that the electric field is uniform, the unpaired charge density follows from Gauss' law.

$$\rho_u = \nabla \cdot \epsilon \mathbf{E} = \nabla \cdot \left(\epsilon \frac{v}{d} \mathbf{i}_y\right) = \frac{v}{d} \frac{\partial \epsilon}{\partial y} = -\frac{\epsilon_a v}{da} \frac{1}{\left(1 + \frac{y}{a}\right)^2} \tag{1}$$

7.2.7 At a radius r, the area of the conductor (and with $r = a$ and $r = b$, of the outer and inner electrodes, respectively) is

$$A = 2\pi r^2 [1 - \cos(\alpha/2)] \tag{1}$$

Consistent with the insulating surfaces of the conductor is the requirement that the current density and associated electric field be radial. Current conservation (fundamentally, the requirement that the current density be solenoidal) then gives as a solution to the field laws

$$\sigma E_r \left[2\pi r^2 \left(1 - \cos\frac{\alpha}{2}\right)\right] = i \tag{2}$$

and it follows that

$$E_r = \frac{i a^2}{2\pi \sigma_o \left(1 - \cos\frac{\alpha}{2}\right)} \frac{1}{r^4} \tag{3}$$

The voltage follows as

$$v = \int_b^a E_r dr = i(a^3 - b^3)/6\pi\sigma_o\left(1 - \cos\frac{\alpha}{2}\right) b^3 a \tag{4}$$

and this relation takes the form $i = vG$, where G is as given.

7.2.8 There can be no current density normal to the interfaces of the conducting material having normals in the azimuthal direction. These boundary conditions are satisfied by an axially symmetric solution in which the current density is purely radial. In that case, both $\mathbf{E}$ and $\mathbf{J}$ are independent of ϕ. Then, the total current is related to the current density and (through Ohm's law) electric field intensity at any radius r by

$$i = 2\pi \alpha d r J_r = 2\pi \alpha d \sigma_o a E_r \tag{1}$$

Thus,

$$E_r = \frac{i}{2\pi \alpha d \sigma_o a} \tag{2}$$

and because

$$\int_b^a E_r dr = \frac{i(a - b)}{2\pi \alpha d \sigma_o a} = v \tag{3}$$

$$G = 2\pi \alpha d a \sigma_o/(a - b) \tag{4}$$

7.3 DISTRIBUTED CURRENT SOURCES AND ASSOCIATED FIELDS

7.3.1 In the conductor, the potential distribution is a particular part comprised of the potential due to the point current soruce, (6) with $i_p \rightarrow I$ and

$$r = \sqrt{x^2 + (y-h)^2 + z^2}$$

In order to satisfy the condition that there be no normal component of $\mathbf{E}$ at the interface, a homogeneous solution is added that amounts to a second source of the same sign in the lower half space. Of course, such a current source could not really exist in the lower region so if the field in the upper region is to be given some equivalent physical situation, it should be pictured as equivalent to a pair of like-signed point current sources in a uniform conductor. In any case, this second source is located at $r = \sqrt{x^2 + (y+h)^2 + z^2}$ and hence the potential in the conductor is as given. In the lower region, the potential must satisfy Laplace's equation everywhere (there are no charges in the lower region). The field in this region is uniquely specified by requiring that the potential be consistent with (a) evaluated at the interface

$$\Phi(x, y=0, z) = \frac{2I}{4\pi\sigma\sqrt{x^2 + h^2 + z^2}} \tag{1}$$

and that it go to zero at infinity in the lower half-space. The potential that matches these conditions is that of a point charge of magnitude $q = 2I\epsilon/\sigma$ located on the y axis at $y = h$, the given potential.

7.3.2 (a) First, what is the potential associated with a uniform line current in a uniform conductor? In the steady state

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = 0 \tag{1}$$

and for a surface S that has radius r from the line current,

$$K_l = 2\pi r J_r = 2\pi r\sigma E_r \Rightarrow E_r = \frac{K_l}{2\pi\sigma r} \tag{2}$$

Within a constant, the associated potential is therefore

$$\Phi = -\frac{K_l}{2\pi\sigma} ln(r) \tag{3}$$

To satisfy the requirement that there be no normal current density in the plane $y = 0$, the potential is that of the line current located at $y = h$ and an image line current of the same polarity located at $y = -h$.

$$\Phi^a = -\frac{K_l}{2\pi\sigma}[ln\sqrt{x^2 + (y-h)^2} + ln\sqrt{x^2 + (y-h)^2}] \tag{4}$$

Note that the normal derivative of this expression in the plane $y = 0$ is indeed zero.

(b) In the lower region, the potential must satisfy Laplace's equation everywhere and match the potential of the conductor in the plane $y = 0$.

$$\Phi^b(y=0) = -\frac{K_l}{\pi\sigma} ln\sqrt{x^2 + h^2} \tag{5}$$

This has the potential distribution of an image line current located at $y = h$. With the magnitude of this line current adjusted so that the potential of (5) is matched at $x = 0$,

$$\Phi^b = -\frac{K_l}{\pi\sigma} ln\sqrt{x^2 + (y-h)^2} \tag{6}$$

the potential is matched at every other value of x as well.

7.3.3 First, the potential due to a single line current is found from the integral form of (2).

$$2\pi\sigma r E_r = K_l \Rightarrow E_r = \frac{K_l}{2\pi\sigma r} \tag{1}$$

Thus, for a single line current,

$$\Phi = -\frac{K_l}{2\pi\sigma} lnr \tag{2}$$

For the pair of line currents, spaced by the distance d,

$$\Phi = -\frac{K_l}{2\pi\sigma}[ln(r - d\cos\phi) - lnr] = -\frac{K_l}{2\pi\sigma} ln\left[1 - \frac{d\cos\phi}{r}\right] \rightarrow \frac{K_l d\cos\phi}{2\pi\sigma r} \tag{3}$$

7.4 SUPERPOSITION AND UNIQUENESS OF STEADY CONDUCTION SOLUTIONS

7.4.1 (a) At $r = b$, there is no normal current density so that

$$J_r(r=b) = 0 \Rightarrow \frac{\partial\Phi}{\partial r}(r=b) = 0 \tag{1}$$

while at $r = a$,

$$J_r = -J_o\cos\theta = -\sigma\frac{\partial\Phi}{\partial r}(r=a) \tag{2}$$

Because the dependence of the potential must be the same as the radial derivative in (2), assume the solution takes the form

$$\Phi = Ar\cos\theta + B\frac{\cos\theta}{r^2} \tag{3}$$

Substitution into (1) and (2) then gives the pair of equations

$$\begin{bmatrix} 1 & -2b^{-3} \\ \sigma & -2\sigma a^{-3} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ J_o \end{bmatrix} \tag{4}$$

from which it follows that

$$A = J_o/\sigma[1-(b/a)^3]; \quad B = J_o b^3/2\sigma[1-(b/a)^3] \tag{5}$$

Substitution into (3) results in the given potential in the conducting region.

(b) The potential inside the hollow sphere is now specified, because we know that the potential on its wall is

$$\Phi(r=b) = 3J_o b/2\sigma[1-(d/a)^3] \tag{6}$$

Here, the origin is included, so the only potential having the required dependence is

$$\Phi = Cr\cos\theta \tag{7}$$

Determination of C by evaluating (7) at $r = b$ and setting it equal to (6) gives C and hence the given interior potential. What we have carried out is an "inside-outside" calculation of the field distribution where the "inside" region is outside and the "outside" region is inside.

7.4.2 (a) This is an example of an inside-outside problem, where the potential is first determined in the conducting material. Because the current density normal to the outer surface is zero, this potential can be determined without regard for the geometry of what may be located outside. Then, given the potential on the surface, the outside potential is determined. Given the ϕ dependence of the normal current density at $r = b$, the potential in the conducting region is taken as having the form

$$\Phi^b = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{1}$$

Boundary conditions are that

$$J_r = -\sigma\frac{\partial\Phi^b}{\partial r} = 0 \tag{2}$$

at $r = a$, which requires that $B = a^3A/2$ and that

$$-\sigma\frac{\partial\Phi^b}{\partial r} = J_o\cos\theta \tag{3}$$

at $r = b$. This condition together with the result of (2) gives $A = J_o/\sigma[(a/b)^3 - 1]$. Thus, the potential in the conductor is

$$\Phi^b = \frac{J_o a}{\sigma}\Big[\frac{r}{a} + \frac{1}{2}(a/r)^2\Big]\cos\theta/[(a/b)^3 - 1] \tag{4}$$

(b) The potential in the outside region must match that given by (4) at $r = a$. To match the θ dependence, a dipole potential is assumed and the coefficient adjusted to match (4) evaluated at $r = a$.

$$\Phi^a = \frac{3J_o a}{2\sigma[(a/b)^3 - 1]}(a/r)^2\cos\theta \tag{5}$$

7.4.3 (a) This is an inside-outside problem, where the region occupied by the conductor is determined without regard for what is above the interface except that at the interface the material above is insulating. The potential in the conductor must match the given potential in the plane $y = -a$ and must have no derivative with respect to y at $y = 0$. The latter condition is satisfied by using the cosh function for the y dependence and, in view of the x dependence of the potential at $y = -a$, taking the x dependence as also being $\cos(\beta x)$. The coefficient is adjusted so that the potential is then the given value at $y = 0$.

$$\Phi^b = V\frac{\cosh\beta y}{\cosh\beta a}\cos\beta x \tag{1}$$

(b) in the upper region, the potential must be that given by (1) in the plane $y = 0$ and must decay to zero as $y \to \infty$. Thus,

$$\Phi^a = V\frac{\cos\beta x}{\cosh\beta a}e^{-\beta y} \tag{2}$$

7.4.4 The potential is zero at $\phi = 0$ and $\phi = \pi/2$, so it is expanded in solutions to Laplace's equation that have multiple zeros in the ϕ direction. Because of the first of these conditions, these are solutions of the form

$$\Phi \propto r^{\pm n}\sin n\theta \tag{1}$$

To make the potential zero at $\phi = \pi/2$,

$$n\frac{\pi}{2} = \pi, 2\pi, \ldots \Rightarrow n = 2, 4, \ldots 2m; \quad m = 1, 2, 3, \ldots \tag{2}$$

Thus, the potential is assumed to take the form

$$\Phi = \sum_{m=1}^{\infty}(A_m r^{2m} + B_m r^{-2m})\sin 2m\phi \tag{3}$$

At the outer boundary there is no normal current density, so

$$\frac{\partial \Phi}{\partial r}(r = a) = 0 \tag{4}$$

and it follows from (3) that

$$2mA_m a^{2m-1} - 2mB_m a^{-2m-1} = 0 \Rightarrow B_m = A_m a^{4m} \tag{5}$$

At $r = b$, the potential takes the form

$$\Phi = \sum_{m=1}^{\infty} V_m \sin 2m\phi = v \tag{6}$$

The coefficients are evaluated as in (5.5.3) through (5.5.9).

$$V_n \frac{\pi}{4} \int_0^{\pi/2} v \sin 2n\theta d\theta = \frac{v}{n}; \quad n \text{ odd}$$

$$\Rightarrow V_m = \frac{4v}{\pi m} = A_m b^{2m} + B_m b^{-2m} = A_m(b^{2m} + a^{4m} b^{-2m}) \tag{7}$$

Thus,

$$A_m = 4v/m\pi b^{2m}[1 - (a/b)^{4m}] \tag{8}$$

Substitution of (8) and (5) into (3) results in the given potential.

7.4.5 (a) To make the ϕ derivative of the potential zero at $\phi = 0$ and $\phi = \alpha$, the ϕ dependence is made $\cos(n\pi\phi/\alpha)$. Thus, solutions to Laplace's equation in the conductor take the form

$$\Phi = \sum_{n=0}^{\infty} \left[A_n (r/b)^{(n\pi/\alpha)} + B_n (b/r)^{(n\pi/\alpha)}\right] \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{1}$$

where $n = 0, 1, 2, \ldots$ To make the radial derivative zero at $r = b$,

$$B_n = A_n \tag{2}$$

so that each term in the series

$$\Phi = \sum_{n=0}^{\infty} A_n \left[(r/b)^{(n\pi/\alpha)} + (b/r)^{(n\pi/\alpha)}\right] \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

satisfies the boundary conditions on the first three of the four boundaries.

(b) The coefficients are now determined by requiring that the potential be that given on the boundary $r = a$. Evaluation of (3) at $r = a$, multiplication by $\cos(m\pi\phi/\alpha)$ and integration gives

$$
\begin{aligned}
-\frac{v}{2}\int_0^{\alpha/2}\cos\Big(\frac{m\pi\phi}{\alpha}\Big)d\phi + \frac{v}{2}\int_{\alpha/2}^{\alpha}\cos\Big(\frac{m\pi}{\alpha}\phi\Big)d\phi \\
= \int_0^{\alpha}\sum_{n=0}^{\infty} A_n\big[(a/b)^{(n\pi/\alpha)} + (b/a)^{(n\pi/\alpha)}\big] \\
\cos\Big(\frac{n\pi\phi}{\alpha}\Big)\cos\Big(\frac{m\pi\phi}{\alpha}\Big)d\phi \\
= -\frac{2\alpha}{m\pi}\sin\Big(\frac{m\pi}{2}\Big)
\end{aligned}
\tag{4}
$$

and it follows that (3) is the required potential with

$$
A_n = -\frac{2v}{n\pi}\sin\Big(\frac{n\pi}{2}\Big)/\big[(a/b)^{n\pi/\alpha} + (b/a)^{n\pi/\alpha}\big]
\tag{5}
$$

7.4.6 To make the potential zero at $\phi = 0$ and $\phi = \pi/2$, the ϕ dependence is made $\sin(2n\phi)$. Then, the r dependence is divided into two parts, one arranged to be zero at $r = a$ and the other to be zero at $r = b$.

$$
\Phi = \sum_{n=1}^{\infty}\{A_n[(r/a)^{2n} - (a/r)^{2n}] + B_n[(r/b)^{2n} - (b/r)^{2n}]\}\sin(2n\phi)
\tag{1}
$$

Thus, when this expression is evaluated on the outer and inner surfaces, the boundary conditions respectively involve only B_n and A_n.

$$
\Phi(r = a) = v_a = \sum_{n=1}^{\infty} B_n[(a/b)^{2n} - (b/a)^{2n}]\sin 2n\phi
\tag{2}
$$

$$
\Phi(r = b) = v_b = \sum_{n=1}^{\infty} A_n[(b/a)^{2n} - (a/b)^{2n}]\sin 2n\phi
\tag{3}
$$

To determine the B_n's, (2) is multiplied by $\sin(2m\phi)$ and integrated

$$
\int_0^{\pi/2} v_a \sin 2m\phi d\phi = \int_0^{\pi/2}\sum_{n=1}^{\infty} B_n[(a/b)^{2n} - (b/a)^{2n}]\sin 2n\phi\sin 2m\phi d\phi
\tag{4}
$$

and it follows that for n even $B_n = 0$ while for n odd

$$
B_n = 4v_a/n\pi[(a/b)^{2n} - (b/a)^{2n}]
\tag{5}
$$

A similar usage of (3) gives

$$A_n = 4v_b/n\pi[(b/a)^{2n} - (a/b)^{2n}] \tag{6}$$

By definition, the mutual conductance is the total current to the outer electrode when its voltage is zero divided by the applied voltage.

$$G = \frac{i_a|_{v_a=0}}{v_b} = -\frac{d\sigma}{v_b}\int_0^{\pi/2}\frac{\partial\Phi}{\partial r}\Big|_{r=a} a\,d\phi = -\frac{d\sigma a}{v_b}\int_0^{\pi/2}\sum_{n=1}^{\infty} A_n\frac{4n}{a}\sin 2n\phi\, d\phi \tag{7}$$

and it follows that the mutual conductance is

$$G = -\frac{d\sigma}{v_b}4\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{A_n}{n} = \frac{16 d\sigma}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{[(a/b)^{2n} - (b/a)^{2n}]n^2} \tag{8}$$

7.5 STEADY CURRENTS IN PIECE-WISE UNIFORM CONDUCTORS

7.5.1 To make the current density the given uniform value at infinity,

$$\Phi \to -\frac{J_o}{\sigma_a} r\cos\theta; \quad r \to \infty \tag{1}$$

At the surface of the sphere, where $r = R$

$$J_r^a = J_r^b \Rightarrow \sigma_a\frac{\partial\Phi^a}{\partial r} = \sigma_b\frac{\partial\Phi^b}{\partial r} \tag{2}$$

and

$$\Phi^a = \Phi^b \tag{3}$$

In view of the θ dependence of (1), select solutions of the form

$$\Phi^a = -\frac{J_o}{\sigma_a} r\cos\theta + A\frac{\cos\theta}{r^2}; \quad \Phi^b = Br\cos\theta \tag{4}$$

Substitution into (2) and (3) then gives

$$A = -\frac{J_o R^3}{\sigma_a}\frac{(\sigma_a - \sigma_b)}{(2\sigma_a + \sigma_b)}; \quad B = -\frac{J_o}{\sigma_a}\frac{3\sigma_a}{(2\sigma_a + \sigma_b)} \tag{5}$$

and hence the given solution.

7.5.2 These are examples of inside-outside approximations where the field in region (a) is determined first and is therefore the "inside" region.

(a) If $\sigma_b \gg \sigma_a$, then

$$\Phi^a(r = R) \approx \text{constant } = 0 \tag{1}$$

(b) The field must be $-(J_o/\sigma_a)\mathbf{i}_z$ far from the sphere and satisfy (1) at $r = R$. Thus, the field is the sum of the potential for the uniform field and a dipole field with the coefficient set to satisfy (1).

$$\Phi^a \approx -\frac{RJ_o}{\sigma_a}\Big[\frac{r}{R} - (R/r)^2\Big]\cos\theta \tag{2}$$

(c) At $r = R$, the normal current density is continuous and approximated by using (2). Thus, the radial current density at $r = R$ inside the sphere is

$$J_r^b(r = R) = J_r^a(r = R) = -\sigma_a \frac{\partial \Phi^a}{\partial r}\Big|_{r=R} = 2J_o\cos\theta \tag{3}$$

A solution to Laplace's equation having this dependence on θ is the potential of a uniform field, $\Phi = Br\cos(\theta)$. The coefficient B follows from (3) so that

$$\Phi^b \approx -\frac{3J_oR}{\sigma_b}(r/R)\cos\theta \tag{4}$$

In the limit where $\sigma_b \gg \sigma_a$, (2) and (4) agree with (a) of Prob. 7.5.1.

(d) In the opposite extreme, where $\sigma_a \gg \sigma_b$,

$$J_r^a(r = R) = 0 \tag{5}$$

Again, the potential is the sum of that due to the uniform field that prevails at infinity and a dipole solution. However, this time the coefficient is adjusted so that the radial derivative is zero at $r = R$.

$$\Phi^a \approx -\frac{RJ_o}{\sigma_a}\Big[\frac{r}{R} + \frac{1}{2}(R/r)^2\Big]\cos\theta \tag{6}$$

To determine the field inside the sphere, potential continuity is used. From (6), the potential at $r = R$ is $\Phi^b = -(3RJ_o/2\sigma_a)\cos\theta$ and it follows that inside the sphere

$$\Phi^b \approx -\frac{3}{2}\frac{RJ_o}{\sigma_a}(r/R)\cos\theta \tag{7}$$

In the limit where $\sigma_a \gg \sigma_b$, (a) of Prob. 7.5.1 agrees with (6) and (7).

7.5.3 (a) The given potential implies a uniform field, which is certainly irrotational and solenoidal. Further, it satisfies the potential conditions at $z = 0$ and $z = -l$ and implies that the current density normal to the top and bottom interfaces is zero. The given "inside" potential is therefore the correct solution.

(b) In the "outside" region above, boundary conditions are that

$$\Phi(x = 0, z) = -vz/l; \quad \Phi(x, 0) = 0;$$

$$\Phi(a, z) = 0; \quad \Phi(-l, x) = v(1 - \frac{x}{a}) \tag{1}$$

The potential must have the given linear dependence on the bottom horizontal interface and on the left vertical boundary. These conditions can be met by a solution to Laplace's equation of the form xz. By translating the origin of the x axis to be at $x = a$, the solution satisfying the boundary conditions on the top and right boundaries is of the form

$$\Phi = A(a - x)z = -\frac{v}{la}(a - x)z \tag{2}$$

where in view of (1a) and (1c), setting the coefficient $A = -v/l$ makes the potential satisfy conditions at the remaining two boundaries.

(c) In the air and in the uniformly conducting slab, the bulk charge density, ρ_u, must be zero. At its horizontal upper interface,

$$\sigma_u = \epsilon_a E_x^a - \epsilon_b E_x^b = -\epsilon_o vz/la \tag{3}$$

Note that $z < 0$ so if $v > 0$, $\sigma_u > 0$ as expected intuitively. The surface charge density on the lower surface of the conductor cannot be specified until the nature of the region below the plane $x = -b$ is specified.

(d) The boundary conditions on the lower "inside" region are homogeneous and do not depend on the "outside" region. Therefore the solution is the same as in (a). The potential in the upper "outside" region is one associated with a uniform electric field that is perpendicular to the upper electrode. To satisfy the condition that the tangential electric field be the same just above the interface as below, and hence the same at any location on the interface, this field must be uniform. If it is to be uniform throughout the air-space, it must be the same above the interface as in the region where the bounding conductors are parallel plates. Thus,

$$\mathbf{E} = \frac{v}{a}\mathbf{i}_x + \frac{v}{l}\mathbf{i}_z \tag{4}$$

The associated potential that is zero at $z = 0$ and indeed on the surface of the electrode where $x = -za/l$ is

$$\Phi = \frac{v}{a}x + \frac{v}{l}z \tag{5}$$

Finally, instead of (3), the surface charge density is now

$$\sigma_u = \epsilon_o v/a$$

7.5.4 (a) Because they are surrounded by either surfaces on which the potential is constrained or by insulating regions, the fields within the conductors are determined without regard for either the fields within the square or outside, where not enough information has been given to determine the fields. The condition that there be no normal current density, and hence no normal electric field intensity on the surfaces of the conductors that interface the insulating regions, is automatically met by having uniform fields in the conductors. Because these fields are normal to the electrodes that terminate these regions, the boundary conditions on these surfaces are met as well. Thus, regardless of what d is relative to a, in the upper conductor,

$$\mathbf{E} = -\mathbf{i}_x\frac{v}{a}; \quad \Phi = \frac{v}{a}x; \quad \mathbf{J} = -\sigma\frac{v}{a}\mathbf{i}_x \tag{1}$$

while in the conductor to the right

$$\mathbf{E} = -\mathbf{i}_y\frac{v}{a}; \quad \Phi = \frac{v}{a}y; \quad \mathbf{J} = -\sigma\frac{v}{a}\mathbf{i}_y \tag{2}$$

(b) In the planes $y = a$ and $x = a$ the potential inside must be the same as given by (1) and (2) in these planes, linear functions of x and of y, respectively. It must also be zero in the planes $x = 0$ and $y = 0$. A simple solution meeting these conditions is

$$\Phi = Axy = \frac{v}{a^2}xy \tag{3}$$

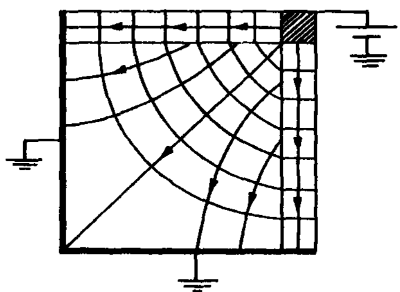


Figure S7.5.4

(c) The distribution of potential and electric field intensity is as shown in Fig. S7.5.4.

7.5.5 (a) Because the potential difference between the plates, either to the left or to the right, is zero, the electric field there must be zero and the potential that of the respective electrodes.

$$\Phi^a \rightarrow 0 \text{ as } x \rightarrow \infty; \quad \Phi^b \rightarrow v \text{ as } x \rightarrow -\infty \tag{1}$$

(b) Solutions that satisfy the boundary conditions on all but the interface at $x = 0$ are

$$\Phi^a = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y \tag{2a}$$

$$\Phi^b = v + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin \frac{n\pi}{a} y \tag{2b}$$

(c) At the interface, boundary conditions are

$$-\sigma_a \frac{\partial \Phi^a}{\partial x} = -\sigma_b \frac{\partial \Phi^b}{\partial x} \tag{3}$$

$$\Phi^a = \Phi^b \tag{4}$$

(d) The first of these requires of (2) that

$$\sigma_a \frac{n\pi}{a} A_n = -\sigma_b \frac{n\pi}{a} B_n \Rightarrow B_n = -\frac{\sigma_a}{\sigma_b} A_n \tag{5}$$

Written using this, the second requires that

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y = v - \sum_{n=1}^{\infty} \frac{\sigma_a}{\sigma_b} A_n \sin \frac{n\pi}{a} y \tag{6}$$

The constant term can also be written as a Fourier series using an evaluation of the coefficients that is essentially the same as in (5.5.3)-(5.5.9).

$$v = \sum_{n=1}^{\infty} \frac{4v}{\pi n} \sin \frac{n\pi}{a} y \tag{7}$$

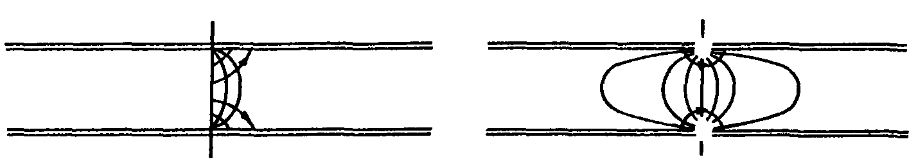
Thus,

$$A_n \left(1 + \frac{\sigma_a}{\sigma_b}\right) = \frac{4v}{n\pi} \tag{8}$$

and it follows that the required potential is

$$\Phi^a = \sum_{n=1}^{\infty} \frac{4v}{n\pi(1 + \sigma_a/\sigma_b)} e^{-\frac{n\pi}{a} x} \sin \frac{n\pi}{a} y$$

$$\Phi^b = v - \sum_{n=1}^{\infty} \frac{\sigma_a}{\sigma_b} \frac{4v e^{\frac{n\pi}{a} x}}{n\pi(1 + \sigma_a/\sigma_b)} \sin \frac{n\pi}{a} y \tag{9}$$



$\sigma_b \gg \sigma_a$ $\sigma_a = \sigma_b$

Figure S7.5.5

(e) In the case where $\sigma_b \gg \sigma_a$, the "inside" region is to the left where boundary conditions are on the potential at the upper and lower surfaces and on its normal derivative at the interface. In this limit, the potential is uniform throughout the region and the interface is an equipotential having $\Phi = v$. Thus, the potential in the region to the right is as shown in Fig. 5.5.3 with the surface at $y = b$ playing the role of the interface and the surface at $y = 0$ at infinity. In the case where the region between electrodes is filled by a uniform conductor, the potential and field distribution are as sketched in Fig. S7.5.5. In the vicinity of the regions where the electrodes abut, the potential becomes that illustrated in Fig. 5.7.2. By symmetry, the plane $x = 0$ is one having the potential $\Phi = v/2$.

(f) The surface at $y = a/2$ is a plane of symmetry in the previous configuration and hence one where $E_y = 0$. Thus, the previous solution applies directly to finding the solution in the conducting layer.

7.6 CONDUCTION ANALOGS

7.6.1 The analogous laws are

$$\mathbf{E} = -\nabla\Phi \qquad \mathbf{E} = -\nabla\Phi \tag{1}$$

$$\nabla\cdot\sigma\mathbf{E} = s \qquad \nabla\cdot\epsilon\mathbf{E} = \rho_u \tag{2}$$

The systems are normalized to different length scales. The conductivity and permittivity are respectively normalized to σ_c and ϵ_ϵ respectively and similarly, the potentials are normalized to the respective voltages V_c and V_ϵ.

$$(x,y,z) = (\underline{x},\underline{y},\underline{z})l_c \qquad (x,y,z) = (\underline{x},\underline{y},\underline{z})l_\epsilon \tag{3}$$

$$\Phi = V_c\underline{\Phi} \qquad \Phi = V_\epsilon\underline{\Phi} \tag{4}$$

$$\mathbf{E} = (V_c/l_c)\underline{\Phi} \qquad \mathbf{E} = (V_\epsilon/l_\epsilon)\underline{\Phi} \tag{5}$$

$$s = (\sigma_c V_c/l_c^2)\underline{s} \tag{6}$$

$$\rho_u = (\epsilon_\epsilon V_\epsilon/l_\epsilon^2)\underline{\rho}_u \tag{7}$$

By definition, the normalized quantities are the same in the two systems

$$\underline{\sigma}(\mathbf{r}) = \underline{\epsilon}(r) \tag{8}$$

$$\underline{s}(r) = \underline{\rho}_u(r) \tag{9}$$

so that both systems are represented by the same normalized laws.

$$\underline{\mathbf{E}} = -\underline{\nabla}\underline{\Phi} \tag{10}$$

$$\underline{\nabla} \cdot \underline{\sigma \mathbf{E}} = \underline{s} \tag{11}$$

Thus, the capacitance and conductance are respectively

$$C = \epsilon_\epsilon l_\epsilon \oint_S \underline{\epsilon \mathbf{E}} \cdot d\underline{a} / \int_C \underline{\mathbf{E}} \cdot d\underline{s} \tag{12}$$

$$G = \sigma_c l_c \oint_S \underline{\sigma \mathbf{E}} \cdot d\underline{a} / \int_C \underline{\mathbf{E}} \cdot d\underline{a} \tag{13}$$

where, again by definition, the normalized integral ratios in (12) and (13) are the same number. Thus,

$$C/G = \frac{\epsilon_\epsilon}{\sigma_c}\frac{l_\epsilon}{l_c} = \frac{\epsilon}{\sigma}\frac{l_\epsilon}{l_c} \tag{14}$$

Note that the deductions summarized by (7.6.3) could be made following the same normalization approach.

7.7 CHARGE RELAXATION IN UNIFORM CONDUCTORS

7.7.1 (a) The charge is given when $t = 0$

$$\rho = \rho_i \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y \tag{1}$$

Given the charge density, none of the bulk or surface conditions needed to determine the field involve time rates of change. Thus, the initial potential distribution is determined from the initial conditions alone.

(b) The properties of the region are uniform, so (3) and hence (4) apply directly. Given the charge is (c) of Prob. 4.1.4 when $t = 0$, the subsequent distribution of charge is

$$\rho = \rho_o(t) \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y; \quad \rho_o = \rho_i e^{-t/\tau}; \quad \tau \equiv \frac{\epsilon}{\sigma} \tag{2}$$

(c) As in (a), at each instant the charge density is known and all other conditions are independent of time rates of change. Thus, the potential and field distributions simply go along with the changing charge density. They follow from (a) and (b) of Prob. 4.1.4 with $\rho_o(t)$ given by (2).

(d) Again, with $\rho_o(t)$ given by (2), the current is given by (6) of Prob. 4.1.4.

7.7.2 (a) The line charge is pictured as existing in the same uniformly conducting material as occupies the surrounding region. Thus, (7.7.3) provides the solution.

$$\lambda_l = \lambda_l(t = 0)e^{-t/\tau}; \quad \tau = \epsilon/\sigma \tag{1}$$

(b) There is no initial charge density in the surrounding region. Thus, the charge density there is zero.

(c) The potential is given by (1) of Prob. 4.5.4 with λ_l given by (1).

7.7.3 (a) With $q < -q_c$, the entire surface of the particle can collect the ions. Equation (7.7.10) becomes simply

$$i = -\mu\rho 6\pi R^2 E_a \int_0^\pi \left(\cos\theta + \frac{q}{q_c}\right)\sin\theta d\theta \tag{1}$$

Integration and the definition of q_c results in the given current.

(b) The current found in (a) is equal to the rate at which the charge on the particle is increasing.

$$\frac{dq}{dt} = -\frac{\mu\rho}{\epsilon}q \tag{2}$$

This expression can either be formally integrated or recognized to have an exponential solution. In either case, with $q(t=0) = q_o$,

$$q = q_o e^{-t/\tau}; \quad \tau \equiv \epsilon/\mu\rho \tag{3}$$

7.7.4 The potential is given by (5.9.13) with q replaced by q_c as defined with (7.7.11)

$$\Phi = -E_a R\cos\theta\left[\frac{r}{R} - (R/r)^2\right] + \frac{12\pi\epsilon_o R^2 E_a}{4\pi\epsilon_o r} \tag{1}$$

The reference potential as $r \to \infty$ with $\theta = \pi/2$ is zero. Evaluation of (1) at $r = R$ therefore gives the particle potential relative to infinity in the plane $\theta = \pi/2$.

$$\Phi = 3RE_a \tag{2}$$

The particle charges until it reaches 3 times a potential equal to the radius of the particle multiplied by the ambient field.

7.8 ELECTROQUASISTATIC CONDUCTION LAWS FOR INHOMOGENEOUS MATERIAL

7.8.1 For $t < 0$, steady conduction prevails, so $\partial(\ )/\partial t = 0$ and the field distribution is defined by (7.4.1)

$$\nabla\cdot(\sigma\nabla\Phi) = -s \tag{1}$$

where

$$\Phi = \Phi_\Sigma \quad \text{on} \quad S'; \qquad -\sigma\nabla\Phi = \mathbf{J}_\Sigma \quad \text{on} \quad S'' \tag{2}$$

To see that the solution to (1) subject to the boundary conditions of (2) is unique, propose different solutions Φ_a and Φ_b and define the difference between these solutions as

$$\Phi_d = \Phi_a - \Phi_b \tag{3}$$

Then it follows from (1) and (2) that

$$\nabla \cdot (\sigma \nabla \Phi_d) = 0 \tag{4}$$

where

$$\Phi_d = 0 \quad \text{on} \quad S'; \qquad -\sigma \nabla \Phi_d = 0 \quad \text{on} \quad S'' \tag{5}$$

Multiplication of (4) by Φ_d and integration over the volume V of interest gives

$$\int_V \Phi_d \nabla \cdot (\sigma \nabla \Phi_d) dv = 0 = \int_V [\nabla \cdot (\Phi_d \sigma \nabla \Phi_d) - \sigma \nabla \Phi_d \cdot \nabla \Phi_d] dv \tag{6}$$

Gauss' theorem converts this expression to

$$\oint_S \Phi_d \sigma \nabla \Phi_d \cdot d\mathbf{a} = \int_V \sigma \nabla \Phi_d \cdot \nabla \Phi_d dv \tag{7}$$

The surface integral can be broken into one on S', where $\Phi_d = 0$ and one on S'', where $\sigma \nabla \Phi_d = 0$. Thus, what is on the left in (7) is zero. If the integrand of what is on the right were finite anywhere, the integral could not be zero, so we conclude that to within a constant, $\Phi_d = 0$ and the steady solution is unique.

For $0 < t$, the steps beginning with (7.8.11) and leading to (7.8.15) apply. Again, the surface integration of (7.8.11) can be broken into two parts, one on S' where $\Phi_d = 0$ and one on S'' where $-\sigma \nabla \Phi_d = 0$. Thus, (7.8.16) and its implications for the uniqueness of the solution apply here as well.

7.9 CHARGE RELAXATION IN UNIFORM AND PIECE-WISE UNIFORM SYSTEMS

7.9.1 (a) In the first configuration, the electric field is postulated to be uniform throughout the gap and therefore the same as though the lossy segment were not present.

$$\mathbf{E} = \mathbf{i}_r v / r ln(a/b) \tag{1}$$

This field is irrotational and solenoidal and integrates to v between $r = b$ and $r = a$. Note that the boundary conditions at the interfaces between the lossy-dielectric and the free space region are automatically met. The tangential electric field (and hence the potential) is indeed continuous and, because there is no normal component of the electric field at these interfaces, (7.9.12) is satisfied as well.

(b) In the second configuration, the field is assumed to take the piece wise form

$$\mathbf{E} = \mathbf{i}_r \mathrm{Re} \frac{1}{r} \begin{Bmatrix} \hat{A} \\ \hat{B} \end{Bmatrix} e^{j\omega t} \qquad \begin{matrix} R < r < a \\ b < r < R \end{matrix} \tag{2}$$

where A and B are determined by the requirements that the applied voltage be consistent with the integration of $\mathbf{E}$ between the electrodes and that (7.9.12) be satisfied at the interface.

$$\hat{A}ln(a/R) + \hat{B}ln(R/b) = \hat{v} \tag{3}$$

$$j\omega\left[\frac{\epsilon_o\hat{A}}{R} - \frac{\epsilon_b\hat{B}}{R}\right] - \frac{\sigma\hat{B}}{R} = 0 \tag{4}$$

It follows that

$$\hat{A} = (j\omega\epsilon_b + \sigma)\hat{v}/\text{Det} \tag{5}$$

$$\hat{B} = j\omega\epsilon_o\hat{v}/\text{Det} \tag{6}$$

where Det is as given and the relations that result from substitution of these coefficients into (2) are those given.

(c) In the first case, the net current to the inner electrode is

$$\hat{i} = j\omega l[(2\pi - \alpha)b\epsilon_o + \alpha b\epsilon]\frac{\hat{v}}{bln(a/b)} + \frac{l\alpha b\sigma\hat{v}}{bln(a/b)} \tag{7}$$

This expression takes the form of the impedance of a resistor in parallel with a capacitor where

$$\hat{i} = \hat{v}G + j\omega C\hat{v} \tag{8}$$

Thus, the C and G are as given in the problem.

In the second case, the equivalent circuit is given by Fig. 7.9.5 which implies that

$$\hat{i} = \frac{\hat{v}(j\omega C_a)(1 + j\omega RC_b)}{1 + j\omega R(C_a + C_b)} \tag{9}$$

In this case, the current to the inner electrode follows from (6) as

$$\hat{i} = \frac{2\pi l\frac{j\omega\epsilon_o}{ln(a/R)}\left(1 + \frac{j\omega\epsilon}{\sigma}\right)}{1 + \frac{j\omega ln(R/b)}{\sigma}\left[\frac{\epsilon_o}{ln(a/R)} + \frac{\epsilon}{ln(r/b)}\right]} \tag{10}$$

Comparison of these last two expressions results in the given parameters.

7.9.2 (a) <u>In the first case</u>, where the interface between materials is conical, the electric field intensity is what it would be in the absence of the material.

$$\mathbf{E} = \frac{v}{r^2}\frac{ab}{(a-b)}\mathbf{i}_r = \text{Re}\,\frac{\hat{v}e^{j\omega t}}{r^2}\frac{ab}{(a-b)}\mathbf{i}_r; \quad \hat{v} \equiv V_o \tag{1}$$

This field is perpendicular to the perfectly conducting electrodes, has a continuous tangential component at the interface and trivially satisfies the condition of charge conservation at the interface.

In the second case, where the interface between materials is spherical, the field takes the form

$$\mathbf{E} = \mathbf{i}_r \mathrm{Re} \begin{cases} \hat{A}/r^2; & R < r < a \\ \hat{B}/r^2; & b < r < R \end{cases} \tag{2}$$

The coefficients are adjusted to satisfy the condition that the integral of $\mathbf{E}$ from $r = b$ to $r = a$ be equal to the voltage,

$$\hat{A}\left(\frac{1}{R} - \frac{1}{a}\right) + \hat{B}\left(\frac{1}{b} - \frac{1}{R}\right) = \hat{v} \tag{3}$$

and conservation of charge at the interface, (7.9.12).

$$-\frac{\sigma}{R^2}\hat{B} + j\omega\left(\epsilon_o \frac{\hat{A}}{R^2} - \epsilon \frac{\hat{B}}{R^2}\right) = 0 \tag{4}$$

Simultaneous solution of these expressions gives

$$\hat{A} = (\sigma + j\omega\epsilon)\hat{v}/\mathrm{Det} \tag{5}$$

$$\hat{B} = j\omega\epsilon_o/\mathrm{Det}$$

where

$$\mathrm{Det} \equiv \sigma\left(\frac{a-R}{aR}\right) + j\omega\left[\epsilon\left(\frac{a-R}{aR}\right) + \epsilon_o\left(\frac{R-b}{bR}\right)\right]$$

which together with (2) give the required field.

(b) In the first case, the inner electrode area subtended by the conical region occupied by the material is $2\pi b^2[1 - \cos(\alpha/2)]$. With the voltage represented as $v = \mathrm{Re}\, \hat{v} \exp(j\omega t)$, the current from the inner spherical electrode, which has the potential v, is

$$\begin{aligned} \hat{i} = {} & 2\pi b^2[1 - \cos(\alpha/2)](\sigma + j\omega\epsilon)\left(\frac{ab}{a-b}\right)\frac{\hat{v}}{b^2} \\ & + 2\pi b^2\left[1 - \cos\left(\frac{2\pi - \alpha}{2}\right)\right] j\omega\epsilon_o\left(\frac{ab}{a-b}\right)\frac{\hat{v}}{b^2} \end{aligned} \tag{6}$$

Equation (6) takes the same form as for the terminal variables of the circuit shown in Fig. S7.9.2a. Thus,

$$\hat{i} = [G + j\omega(C_a + C_b)]\hat{v} \tag{7}$$

$$G = 2\pi[1 - \cos(\alpha/2)]\frac{ab\sigma}{a-b}$$

$$C_a = 2\pi\left[1 - \cos\left(\frac{2\pi - \alpha}{2}\right)\right]\frac{ab\epsilon_o}{a-b} \tag{8}$$

$$C_b = 2\pi[1 - \cos(\alpha/2)]\frac{ab\epsilon}{a-b}$$

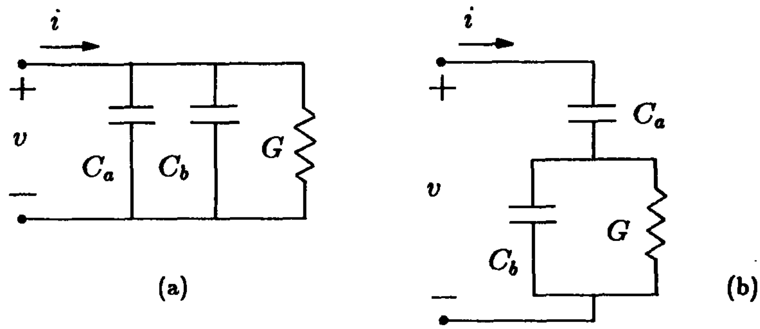


Figure S7.9.2

In the second case, the current from the inner electrode is

$$\hat{i} = 4\pi b^2\left(\frac{\sigma\hat{B}}{b^2} + j\omega\epsilon\frac{\hat{B}}{b^2}\right) \tag{9}$$

$$= \frac{j\omega\left(\frac{4\pi aR\epsilon_o}{a-R}\right)\left(\frac{4\pi Rb\sigma}{R-b} + j\omega\frac{4\pi Rb}{R-b}\epsilon\right)}{\frac{4\pi Rb\sigma}{R-b} - j\omega\left(\frac{4\pi Rb\epsilon}{R-b} + \frac{4\pi aR\epsilon_o}{a-R}\right)} \tag{10}$$

This takes the same form as the relationship between the terminal voltage and current for the circuit shown in Fig. S7.9.2b.

$$\hat{i} = \frac{j\omega C_a(G + j\omega C_b)}{G + j\omega(C_a + C_b)}\hat{v} \tag{11}$$

Thus, the elements in the equivalent circuit are

$$G = 4\pi\frac{Rb\sigma}{R-b}; \quad C_a = \frac{4\pi Rb\epsilon_o}{R-b}; \quad C_b = \frac{4\pi Rb\epsilon}{R-b} \tag{12}$$

7.9.3 In terms of the potential, v, of the electrode, the potential distribution and hence field distribution are

$$\Phi = v(a/r) \Rightarrow \mathbf{E} = \mathbf{i}_r\frac{va}{r^2} \tag{1}$$

The total current into the electrode is then equal to the sum of the rate of increase of the surface charge density on the interface between the electrode and the media and the conduction current from the electrode into the media.

$$i = \int_S \left[\frac{\partial}{\partial t}\epsilon E_r + \sigma E_r\right]da \tag{2}$$

In view of (1), this expression becomes

$$i = 2\pi a^2\left(\frac{\epsilon a}{a^2}\frac{dv}{dt} + \frac{\sigma a}{a^2}v\right) = 2\pi\epsilon a\frac{dv}{dt} + (2\pi\sigma a)v \tag{3}$$

The equivalent parameters are deduced by comparing this expression to one describing the current through a parallel capacitance and resistance.

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7.9.4 (a) With A and B functions of time, the potential is assumed to have the same ϕ dependence as the applied field.

$$\Phi^a = -Er\cos\phi + A\frac{\cos\phi}{r} \tag{1}$$

$$\Phi^b = Br\cos\phi \tag{2}$$

The coefficients are determined by continuity of potential at $r = a$

$$\Phi^a(r = a) = \Phi^b(r = b) \tag{3}$$

and the combination of charge conservation and Gauss' continuity condition, also at $r = a$

$$(\sigma_a E_r^a - \sigma_b E_r^b) + \frac{\partial}{\partial t}(\epsilon_a E_r^a - \epsilon_b E_r^b) = 0 \tag{4}$$

Substitution of (1) and (2) into (3) and (4) gives

$$\frac{A}{a} - Ba = Ea \Rightarrow B = \frac{A}{a^2} - E \tag{5}$$

$$\sigma_a\Big(E - \frac{A}{a^2}\Big) + \sigma_b B + \frac{d}{dt}\Big[\epsilon_a\Big(E + \frac{A}{a^2}\Big) + \epsilon_b B\Big] = 0 \tag{6}$$

and from these relations,

$$(\epsilon_a + \epsilon_b)\frac{dA}{dt} + (\sigma_a + \sigma_b)A = (\sigma_b - \sigma_a)a^2 E + (\epsilon_b - \epsilon_a)a^2\frac{dE}{dt} \tag{7}$$

With E_o the magnitude of a step in $E(t)$, integration of (7) from $t = 0^-$ when $A = 0$ to $t = 0^+$ shows that

$$A(0^+) = \frac{\epsilon_b - \epsilon_a}{\epsilon_b + \epsilon_a}a^2 E_o \tag{8}$$

A particular solution to (7) for $t > 0$ is

$$A = \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)a^2 E_o \tag{9}$$

while a homogeneous solution is $\exp(-t/\tau)$, where

$$\tau \equiv \frac{\epsilon_a + \epsilon_b}{\sigma_a + \sigma_b} \tag{10}$$

Thus, the required solution takes the form

$$A = A_1 e^{-t/\tau} + \left(\frac{\sigma_b - \sigma_a}{\sigma_a + \sigma_b}\right)a^2 E_o$$

where the coefficient A_1 is determined by the initial condition, (8). Thus,

$$A = \left[\frac{(\epsilon_b - \epsilon_a)}{(\epsilon_b + \epsilon_a)} - \frac{(\sigma_b - \sigma_a)}{(\sigma_b + \sigma_a)}\right] a^2 E_o e^{-t/\tau} + \frac{(\sigma_b - \sigma_a)}{(\sigma_b + \sigma_a)} a^2 E_o \tag{12}$$

The coefficient B follows from (5).

$$B = \frac{A}{a^2} - E_o \tag{13}$$

In view of this last relation, and then (12), the unpaired surface charge density is

$$\begin{aligned} \sigma_{su} &= \epsilon_a\left(E_o + \frac{A}{R^2}\right) + \epsilon_b\left(\frac{A}{R^2} - E_o\right) \\ &= \frac{2(\epsilon_a\sigma_b - \epsilon_b\sigma_a)}{\sigma_a + \sigma_b} E_o(1 - e^{-t/\tau}) \end{aligned} \tag{14}$$

(b) In the sinusoidal steady state, the drive in (7) takes the form Re $\hat{E}\exp(j\omega t)$ and the resonse is of the form Re $\hat{A}\exp(j\omega t)$. Thus, (7) shows that

$$\hat{A} = \frac{[(\sigma_b - \sigma_a) + j\omega(\epsilon_b - \epsilon_a)]}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} a^2 \hat{E}_p \tag{15}$$

and in turn, from (5),

$$\hat{B} = -\frac{2(\sigma_a + j\omega\epsilon_a)}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} \tag{16}$$

This expressions can then be used to show that the complex amplitude of the unpaired surface charge density is

$$\hat{\sigma}_{su} = \frac{2(\sigma_b\epsilon_a - \sigma_a\epsilon_b)}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} \tag{17}$$

(c) From (1) and (2) it is clear that the plane $\phi = \pi/2$ is one of zero potential, regardless of the values of the drive $E(t)$ or of A or B. Thus, the $x = 0$ plane can be replaced by a perfect conductor. In the limit where $\sigma_a \to 0$ and $\omega(\epsilon_a + \epsilon_b)/\sigma_b \ll 1$, (15) and (16) become

$$\hat{A} \to a^2 E_p \tag{18}$$

$$\hat{B} \to -\frac{2j\omega\epsilon_a}{\sigma_b}\hat{E}_p \tag{19}$$

Substitution of these coefficients into the sinusoidal steady state versions of (1) and (2) gives

$$\Phi^a = -\text{Re } \hat{E}_p a\left[\frac{r}{a} - \frac{a}{r}\right]\cos\phi e^{j\omega t} \tag{20}$$

$$\Phi^b = -\text{Re } \frac{2j\omega\epsilon_a}{\sigma_b}\hat{E}_p e^{j\omega t} \tag{21}$$

These are the potentials that would be obtained under sinusoidal steady state conditions using (a) and (b) of Prob. 7.9.5.

7.9.5 (a) This is an example of an "inside-outside" situation. The "inside" region is the one where the excitation is applied, namely region (a). In so far as the field in the exterior region is concerned, the surface is essentially an equipotential. Thus, the solution given by (a) must be constant at $r = a$ (it is zero), must become the uniform applied field at infinity (which it does) and must be comprised of solutions to Laplace's equation (which certainly the uniform and dipole fields are).

(b) To approximate the interior field, note that in general charge conservation and Gauss' law (7.9.12) require that

$$\frac{\partial}{\partial t}(\epsilon_o E_r^a - \epsilon E_r^b) - \sigma E_r^b = 0 \tag{1}$$

So long as the interior field is much less than that applied, this expression can be approximated by

$$\sigma E_r^b = \frac{\partial \epsilon_o E_r^a}{\partial t} = \epsilon_o 2 \cos\phi \frac{dE}{dt} \tag{2}$$

which, in view of (a), is a prescription for the normal conduction current density inside the cylinder. This is then the boundary condition on the potential in region (b), the interior of the cylinder, and it follows that the potential within is

$$\Phi^b = Ar\cos\phi = -\frac{2\epsilon_o}{\sigma} r\cos\phi \frac{dE}{dt} \tag{3}$$

Note that the approximation made in going from (1) to (2) is valid if

$$\epsilon_o E_r^a \gg \epsilon E_r^b \Rightarrow \epsilon_o 2\cos\theta E \gg \frac{\epsilon}{\sigma}\epsilon_o 2\cos\phi \frac{dE}{dt} \tag{4}$$

Thus, if $E(t) = E_o \cos\omega t$, the approximation is valid provided

$$1 \gg \frac{\omega\epsilon}{\sigma} \tag{5}$$

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7.9.6 (a) Just after the step, there has been no time for the relaxation of unpaired charge, so the system is still behaving as if the conductivity were zero. In any case, piece-wise solutions to Laplace's equation, having the same θ dependence as the dipole potential and having the dipole potential in the neighborhood of the origin are

$$\Phi^a = A\frac{\cos\theta}{r^2} \tag{1}$$

$$\Phi^b = \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} + Br\cos\theta \tag{2}$$

At $r = a$,

$$\Phi^a = \Phi^b \tag{3}$$

$$-\epsilon\frac{\partial\Phi^a}{\partial r} = -\epsilon_o\frac{\partial\Phi^b}{\partial r} \tag{4}$$

Substitution of (1) and (2) into these relations gives

$$\begin{bmatrix} \frac{1}{a^2} & -a \\ \frac{2\epsilon}{a^3} & \epsilon_o \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{P}{4\pi\epsilon_o a^3}\begin{bmatrix} a \\ 1 \end{bmatrix} \tag{5}$$

Thus, the desired potentials are (1) and (2) evaluated using A and B found from (5) to be

$$A = \frac{3p}{4\pi(\epsilon_o + 2\epsilon)} \tag{6}$$

$$B = \frac{2(\epsilon_o - \epsilon)p}{4\pi\epsilon_o a^3(\epsilon_o + 2\epsilon)} \tag{7}$$

(b) After a long time, charge relaxes to the interface to render it an equipotential. Thus, the field outside is zero and that inside is determined by making B in (2) satisfy the condition that $\Phi^b(r = a) = 0$.

$$A = 0 \tag{8}$$

$$B = -\frac{p}{4\pi\epsilon_o a^3} \tag{9}$$

(c) In the general case, (4) is replaced by

$$\sigma\frac{\partial\Phi^a}{\partial r} + \frac{\partial}{\partial t}(\epsilon\frac{\partial\Phi^a}{\partial r} - \epsilon_o\frac{\partial\Phi^b}{\partial r}) = 0 \tag{10}$$

and substitution of (1) and (2) gives

$$2\sigma A + \frac{d}{dt}\left[2\epsilon A + \epsilon_o(\frac{-2p}{4\pi\epsilon_o} + Ba^3)\right] = 0 \tag{11}$$

With B replaced using (5a),

$$\frac{dA}{dt} + \frac{A}{\tau} = \frac{3}{4\pi(2\epsilon + \epsilon_o)}\frac{dp}{dt}; \quad \tau \equiv \frac{2\epsilon + \epsilon_o}{2\sigma} \tag{12}$$

With p a step function, integration of this expression from $t = 0^-$ to $t = 0^+$ gives

$$A(0^+) = \frac{2p_o}{4\pi(2\epsilon + \epsilon_o)} \tag{13}$$

It follows that

$$A = \frac{3p_o}{4\pi(2\epsilon + \epsilon_o)}e^{-t/\tau} \tag{14}$$

and in turn that

$$B = \frac{p_o}{4\pi a^3}\left(\frac{3e^{-t/\tau}}{2\epsilon + \epsilon_o} - \frac{1}{\epsilon_o}\right) \tag{15}$$

As $t \to 0$, these expressions become (6) and (7) while as $t \to \infty$, they are consistent with (8) and (9).

7.9.7 (a) This is an "inside-outside" situation where the layer of conductor is the "inside" region. The potential is constrained at the lower surface by the electrodes and the y derivative of the potential must be zero at the upper surface. This potential follows as

$$\Phi^b = V\frac{\cosh\beta y}{\cosh\beta d}\cos\beta x \tag{1}$$

The potential must be continuous at the upper interface, where it follows from (1) with $y = 0$ that it is

$$\Phi^a(y=0) = V\frac{\cos\beta x}{\cosh\beta d} \tag{2}$$

The potential that matches this condition in the plane $y = 0$ and goes to zero as y goes to infinity is

$$\Phi^a = V\frac{\cos\beta x}{\cosh\beta d}e^{-\beta y} \tag{3}$$

Thus, before $t = 0$, the surface charge density is

$$\sigma_{su} = -\left[\epsilon_o\frac{\partial\Phi^a}{\partial y} - \epsilon\frac{\partial\Phi^b}{\partial y}\right]_{y=0} = \frac{\epsilon_o\beta V\cos\beta x}{\cosh\beta d} \tag{4}$$

(b) Once the potential imposed by the lower electrodes is zero, the potentials in the respective regions take the form

$$\Phi^a = Ae^{-\beta y}\cos\beta x \tag{5a}$$

$$\Phi^b = A\frac{\sinh\beta(y+d)}{\sinh\beta d}\cos\beta x \tag{5b}$$

Here, the coefficients have been adjusted so that the potential is continuous at $y = 0$. The remaining condition to be satisfied at this interface is (7.9.12).

$$\frac{\partial}{\partial t}(\epsilon_o E_y^a - \epsilon E_y^b) - \sigma E_y^b = 0 \tag{6}$$

Substitution from (5) shows that

$$\frac{\partial}{\partial t}[(\epsilon_o\beta + \epsilon\coth\beta d)\cos\phi A] + \sigma\beta\coth\beta d\cos\phi A = 0 \tag{7}$$

The term inside the time derivative is the surface charge density. Thus, (7) can be converted to a differential equation for the surface charge density

$$\frac{d\sigma_{su}}{dt} + \frac{\sigma_{su}}{\tau} = 0 \tag{8}$$

where

$$\tau = (\epsilon_o\tanh\beta d + \epsilon)/\sigma$$

Thus, given the initial condition from part (a), the surface charge density is

$$\sigma_{su} = \frac{\epsilon_o\beta V\cos\beta x}{\cosh\beta d}e^{-t/\tau} \tag{9}$$

7.9.8 (a) Just after Q has been turned on, there is still no surface charge on the interface. Thus, when $t = 0^+$,

$$\Phi^a(y = 0) = \Phi^b(y = 0) \tag{1}$$

$$\epsilon_o \frac{\partial \Phi^a}{\partial y}(y = 0) = \epsilon \frac{\partial \Phi^b}{\partial y}(y = 0) \tag{2}$$

It follows from the postulated solutions that

$$Q - q_b = q_a \tag{3}$$

$$-\epsilon_o Q - \epsilon_o q_b = -\epsilon q_a \tag{4}$$

and finally that $q_a(0^+)$ and $q_b(0^+)$ have the given values.

(b) As $t \to \infty$, the interface becomes an equipotential. It follows from the postulated solution evaluated at the interface, where the potential must be what it is at infinity, namely zero, that

$$q_b = Q \tag{5}$$

(c) Throughout the transient, (1) must hold. However, the condition of (2) is generalized to represent the buildup of the surface charge density, (7.9.12). At $y = 0$

$$\frac{\partial}{\partial t}\left[-\epsilon_o \frac{\partial \Phi^a}{\partial y} + \epsilon \frac{\partial \Phi^b}{\partial y}\right] + \sigma \frac{\partial \Phi^b}{\partial y} = 0 \tag{6}$$

When $t > 0$, Q is a constant. Thus, evaluation of (6) with the postulated solutions gives

$$\epsilon_o \frac{dq_b}{dt} - \epsilon \frac{dq_a}{dt} - \sigma q_a = 0 \tag{7}$$

Using (1) to eliminate q_b, this expression becomes

$$\frac{dq_a}{dt} + \frac{q_a}{\tau} = 0 \tag{8}$$

where $\tau = (\epsilon_o + \epsilon)/\sigma$. The solution to this expression is $A\exp(-t/\tau)$, where A is the initial value found in part (a). The other image charge, q_b, is then given by using (1).

7.9.9 (a) As $t \to \infty$, the surface at $x = 0$ requires that there be no normal current density and hence electric field intensity on the (b) side. Thus, all boundary conditions in region (b) and Laplace's equation are satisfied in region (b) by a uniform electric field and a linear potential.

$$\mathbf{E}^b = \frac{v}{a}\mathbf{i_y}; \quad \Phi^b = \frac{v}{a}(a - y) \tag{1}$$

The field in region (a) can then be found. It has a potential that is zero on three of the four boundaries. On the fourth, where $x = 0$, the potential must be the same as given by (1)

$$\Phi^a(x=0) = \Phi^b(x=0) = \frac{v}{a}(a-y) \tag{2}$$

To match these boundary conditions, we take the solution to Laplace's equation to be an infinite sum of modes that satisfy the first three boundary conditions.

$$\Phi^a = -\sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi}{a}(x-b)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right) \tag{3}$$

The coefficients are determined by requiring that this sum satisfy the last boundary condition at $x = 0$.

$$\frac{v}{a}(a-y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right) \tag{4}$$

Multiplication by $\sin(m\pi y/a)$ and integration from $x = 0$ to $x = a$ gives

$$A_m = \frac{2}{a}\int_0^a \frac{v}{a}(a-y)\sin\left(\frac{m\pi y}{a}\right)dy = \frac{2v}{m\pi} \tag{5}$$

Thus, it follows that the potential in region (a) is

$$\Phi^a = -\sum_{n=1}^{\infty} \frac{2v}{n\pi} \frac{\sinh \frac{n\pi}{a}(x-b)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right) \tag{6}$$

(b) During the transient, the two regions are coupled by the temporal and spatial evoluation of unpaired charge at the interface, where $x = 0$. So, in region (b) we add to the asymptotic solution, which satisfies the conditions on the potential at $y = 0, y = a$ and as $x \to -\infty$, one that term-by-term is zero on these boundaries and as $x \to -\infty$ and that term-by-term satisfies Laplace's equation.

$$\Phi^b = \frac{v}{a}(a-y) + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \tag{7}$$

The result of (4)-(5) shows that the first term on the right can just as well be represented by the same Fourier series for its y dependence as the last term.

$$\Phi^b = \sum_{n=1}^{\infty} \frac{2v}{n\pi} \sin\left(\frac{n\pi y}{a}\right) + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \tag{8}$$

The potential in region (a) can generally take the form of (3). There remains finding $A_n(t)$ and $B_n(t)$ such that the continuity conditions at $x = 0$ on the potential and representing Gauss plus charge conservation are met. Evaluation of (3) and (8) at $x = 0$ shows that the potential continuity condition can be satisfied term-by-term if

$$A_n = \frac{2v}{n\pi} + B_n \tag{9}$$

The second condition brings in the dynamics, (7.9.12) at $x = 0$,

$$\left[\sigma_a \frac{\partial \Phi^a}{\partial x} - \sigma_b \frac{\partial \Phi^b}{\partial x}\right] + \frac{\partial}{\partial t}\left(\epsilon_a \frac{\partial \Phi^a}{\partial x} - \epsilon_b \frac{\partial \Phi^b}{\partial x}\right) = 0 \tag{10}$$

Substitution from (3) and (8) gives an expression that can also be satisfied term-by-term if

$$\begin{aligned}-\sigma_a \frac{n\pi}{a}\coth\left(\frac{n\pi b}{a}\right)A_n - \sigma_b \frac{n\pi}{a} B_n - \epsilon_a \frac{n\pi}{a}\coth\left(\frac{n\pi b}{a}\right)\frac{dA_n}{dt}\\ - \epsilon_b \frac{n\pi}{b}\frac{dB_n}{dt} = 0\end{aligned} \tag{11}$$

Substitution for B_n from (9) then gives one expression that describes the temporal evolution of $A(t)$.

$$\frac{dA_n}{dt} + \frac{A_n}{\tau} = \frac{2}{n\pi}\left(\epsilon_b \frac{dv}{dt} + \sigma_b v\right) / \left(\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b\right) \tag{12}$$

where

$$\tau \equiv \frac{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}{\sigma_a \coth\left(\frac{n\pi b}{a}\right) + \sigma_b}$$

To find the response to a step, the volue of A_n when $t = 0$ is found by integrating (12) from $t = 0^-$ when $A_n = 0$ to $t = 0^+$.

$$A_n(0^+) = \epsilon_b V_o\left(\frac{2}{n\pi}\right) / \left(\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b\right) \tag{13}$$

The solution to (12), which takes the form of a homogeneous solution $\exp(-t/\tau)$ and a constant particular solution, must then satisfy this initial condition.

$$A_n = A_{n1} e^{-t/\tau} + \sigma_b\left(\frac{2}{n\pi}\right)\frac{V_o}{\left(\sigma_a \coth\frac{n\pi b}{a} + \sigma_b\right)} \tag{14}$$

The coefficient of the homogeneous term is adjusted to satisfy (13), and (14) becomes

$$\begin{aligned}A_n = V_o\left(\frac{2}{n\pi}\right)\Bigg\{\left[\frac{\sigma_b}{\sigma_a \coth\left(\frac{n\pi b}{a}\right) + \sigma_b} - \frac{\epsilon_b}{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}\right]\left(1 - e^{-t/\tau}\right)\\ + \frac{\epsilon_b}{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}\Bigg\}\end{aligned} \tag{15}$$

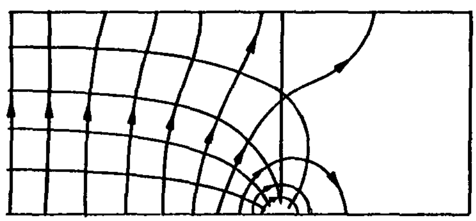
There is some insight gained by writing this expression in the alternative form

$$A_n = V_o\Big(\frac{2}{n\pi}\Big)\Bigg[\frac{(\epsilon_a\sigma_b - \epsilon_b\sigma_a)\cosh\big(\frac{n\pi b}{2}\big)\big(1 - e^{-t/\tau}\big)}{\sigma_a \coth\big(\frac{n\pi b}{a}\big) + \sigma_b} + \epsilon_b\Bigg]\Big/\Big(\epsilon_a \coth\frac{n\pi b}{a} + \epsilon_b\Big) \tag{16}$$

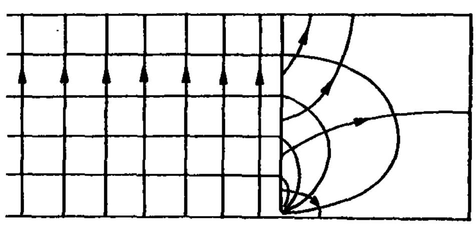
Given this expression for A_n, B_n follows from (9). In this specific situation these expressions are satisfied with $\sigma_a = 0$ and $\epsilon_a = \epsilon_o$.

(c) In this limit, it follows from (15) that as $t \to \infty$, $A_n \to V_o(2/n\pi)$ and this is consistent with what was found for this limit in part (a), (5).

With the permittivities equal, the potential and field distributions just after the potential has been turned on and therefore as there has been no time for unpaired charge to accumulate at the interface, is as shown in Fig. S7.9.9a. To make this sketch, note that far to the left, the equipotentials are equally spaced straight lines (surfaces) running parallel to the boundaries, which are themselves equipotentials. All of these must terminate in the gap at the origin. In the neighborhood of that gap, the potential has the form familiar from Fig. 5.7.2 (except that the equipotential $\Phi = V$ is at $\phi = \pi$ and not at $\phi = 2\pi$).



(a)



(b)

Figure S7.9.9

In the limit where $t \to \infty$, the uniform equipotentials in region (b) extend up to the interface. Just as we could solve for the field in region (b) and then for that in region (a), we can also draw the fields in that order. In region (a), the potential is linear in y in the plane $x = 0$ and zero on the other two boundaries. Thus, the equipotentials that originate on the boundary at

$x = 0$ at equal distances, must terminate in the gap, where they converge like equally spaced spokes on the hub of a wheel.

The transient that we have described takes the field distribution from that of Fig. S7.9.9a, where there is a conduction current normal to the interface from the (b) region side supplying surface charge to the interface, to that of Fig. S7.9.9b, where the current density normal to the surface has subsided because charges on the interface have created just that field necessary to null the normal field in region (b).

SOLUTIONS TO CHAPTER 8

8.1 THE VECTOR POTENTIAL AND THE VECTOR POISSON EQUATION

8.1.1 (a) Ampère's differential law inside the solenoid gives

$$\nabla \times \mathbf{H} = 0 \tag{1}$$

The continuity law of magnetic flux gives

$$\nabla \cdot \mu_o \mathbf{H} = 0 \tag{2}$$

Therefore, $\mathbf{H}$ is the gradient of a Laplacian potential. A uniform field is, of course, one special case of such a field. At the boundary, representing the coil as a surface current

$$\mathbf{K} = \mathbf{i}_\phi \frac{Ni}{d} \tag{3}$$

we have

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \tag{4}$$

where $\mathbf{n} = -\mathbf{i_r}$, the outside region is (b). Further we have

$$\mathbf{n} \cdot \mu_o (\mathbf{H}^a - \mathbf{H}^b) = 0 \tag{5}$$

(b) An axial z-directed uniform field inside, zero field outside, automatically satisfies (1), (2) and (5). On the surface we get from (4)

$$-\mathbf{i_r} \times H_z^a \mathbf{i_z} = \mathbf{i}_\phi \frac{Ni}{d}$$

and since

$$\mathbf{i_r} \times \mathbf{i_z} = -\mathbf{i}_\phi$$

$$H_z^a = \frac{Ni}{d}$$

(c) $\mathbf{A}$ is ϕ directed by symmetry. From the integral form of $\nabla \times \mathbf{A} = \mu_o \mathbf{H}$ we obtain

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S \mu_o \mathbf{H} \cdot d\mathbf{a}$$

Taking a radius r we find

$$2\pi r A_\phi(r) = \begin{cases} \pi r^2 \mu_o H_z^a & \text{for } r < a \\ \pi a^2 \mu_o H_z^a & \text{for } r > a \end{cases}$$

Therefore

$$A_\phi = \begin{cases} \frac{r}{2} \mu_o \frac{Ni}{d} & \text{for } r < a \\ \frac{a^2}{2r} \mu_o \frac{Ni}{d} & \text{for } r > a \end{cases}$$

8.1.2 Using the coordinates defined in Fig. P4.4.3, superposition of line current vector potentials (8.1.16) gives

$$A_z = -\frac{\mu_o i}{2\pi} ln\left[\frac{r_1 r_3}{r_2 r_4}\right] \tag{1}$$

where

$$r_1 = \sqrt{x^2 + (y - \frac{d}{2})^2}; \quad r_3 = \sqrt{x^2 + (y + \frac{d}{2})^2}$$

$$r_2 = \sqrt{(x + \frac{d}{2})^2 + y^2}; \quad r_4 = \sqrt{(x - \frac{d}{2})^2 + y^2}$$

To linear terms in $(d/2)^2$, the numerator of this expression is

$$r_1 r_3 = \sqrt{(x^2 + y^2) + 2(x^2 - y^2)(d/2)^2} = r^2\sqrt{1 + \frac{2(x^2 - y^2)}{r^2}\left(\frac{d}{2r}\right)^2} \tag{2}$$

where

$$r = \sqrt{x^2 + y^2}$$

Similarly, the denominator is

$$r_2 r_4 = r^2\sqrt{1 - \frac{2(x^2 - y^2)}{r^2}\left(\frac{d}{2r}\right)^2} \tag{3}$$

Thus, to linear terms in $(d/2r)^2$, (1) becomes

$$\begin{aligned} A_z &\sim -\frac{\mu_o}{2\pi} i \frac{1}{2} ln\left[\frac{1 + 2\frac{(x^2-y^2)}{r^2}\left(\frac{d}{2r}\right)^2}{1 - 2\frac{(x^2-y^2)}{r^2}\left(\frac{d}{2r}\right)^2}\right] \\ &\simeq -\frac{\mu_o i}{4\pi} ln\left[1 + \frac{4(x^2 - y^2)}{r^2}\left(\frac{d}{2r}\right)^2\right] \end{aligned} \tag{4}$$

Observe that

$$\frac{x}{r} = \cos\phi; \quad \frac{y}{r} = \sin\phi; \quad \frac{x^2 - y^2}{r^2} = \cos^2\phi - \sin^2\phi = \cos 2\phi \tag{5}$$

and it follows that (4) is the given vector potential.

8.1.3 We can take advantage of the analog of a solution of Poisson's equation for a two dimensional charge problem, and for a two dimensional current problem (because the structure is long, $l \gg w$ and $l \gg d$ we treat it as two dimensional). The analog charge problem is one with two charge sheets of opposite signs, producing a uniform field, and a potential $\Phi \propto y$. Thus (see Fig. S8.1.3)

$$A_z = A_o(y/d)$$

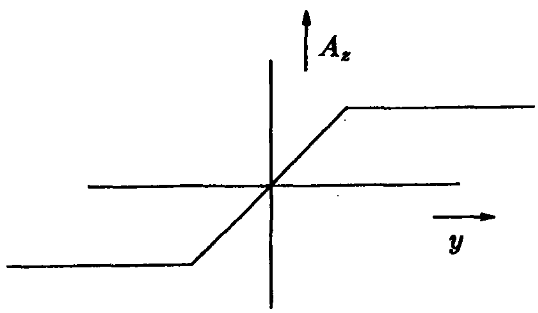


Figure S8.1.3

inside, $A_z =$ const outside, and we adjust A so that we get the proper discontinuity of $\partial A_z/\partial y$ to account for the discontinuity of H_x

$$\mu_o H_x = \frac{\partial A_z}{\partial y} = \mu_o K = \mu_o \frac{Ni}{w}$$

Therefore

$$\frac{A_o}{d} = \mu_o \frac{Ni}{w}$$

and

$$\begin{aligned} A_z &= \mu_o \frac{Ni}{w} y \qquad \text{inside} \\ &= \pm \mu_o \frac{Ndi}{2w} \begin{cases} \text{top} \\ \text{bottom} \end{cases} \end{aligned}$$

8.2 THE BIOT-SAVART SUPERPOSITION INTEGRAL

8.2.1 The Biot-Savart integral, (7), is evaluated recognizing that

$$(\mathbf{i}_\phi \times \mathbf{i}_{r'r})_z = \frac{r}{\sqrt{z^2 + r^2}} \tag{1}$$

Thus,

$$H_z = \frac{J_o}{4\pi}\int_0^{\Delta}\int_0^{2\pi}\int_b^a \frac{r}{\sqrt{z^2+r^2}}\frac{dzrd\phi dr}{(z^2+r^2)} \tag{2}$$

The integration on z amounts to a multiplication by Δ while that on ϕ is simply a multiplication by 2π. Thus, (2) becomes

$$H_z = \frac{\Delta J_o}{2}\int_b^a \frac{r^2 dr}{(z^2+r^2)^{3/2}} \tag{3}$$

and integration gives

$$H_z = \frac{\Delta J_o}{2}\left[\frac{-r}{\sqrt{r^2+z^2}} + ln(r+\sqrt{r^2+z^2})\right]_b^a \tag{4}$$

which is the given result.

8.2.2 We use the Biot-Savart law,

$$\mathbf{H} = \frac{i}{4\pi}\oint \frac{d\mathbf{s}\times \mathbf{i}_{r'r}}{|\mathbf{r}-\mathbf{r}'|^2} \tag{1}$$

The field due to the turns within the width $Rd\theta$, and length $\sin\theta Rd\phi$ which produce a differential current $ids = K_o\sin^2\theta R^2 d\theta d\phi$, is (Note: $\mathbf{i}_{r'r} = -\mathbf{i}_{r}$.)

$$dH_\theta = -\frac{K_o\sin^2\theta R^2 d\theta}{4\pi R^2}d\phi \tag{2}$$

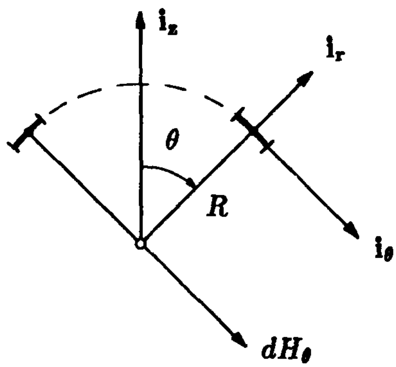


Figure S8.2.2

The field along the axis adds as one integrates around one turn, the components normal to the axis cancel

$$dH_z = -\sin\theta\int_0^{2\pi} dH_\theta = \frac{K_o d\theta}{2}\sin^3\theta \tag{3}$$

The total field is obtained by adding over all the currents

$$H_z = \frac{K_o}{2}\int_{\theta=0}^{\pi}\sin^3\theta d\theta = \frac{2K_o}{3} \tag{4}$$

8.2.3 We replace $K_o \sin\theta$ by K_o in Prob. 8.2.2. We can start with the integral in (4), where we drop one factor of $\sin\theta$. We get

$$H_z = \frac{K_o}{2} \int_{\theta=0}^{\pi} \sin^2\theta d\theta = \frac{\pi K_o}{4}$$

8.2.4 We can use the result of 8.2.3 for a single shell. The total current distribution can be thought of as produced by a concentric set of shells. Each shell produces the field $\frac{\pi}{4} J_o dR$. Thus the net field at the center is

$$H_z = \frac{\pi}{4} J_o \int_0^a dR = \frac{\pi}{4} J_o R$$

8.2.5 No matter where the vertices of the loop, (8.2.22) can be used to determine the field. However, the algebra is simplified by recognizing that the triangle not only has sides of equal length, d, but that the z axis is at the center of the triangle. Thus, each leg makes the same contribution to the z component of the field along the z axis, and along that axis the x and y components cancel. To see that the sides are of length equal to that of the one paralleling the x axis, note that the distance from the center of the leg to the vertex on the y axis is $\sqrt{3/4}d$ and that based on the base $d/2$ and this distance, either of the other leg lengths must be of length $\sqrt{(d/2)^2 + (\sqrt{3/4}d)^2} = d$. Further, if the z axis is at the center of the triangle, then the distance from the origin to either of the legs not parallel to the x axis must be the distance to the parallel leg, $\sqrt{3/4}d/3$. Thus, we should have $2\sqrt{3/4}d/3 = \sqrt{(d/2)^2 + (\sqrt{3/4}d/3)^2}$, as indeed we do.

For the leg parallel to the x axis,

$$\begin{aligned}
\mathbf{a} &= d\mathbf{i_x} \\
\mathbf{b} &= -\frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}} d\mathbf{i_y} - z\mathbf{i_z} \\
\mathbf{c} &= \frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}} d\mathbf{i_y} - z\mathbf{i_z}
\end{aligned} \tag{1}$$

Thus,

$$\begin{aligned}
\mathbf{c} \times \mathbf{a} &= -zd\mathbf{i_y} + \frac{1}{3}\sqrt{\frac{3}{4}} d^2\mathbf{i_z} \\
&\Rightarrow |\mathbf{c} \times \mathbf{a}| = d\Big(\frac{1}{12}d^2 + z^2\Big)^{1/2} \\
\mathbf{a} \cdot \mathbf{c} &= d^2/2 \quad |c| = (d^2/3 + z^2)^{1/2} \\
\mathbf{a} \cdot \mathbf{b} &= -d^2/2 \quad |b| = |c|
\end{aligned} \tag{2}$$

and the given result follows from (8.2.22), multiplied by 3 to reflect the contributions from the other two legs. This same result is obtained using either of the other legs. For example, using the back leg,

$$\begin{aligned} \mathbf{a} &= -\frac{d}{2}\mathbf{i_x} - \sqrt{\frac{3}{4}}d\mathbf{i_y} \\ \mathbf{b} &= \frac{2}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z} \\ \mathbf{c} &= -\frac{d}{2}\mathbf{i_x} - \frac{1}{3}\sqrt{\frac{3}{4}}d\mathbf{i_y} - z\mathbf{i_z} \end{aligned} \tag{3}$$

8.2.6 From (8.2.22)

$$\mathbf{H} = \frac{i}{4\pi}\frac{\mathbf{c}\times\mathbf{a}}{|\mathbf{c}\times\mathbf{a}|^2}\left(\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{c}|} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)$$

we can find the H-field produced by a current stick! We look at one stick in the bottom layer of wires, extending from the position vector

$$\mathbf{b} = (x' - x)\mathbf{i_x} - \frac{d}{2}y\mathbf{i_y} - \frac{l}{2}\mathbf{i_z}$$

to the position vector

$$\mathbf{c} = (x' - x)\mathbf{i_x} - \frac{d}{2}y\mathbf{i_y} + \frac{l}{2}\mathbf{i_z}$$

with

$$\mathbf{a} \equiv \mathbf{c} - \mathbf{b} = l\mathbf{i_z}$$

Thus

$$\mathbf{c}\times\mathbf{a} = l[(x' - x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]$$

$$|\mathbf{c}\times\mathbf{a}|^2 = l^2[(x' - x)^2 + (d/2)^2]$$

$$|\mathbf{b}| = |\mathbf{c}| = \sqrt{(x' - x)^2 + (d/2)^2 + (l/2)^2}$$

$$\mathbf{a}\cdot\mathbf{c} = \frac{l^2}{2} \qquad \mathbf{a}\cdot\mathbf{b} = -\frac{l^2}{2}$$

Therefore, $\mathbf{H}$ due to one stick, carrying the differential current $\frac{Ni}{w}dx'$ is

$$\begin{aligned} \mathbf{H} &= \frac{Nidx'}{4\pi w}l\frac{[(x' - x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]}{l^2[(x' - x)^2 + (d/2)^2]}\frac{l^2}{\sqrt{(x' - x) + (d/2)^2 + (l/2)^2}} \\ &\simeq \frac{Nidx'}{2\pi w}\frac{[(x' - x)\mathbf{i_y} + \frac{d}{2}\mathbf{i_x}]}{(x' - x)^2 + (d/2)^2} \end{aligned}$$

in the limit when l is very long compared with d and w. This very same result could have been obtained from Ampère's law and symmetry considerations for an infinitely long wire (see Fig. S8.2.6)

$$\mathbf{H} = -\frac{Nidx'}{w}\frac{1}{2\pi r}\mathbf{i}_\phi = \frac{Nidx}{2\pi w}\frac{(x'-x)\mathbf{i}_\mathbf{y} + \frac{d}{2}\mathbf{i}_\mathbf{x}}{(x'-x)^2 + (d/2)^2}$$

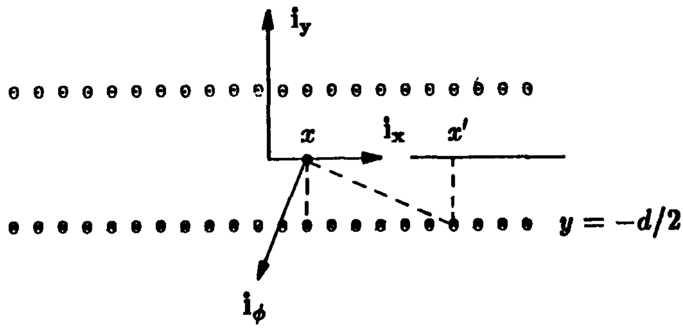


Figure S8.2.6

The total field is obtained by adding the contribution from a symmetrically located set of wires at the top, which cancels the y-component and doubles the x-component, and by integrating over the length of the coil

$$\begin{aligned} H_x &= \int_{-w/2}^{w/2} \frac{Nidx'}{2\pi w}\frac{d}{(x'-x)^2 + (d/2)^2} \\ &= \frac{Ni}{\pi w}\tan^{-1}\left[\frac{2}{d}\left(\frac{w}{2} - x\right)\right] + \tan^{-1}\left[\frac{2}{d}\left(\frac{w}{2} + x\right)\right] \end{aligned}$$

since

$$\int \frac{dx}{x^2 + (d/2)^2} = \frac{2}{d}\tan^{-1}(2x/d)$$

We may test this result by having $w \to \infty$. Then

$$H_x = \frac{Ni}{w}$$

QED as is correct for sheets of an infinite set.

8.2.7 From (8.1.8) integrated over the cross-section of the stick,

$$\mathbf{A} = \frac{\mu_o}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')dv'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_o}{4\pi}i\int_{\xi_b}^{\xi_c} \frac{\mathbf{a}}{|a|}\frac{d\xi}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

where $\mathbf{a}/|a|$ is a unit vector in the direction of the stick and hence $[\mathbf{a}/|a|]d\xi$ is a differential length along the stick. Using the expression for $|\mathbf{r}-\mathbf{r}'|$ following (8.2.17), (1) is converted to an expression ready for integration.

$$\mathbf{A} = \frac{\mu_o}{4\pi}i\frac{\mathbf{a}}{|a|}\int_{\xi_b}^{\xi_c} \frac{d\xi}{\sqrt{\xi^2 + r_o^2}} \tag{2}$$

Integration gives

$$\mathbf{A} = \frac{\mu_o}{4\pi} i \frac{\mathbf{a}}{|a|} ln \left[\frac{\xi_c + \sqrt{\xi_c^2 + r_o^2}}{\xi_b + \sqrt{\xi_b^2 + r_o^2}} \right] \tag{3}$$

Finally, substitution from (8.2.21) makes this expression the given result.

8.3 THE SCALAR MAGNETIC POTENTIAL

8.3.1 From the Biot-Savart law

$$\mathbf{H} = \frac{i}{4\pi} \oint \frac{d\mathbf{s}' \times \mathbf{i}_{\mathbf{r}'\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|^2}$$

we find the axial field H_z

$$\begin{aligned} H_z &= \frac{i}{4\pi} \int_0^{2\pi} \frac{R d\phi'}{R^2 / \sin^2\theta} \sin\theta = \frac{i}{4\pi} \frac{2\pi}{R} \sin^3\theta \\ &= \frac{i}{2R} \frac{R^3}{\sqrt{R^2 + z^2}^3} \end{aligned}$$

For large z,

$$H_z \simeq 2 \frac{i\pi R^2}{4\pi z^3}$$

which is consistent with the axial field of a dipole (see Fig. S8.3.1).

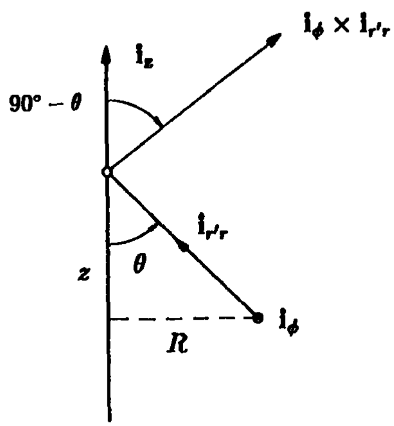


Figure S8.3.1

8.3.2 The potential of one wire carrying the current i in the $+z$ direction is

$$\psi = -\frac{i}{2\pi}\phi \tag{1}$$

The superposition gives (Fig. S8.3.2)

$$\Psi = -\frac{i}{2\pi}(\phi_1 - \phi_2) \tag{2}$$

The lines $\Psi =$ const are described by

$$\tan(\phi_1 - \phi_2) = \text{const} = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2} = \frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{x^2 - a^2}}$$

Therefore

$$2ya = \text{const}[x^2 - a^2 + y^2]$$

This is the equation of circles that go through the points $x = \pm a, y = 0$.

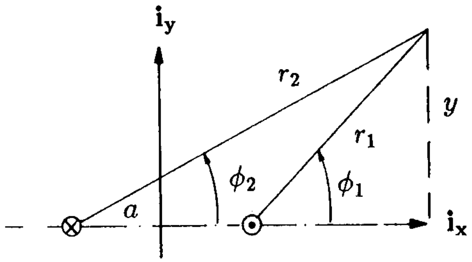


Figure S8.3.2

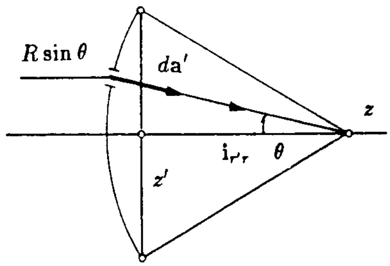


Figure S8.3.3

8.3.3 Assume that the coil extends from $z = -l/2$ to $z = +l/2$. The potential of a loop is

$$\Psi(\mathbf{r}) = \frac{i}{4\pi}\Omega$$

$$\Omega = \int_0^\theta \frac{2\pi R\sin\theta R d\theta}{R^2} = 2\pi(1 - \cos\theta) = 2\pi\left(1 - \frac{(z - z')}{\sqrt{(z - z')^2 + R^2}}\right)$$

The individual differential loops of length dz' carry currents $\frac{Ni}{l}dz'$. Therefore the total potential is

$$\begin{aligned}\Psi(x) &= \frac{Ni}{2l}\int_{z'=-l/2}^{z'=l/2} dz'\left(1 - \frac{(z - z')}{\sqrt{(z - z')^2 + R^2}}\right) \\ &= \frac{Ni}{2l}\left[l + \sqrt{(\frac{l}{2} - z)^2 + R^2} - \sqrt{(\frac{l}{2} + z)^2 + R^2}\right]\end{aligned}$$

We can check the result for a long coil, $l \to \infty$. Then

$$\sqrt{(\frac{l}{2} \mp z)^2 + R^2} = \frac{l}{2}\sqrt{(1 \mp \frac{2z}{l})^2 + (\frac{2R}{l})^2} \simeq \frac{l}{2}(1 \mp \frac{2z}{l})$$

and we find

$$\Psi(z) = \frac{Ni}{2l}[l - 2z]$$

giving a field

$$-\frac{\partial \Psi}{\partial z} = H_z = \frac{Ni}{l}$$

which is correct.

8.4 MAGNETOQUASISTATIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

√

8.4.1 From (8.3.13),

$$\Psi(r \to 0) \to \frac{\pi i R^2}{4\pi}\frac{\cos\theta}{r^2} \tag{1}$$

and at $r = b$

$$\left.\frac{\partial \Psi}{\partial r}\right|_{r=b} = 0 \tag{2}$$

To meet these conditions, take the solutions to Laplace's equation

$$\Psi = \frac{\pi i R^2}{4\pi}\frac{\cos\theta}{r^2} + Ar\cos\theta \tag{3}$$

where the first automatically satisfies (1) and the coefficient A of the second is determined by requiring (2). Thus,

$$\Psi = \frac{i\pi R^2}{4\pi}(\frac{1}{r^2} + \frac{2r}{b^3})\cos\theta \tag{4}$$

The negative gradient of this magnetic potential is the given field intensity.

8.4.2 The magnetic field of the dipole is given by (8.1.21)

$$\mathbf{H} = \frac{id}{2\pi r^2}(-\sin\phi\mathbf{i_r} + \cos\phi\mathbf{i_\phi})$$

This corresponds to a scalar potential of

$$\Psi_d = -\frac{id}{2\pi r}\sin\phi$$

The conductor acts like a perfect conductor cancelling the normal component of $\mathbf{H}$, H_r. Thus we must have the total scalar potential

$$\Psi = -\frac{id}{2\pi a}\sin\phi\left(\frac{a}{r}+\frac{r}{a}\right)$$

with the field

$$\mathbf{H} = -\frac{id}{2\pi a^2}\left[\sin\phi\left(\frac{a^2}{r^2}-1\right)\mathbf{i_r} - \cos\phi\left(\frac{a^2}{r^2}+1\right)\mathbf{i}_\phi\right]$$

8.4.3 (a) Far from the half-cylinder, the magnetic potential must become that of a uniform magnetic field in the $-z$ direction.

$$\mathbf{H}(z \to \pm\infty) = -H_o\mathbf{i_z} \Rightarrow \Psi = H_o z = H_o r\cos\phi \tag{1}$$

Thus, to satisfy the condition that there be no normal component of the field intensity at the surface of the half-cylinder, a second solution is added to this one having the same azimuthal dependence.

$$\Psi = H_o r\cos\phi + A\frac{\cos\phi}{r} \tag{2}$$

Adjusting A so that

$$\frac{\partial\Psi}{\partial r}(r=R) = 0 \tag{3}$$

results in the given potential.

(b) As suggested, the field intensity shown in Fig. 8.4.2 satisfies the requirement of being tangential to the perfectly conducting surfaces. Note that the surface current density has the polarity required to exclude the magnetic field from the perfectly conducting regions, in accordance with (3).

8.4.4 The potential Ψ of the uniform field is

$$\Psi_o = H_o r\cos\theta$$

The sphere causes H to be tangential. The normal component H_r must be cancelled:

$$\Psi = H_o R\cos\theta\left[\frac{r}{R}+\frac{1}{2}\left(\frac{R}{r}\right)^2\right]$$

We obtain for the H field

$$\mathbf{H} = -\nabla\Psi = -H_o\left\{\cos\theta\left[1-\left(\frac{R}{r}\right)^3\right]\mathbf{i_r} - \sin\theta\left[1+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]\mathbf{i}_\phi\right\}$$

8.4.5 (a) An image current is used to satisfy the condition that there be no normal component of the field intensity in the plane $y = 0$. Thus, the solution in region $y < 0$ is composed of a particular part due to the line current at $x = 0, y = -h$ and a homogeneous part equivalent to the field of a line current at $x = 0, y = h$ flowing in the opposite direction. To write these fields, first note that for a line current on the z axis,

$$\begin{aligned}\mathbf{H} = \mathbf{i}_\phi\Big(\frac{i}{2\pi r}\Big) &= \frac{i}{2\pi}\Big(-\frac{\mathbf{i}_x \sin\phi + \mathbf{i}_y \cos\phi}{\sqrt{x^2+y^2}}\Big)\\ &= \frac{i}{2\pi}\Big(-\frac{\mathbf{i}_x y}{x^2+y^2} + \frac{\mathbf{i}_y x}{x^2+y^2}\Big)\end{aligned} \tag{1}$$

Translation of this field to represent first the actual and then in addition the image line current then results in the given field intensity.

(b) The surface current density that must exist at $y = 0$ if the region above sustains no field intensity is

$$\mathbf{K} = \mathbf{n} \times \mathbf{H} \Rightarrow K_z = H_x(y = 0) \tag{2}$$

This is the given function.

8.4.6 (a) The scalar potential produced by one segment of length dx' is

$$d\Psi = -\frac{K_o dx'}{2\pi} \tan^{-1}\Big(\frac{y}{x - x'}\Big) = \frac{K_o dx'}{2\pi} \cot^{-1}\Big(\frac{x' - x}{y}\Big) \tag{1}$$

The integral over the strip is

$$\begin{aligned}\Psi = \int_{x'=b}^{x'=a} d\Psi = \frac{K_o}{2\pi}\Big\{&(a - x)\cot^{-1}\Big(\frac{a-x}{y}\Big)\\ &- (b - x)\cot^{-1}\Big(\frac{b-x}{y}\Big) + \frac{y}{2}\log\Big[1 + \Big(\frac{a-x}{y}\Big)^2\Big]\\ &- \frac{y}{2}\log\Big[1 + \Big(\frac{b-x}{y}\Big)^2\Big]\Big\}\end{aligned} \tag{2}$$

where the integral is taken from: B. O. Pearce, R. M. Foster, **A Short Table of Integrals**, 4th Ed., Ginn and Co. (1956). To this potential must be added an image potential that causes $\partial\Psi/\partial x = 0$ at $x = 0$. This is achieved by adding to (2) a potential with the replacements

$$K_o \rightarrow -K_o \quad a \rightarrow -a, \quad b \rightarrow -b$$

(b) The field $\mathbf{H} = -\nabla\Psi$ and thus from (2)

$$\begin{aligned}
H_x = &-\frac{K_o}{2\pi}\Big[-\cot^{-1}\big(\frac{a-x}{y}\big) + \cot^{-1}\big(\frac{b-x}{y}\big) \\
&+ \frac{(a-x)/y}{1+\big(\frac{a-x}{y}\big)^2} - \frac{(b-x)/y}{1+\big(\frac{b-x}{y}\big)^2} \\
&- \frac{(a-x)/y}{1+\big(\frac{a-x}{y}\big)^2} + \frac{(b-x)/y}{1+\big(\frac{b-x}{y}\big)^2}\Big] \\
= &-\frac{K_o}{2\pi}\Big[-\cot^{-1}\big(\frac{a-x}{y}\big) + \cot^{-1}\big(\frac{b-x}{y}\big)\Big] \\
= &\frac{K_o}{2\pi}\Big[\tan^{-1}\big(\frac{y}{a-x}\big) - \tan^{-1}\big(\frac{y}{b-x}\big)\Big]
\end{aligned}$$

To this field we add

$$H_x = \frac{K_o}{2\pi}\Big[\tan^{-1}\big(\frac{y}{x+a}\big) - \tan^{-1}\big(\frac{y}{b+x}\big)\Big]$$

8.5 PIECE-WISE MAGNETIC FIELDS

8.5.1 (a) The surface current density is

$$\mathbf{K} = \frac{N}{2R} i \sin\phi \mathbf{i_z} \tag{1}$$

so that the continuity conditions at the cylinder surface where $r = R$ are

$$H_\phi^a - H_\phi^b = \frac{Ni}{2R}\sin\phi \tag{2}$$

$$\mu_o H_r^a - \mu_o H_r^b = 0 \tag{3}$$

Looking forward to satisfying (2), the ϕ dependence of the scalar potential is taken to be $\cos\phi$. Thus, the appropriate solutions to Laplace's equation are

$$\Psi^a = A\frac{\cos\phi}{r} \tag{4}$$

$$\Psi^b = Cr\cos\phi \tag{5}$$

so that the field intensities are

$$\mathbf{H}^a = A\big(\frac{\cos\phi}{r^2}\mathbf{i_r} + \frac{\sin\phi}{r^2}\mathbf{i_\phi}\big) \tag{6}$$

$$\mathbf{H}^b = -C(\cos\phi \mathbf{i}_r - \sin\phi \mathbf{i}_\phi) \tag{7}$$

Substitution of these fields into (2) and (3) then gives

$$\frac{A}{R^2} - C = \frac{Ni}{2R} \tag{8}$$

$$\frac{A}{R^2} + C = 0 \tag{9}$$

from which it follows that

$$A = \frac{RNi}{4}; \quad C = -\frac{Ni}{4R} \tag{10}$$

Substitution of these coefficients into (4)-(7) results in the given expressions for the magnetic scalar potential and field intensity.

(b) Because the flux density is uniform over the interior of the cylinder, the flux linked by a turn in the plane $x = x' = R\cos\phi'$ is

$$\Phi_\lambda = \mu_o H_x 2R \sin\phi' = \mu_o \frac{Ni}{4R} 2R\sin\phi' \tag{11}$$

Thus, the total flux is

$$\begin{aligned}\lambda &= i\int_0^\pi \frac{\mu_o N}{2}\sin\phi' \big(\frac{N}{2R}\big)\sin\phi' R d\phi' \\ &= i\frac{\mu_o N^2}{4}\int_0^\pi \sin^2\phi' d\phi' = \big[\frac{\mu_o N^2 \pi}{8}\big] i\end{aligned} \tag{12}$$

and thus the inductance is identified as that given.

8.5.2 (a) At $r = b$, there is a jump in tangential $\mathbf{H}$:

$$\mathbf{n}\times(\mathbf{H}^{(a)} - \mathbf{H}^{(b)}) = \mathbf{K} \tag{1}$$

with region (a) outside, (b) inside the cylinder carrying the windings. Thus $\mathbf{n} = \mathbf{i}_r$ and at $r = b$

$$-\frac{1}{b}\frac{\partial\Psi^{(a)}}{\partial\phi} + \frac{1}{b}\frac{\partial\Psi^{(b)}}{\partial\phi} = K_z(\phi) \tag{2}$$

Further the normal component of Ψ must be continous at $r = b$.

$$-\frac{\partial\Psi^{(a)}}{\partial r} + \frac{\partial\Psi^{(b)}}{\partial r} = 0 \tag{3}$$

At $r = a$, the normal component of $\mathbf{H}$ has to vanish:

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a} = 0 \tag{4}$$

(b) We have a "square-wave" for the current distribution. Therefore, we need an infinite sum of terms for Ψ:

$$\Psi^{(b)} = \sum_{n=1}^{\infty} A_n (r/b)^n \cos n(\phi - \phi_o); \quad 0 < r < b$$

$$\Psi^{(a)} = \sum_{n=1}^{\infty} B_n (a/r)^n \cos n(\phi - \phi_o); \quad b < r < a \tag{5}$$
$$+ \sum_{n=0}^{\infty} C_n (r/a)^n \cos n(\phi - \phi_o)$$

We picked the normalization of the coefficients so that the boundary conditions are most simply stated. From (4) we have

$$-nB_n \frac{a^n}{a^{n+1}} + nC_n \frac{a^{n-1}}{a^n} = 0$$

and thus

$$C_n = B_n \tag{6}$$

From (3) we have

$$nB_n \frac{a^n}{b^{n+1}} - nC_n \frac{b^{n-1}}{a^n} + nA_n \frac{1}{b} = 0 \tag{7}$$

and using (6)

$$A_n = C_n\left[(b/a)^n - (a/b)^n\right] \tag{8}$$

From (2) we obtain:

$$n\left[B_n (a/b)^n + C_n (b/a)^n\right] \sin n(\phi - \phi_o) - nA_n \sin n(\phi - \phi_o) = bK_z(\phi) \tag{9}$$

The expansion of the square wave $K_z(\phi)$ is

$$K_z(\phi) = K_o \sum_{\substack{n \\ n-\text{odd}}} \frac{4}{n\pi} \sin n(\phi - \phi_o) \tag{10}$$

Thus, using (6), (8) and (10) in (9) we obtain, for n odd:

$$nC_n\left[(a/b)^n + (b/a)^n\right] - nC_n\left[(b/a)^n - (a/b)^n\right] = \frac{4}{n\pi} K_o$$

and $C_n = 0$ for n even. Thus

$$C_n = \frac{2b}{n^2\pi} K_o (b/a)^n = B_n \tag{11}$$

and

$$A_n = \frac{2b}{n^2\pi}K_o\left[(b/a)^{2n} - 1\right] \tag{12}$$

for n odd, zero for n even. We should check a few limits right away. When $a \to \infty$, we get (for n odd)

$$A_n = -\frac{2b}{n^2\pi}K_o$$

and

$$\Psi^{(b)} = -\sum_{n-\mathrm{odd}} \frac{2b}{n^2\pi}K_o(r/b)^n \cos n(\phi - \phi_o)$$

$$\Psi^{(a)} = \sum_{n-\mathrm{odd}} \frac{2b}{n^2\pi}K_o(b/r)^n \cos n(\phi - \phi_o)$$

which gives the field due to the cylinder alone. For $a \to b$, we get $A_n = 0$

$$\Psi^{(a)} \simeq \sum_{n-\mathrm{odd}} \frac{2b}{n^2\pi}K_o\left[(a/r)^n + (r/a)^n\right]\cos n(\phi - \phi_o)$$

$$\simeq \sum_{n-\mathrm{odd}} \frac{4b}{n^2\pi}K_o \cos n(\phi - \phi_o)$$

There is a ϕ directed field in the region between the coil and the shield of magnitude

$$H_\phi \simeq -\frac{1}{a}\frac{\partial \Psi}{\partial \phi} \simeq \sum_{n-\mathrm{odd}} \frac{4K_o}{n\pi}\sin n(\phi - \phi_o)$$

which is approximately square-wave-like. These checks confirm the correctness of the solution.

(c) The inductance of the rotor coil is computed from the flux linkage of an individual wire-loop,

$$\Phi_\lambda = l\int_{\phi=-\phi'}^{-\phi'+\pi} \mu_o H_r b d\phi\Big|_{r=b} = \sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} -l\mu_o\frac{nA_n}{b}\int_{\phi=-\phi'}^{-\phi'+\pi} \cos n(\phi - \phi_o) b d\phi$$

$$= \sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} l\mu_o\frac{4K_o b}{n\pi}\left[1 - \left(\frac{b}{a}\right)^{2n}\right]\sin n(\phi' - \phi_o)$$

where l is the length of the system. The flux linkage is obtained by taking the number of wires per unit circumference $N/\pi b$, multiplying them by Φ_λ and integrating from $\phi' = \phi_o$ to $\phi' = \phi_o + \pi$

$$\lambda = \int \frac{N}{\pi b} b d\phi' \Phi_\lambda = l\frac{N}{\pi}\sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} \mu_o\frac{4K_o b}{n\pi}\left[1 - \left(\frac{b}{a}\right)^{2n}\right]\int d\phi' \sin n(\phi' - \phi_o)$$

$$= l\frac{8N^2}{2\pi}\mu_o i\left(\sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} \frac{1}{n^2}\left[1 - \left(\frac{b}{a}\right)^{2n}\right]\right)$$

where we use the fact that

$$K_o = \frac{Ni}{2b}$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{8N^2}{2\pi}\mu_o l \sum_{\substack{n=1 \\ n-\text{odd}}}^{\infty} \frac{1}{n^2}\left[1 - \left(\frac{b}{a}\right)^{2n}\right]$$

The inductance is, of course, ϕ_o independent because the field is "tied" to the rotor and moves with ϕ_o.

8.6 VECTOR POTENTIAL AND THE BOUNDARY VALUE POINT OF VIEW

8.6.1 (a) For the two-dimensional situation under consideration, the magnetic field intensity is found from the vector potential using (8.1.17)

$$\mathbf{H} = \frac{1}{\mu_o}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi}\mathbf{i}_r - \frac{\partial A_z}{\partial r}\mathbf{i}_\phi\right) \tag{1}$$

Thus, if the vector potential were discontinuous at $r = R$, the azimuthal magnetic field intensity would be infinite there.

(b) Integration of (1) using the fields given by (1.4.7) gives

$$A_z = -\mu_o \int H_\phi dr + f(\phi) = -\mu_o J_o \begin{cases} R^3/9R + f_1; & r < R \\ \frac{R^2}{3} ln(r/R) + f_2; & R < r \end{cases} \tag{2}$$

$$A_z = \begin{cases} g_1(r); & r < R \\ g_2(r); & R < r \end{cases} \tag{3}$$

Because the integrations are performed holding r and ϕ-constant, respectively, the integration "constants" are actually functions of the "other" independent variable, as indicated. From (3) it is clear, however, that there is no dependence of f_1 and f_2 on ϕ. Given that the vector potential is zero at $r = 0$ and that A_z is continuous at $r = R$, $f_1 = 0$ and $f_2 = R^2/9$. Thus, the vector potential is as given.

(c) In terms of the vector potential, the flux is given by (8.4.12). Because there are no contributions on the radial legs and because $A_z(r = 0)$ has been defined as zero,

$$\begin{aligned} \lambda &= \oint_{C'} \mathbf{A} \cdot d\mathbf{s} = l[A_z(0) - A_z(a)] = -lA_z(a) \\ &= \frac{\mu_o l R^2 J_o}{3}\left[ln(a/R) + \frac{1}{3}\right] \end{aligned} \tag{4}$$

This illustrates how the use of $\mathbf{A}$ to represent the field makes it possible to evaluate the flux linkage without carrying out an integration.

8.6.2 **A** must be z-directed and must obey Poisson's equation

$$\nabla^2 A_z = -\mu_o J_z \tag{1}$$

Now

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

in the special symmetry of the problem. Thus

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)A_z = -\mu_o J_z \tag{2}$$

and

$$A_z = -\mu_o J_z \frac{r^2}{4} \qquad r < b \tag{3}$$

Outside this region $b < r < a$, A_z obeys Laplace's equation

$$A_z \propto C ln(r/b) + \text{const}$$

At $r = b$ we must have continuous A_z and dA_z/dr (continuous H_ϕ). Thus,

$$\text{const} = -\mu_o J_z \frac{b^2}{4}$$

and

$$\frac{C}{b} = -\mu_o J_z \frac{b}{2}$$

Thus

$$A_z = -\mu_o J_z \frac{b^2}{4}[2ln(r/b) + 1]; \quad b < r < a$$

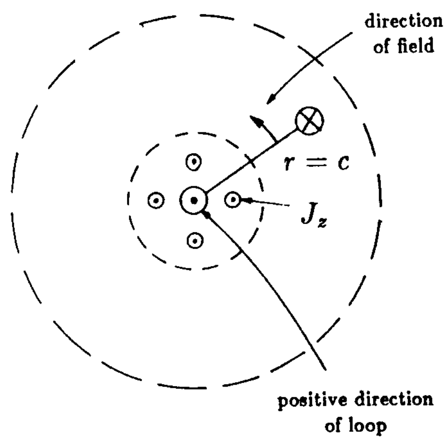


Figure S8.6.2

The flux is, according to (8.6.5) [see Fig. S8.6.2]

$$\lambda = l(A_z^a - A_z^b)$$

and thus

$$\lambda = -lA_z^b$$

because

$$A_z^a = 0$$

For $c < b$

$$\lambda = l\mu_o J_z \frac{c^2}{4}$$

For $c > b$

$$\lambda = l\mu_o J_z \frac{b^2}{4}[1 + 2ln(c/b)]$$

Note that $A_z \neq 0$ for $r > 0$. This should be remedied by adding a constant to A_z. It does not affect the flux linkage.

8.6.3 (a) In cylindrical coordinates where there is no ϕ dependence, the vector potential has only a θ component

$$\mathbf{A} = A_\theta(r, z)\mathbf{i}_\theta \tag{1}$$

and the flux density is found from

$$\mu_o\mathbf{H} = \nabla \times \mathbf{A} \Rightarrow \mu_o\mathbf{H} = \mathbf{i_r}\Big(-\frac{\partial A_\theta}{\partial z}\Big) + \mathbf{i_z}\Big[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\Big] \tag{2}$$

For reasons that are apparent in part (b), it is convenient to write $\mathbf{A}$ as

$$\mathbf{A} = \frac{\Lambda_c(r,t)}{r} \tag{3}$$

in which case, (2) becomes

$$\mu_o\mathbf{H} = \frac{1}{r}\Big[-\frac{\partial \Lambda_c}{\partial z}\mathbf{i_r} + \frac{\partial \Lambda_c}{\partial r}\mathbf{i_z}\Big] \tag{4}$$

(b) For any surface S enclosed by the contour C, the net flux can be found from the vector potential by

$$\lambda = \oint_C \mathbf{A} \cdot d\mathbf{s} \tag{5}$$

In particular, consider a surface enclosed by a contour C having as the first of four segments a contour spanning $0 < \phi < 2\pi$ at the radius, a, from the z axis. The second segment connects that circular contour with a second at the radius b by a segment connecting the two in a plane of constant ϕ. The contour is closed by a second contour in an adjacent ϕ = constant plane joining these circular segments. Integration of (5) gives contributions only from the circular contours. The segments joining the circular contours are perpendicular to the direction of $\mathbf{A}$, and in any case make compensating contributions because they are in essentially the same ϕ = constant planes. Thus, the flux through the surface having outer and inner radii, a and b respectively, is as given.

8.6.4 (a) The vector potential, A_z, satisfies Laplace's equation. The first three conditions of (8.6.18) are met by the solution

$$A_z = A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x \tag{1}$$

The last condition is met by superimposing these solutions

$$A_z = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x \tag{2}$$

and evaluating the coefficients by requiring that this function satisfy the fourth boundary condition of (8.6.18).

$$\Lambda = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} b \sin \frac{n\pi}{a} x \tag{3}$$

Multiplication by $\sin(m\pi x/a)$ and integration gives

$$-\frac{\Lambda a}{m\pi} \cos \frac{m\pi}{a} x \bigg]_0^a = \frac{A_m a}{2} \sinh \frac{m\pi}{a} b \tag{4}$$

which therefore gives the coefficients as

$$A_m = \frac{2\Lambda}{m\pi \sinh \frac{m\pi}{a} b} [-\cos m\pi + 1] \tag{5}$$

so that (2) becomes the given solution.

(b) The total current in the lower plate is

$$i = \int_0^a K_z dx = -\int_0^a H_x(y=0) dx = -\int_0^a \frac{1}{\mu_o} \frac{\partial A_z}{\partial y}\bigg|_{y=0} dx \tag{6}$$

Evaluation using the given vector potential gives

$$i = -\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{8\Lambda}{\mu_o n\pi \sinh\left(\frac{n\pi b}{a}\right)} = -\sum_{n=1}^{\infty} \frac{I \sin \omega t}{2n \sinh\left(\frac{n\pi b}{a}\right)} \tag{7}$$

(c) In the limit where $b/a \gg 1$,

$$\sinh\left(\frac{n\pi b}{a}\right) \to \frac{1}{2} e^{n\pi b/a} \tag{8}$$

and (7) becomes

$$i \to -\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{I}{n} e^{-n\pi b/a} \sin\omega t \to -Ie^{-\pi b/a} \sin\omega t \tag{9}$$

Taking ln of the magnitude of this expression gives

$$ln\left(\frac{|i|}{I}\right) = -\pi(b/a) \tag{10}$$

which is the straight line portion of the plotted function.

(d) In the limit $b/a \ll 1$, (7) becomes

$$i \to -\frac{1}{\mu_o}\frac{8\Lambda}{\pi^2}\frac{a}{b}\sum\frac{1}{n^2} = -\frac{1}{\mu_o}\Lambda\frac{a}{b} \tag{11}$$

This is the same as what is obtained if it is assumed that the field is uniform and simply $H_x \to \Lambda/b\mu_o$ so that

$$K_z \to -H_x \Rightarrow i \to K_z a \to -a\Lambda/b\mu_o \tag{12}$$

8.6.5 The perfectly conducting electrodes force H to be tangential to the electrodes. Thus $\partial A_z/\partial x = -\mu_o H_y$ vanishes at $y = 0, y = d$ except for the gap at $x = 0$ and $\partial A_z/\partial y = \mu_o H_x$ vanishes at $x = \pm a$. The magnetic vector potential jumps by Λ as one goes from $x = 0_-$ to $x = 0_+$, at $y = 0$ and $y = d$. Thus A_z is constant around the $\subset$ shaped contour as well as the $\supset$ shaped one. Denoting by the superscripts (a) and (b) these two regions respectively, we have for Laplacian solutions of A_z

$$A_z^{(a)} = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{d}(x+a)\sin\frac{n\pi}{d}y + A_o(x+a)$$

$$A_z^{(b)} = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{d}(x-a)\sin\frac{n\pi}{d}y + B_o(x-a)$$

At $x = 0$, the constants A_o and B_o account for the jump of $A_z, B_o = -\Lambda/2 = A_o$. The vector potential and its curl must be continuous for $0 < y < d$ at $x = 0$. We thus have $A_n = -B_n$ for all n except $n = 0$. The sinusoidal series has to cancel that jump for $0 < y < d$. We must have

$$\sum_n A_n \sinh\frac{n\pi}{d}a \sin\frac{n\pi}{d}y = -\sum_{n-\text{odd}}\frac{4A_o}{n\pi}\sin\frac{n\pi}{d}y$$

and similarly for the series in region b. We obtain

$$A_z^{(a)} = \sum_{n-\text{odd}}\frac{2\Lambda}{n\pi}\frac{\sinh\frac{n\pi}{d}(x+a)}{\sinh\frac{n\pi}{d}a}\sin\frac{n\pi}{d}y - \frac{\Lambda}{2}(x+a)$$

$$A_z^{(b)} = \sum_{n-\text{odd}} \frac{2\Lambda}{n\pi} \frac{\sinh \frac{n\pi}{d}(x-a)}{\sinh \frac{n\pi}{d} a} \sin \frac{n\pi}{d} y - \frac{\Lambda}{2}(x-a)$$

(b) See Fig. S8.6.5.

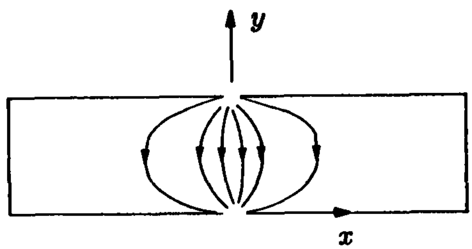


Figure S8.6.5

8.6.6 (a) We must satisfy Poisson's equation for the vector potential everywhere inside the perfectly conducting boundaries

$$\nabla^2 A_z = \mu_o i n_o \sin\left(\frac{\pi x}{a}\right) \tag{1}$$

and make the normal flux density and hence A_z zero on the boundaries.

$$A_z = 0 \quad \text{at} \quad x = \pm a, y = 0, y = b \tag{2}$$

A particular solution to (1) follows by looking for one that depends only on x.

$$\frac{\partial^2 A_{zp}}{\partial x^2} = \mu_o i n_o \sin\left(\frac{\pi x}{a}\right) \Rightarrow A_{zp} = -\mu_o i n_o \frac{a^2}{\pi^2} \sin \frac{\pi x}{a} \tag{3}$$

Then the homogeneous solution must satisfy Laplace's equation and the conditions

$$A_{zh} = 0 \quad \text{at} \quad x = \pm a; \tag{4a}$$

$$A_{zh} = \mu_o i n_o \frac{a^2}{\pi^2} \sin \frac{\pi x}{a} \quad \text{at} \quad y = 0, b \tag{4b}$$

The first of these conditions, can be met by making the x dependence $\sin(\pi x/a)$. Then, the y dependence must be comprised of a linear combination of $\exp(+ky)$ and $\exp(-ky)$. If the y coordinate were at $y = b/2$, the second of the conditions of (4) would be even in y. So, make the linear combination $\cosh k(y - \frac{b}{2})]$ and for convenience adjust the coefficient so that the second of conditions (4) are met, divide this function by its value at $y = b/2$. This makes it clear that the coefficient is the value given on the boundary from (4). Thus, the desired solution, the sum of the particular and homogeneous parts, is

$$A_z = A_{zp} + A_{zh} = \frac{\mu_o i n_o a^2}{\pi^2} \left[\frac{\cosh \frac{\pi}{a}\left(y - \frac{b}{2}\right)}{\cosh\left(\frac{\pi b}{2a}\right)} - 1 \right] \sin\left(\frac{\pi x}{a}\right) \tag{5}$$

(b) The flux linked by one turn is

$$\begin{aligned}\Phi_\lambda &= -l[A_z(x,y) - A_z(-x,y)] \\ &= -\frac{2\mu_o i n_o a^2 l}{\pi^2}\left[\frac{\cosh\frac{\pi}{a}(y-\frac{b}{2})}{\cosh\left(\frac{\pi b}{2a}\right)} - 1\right]\sin\frac{\pi x}{a}\end{aligned} \tag{6}$$

and the total flux of all of the windings in series is

$$\begin{aligned}\lambda &= \int_0^a\int_0^b \Phi_\lambda n_o \sin\left(\frac{\pi x}{a}\right)dy dx \\ &= \frac{2\mu_o n_o^2 a^4 l}{\pi^3}\left[\frac{\pi b}{2a} - \tanh\left(\frac{\pi b}{2a}\right)\right]\end{aligned} \tag{7}$$

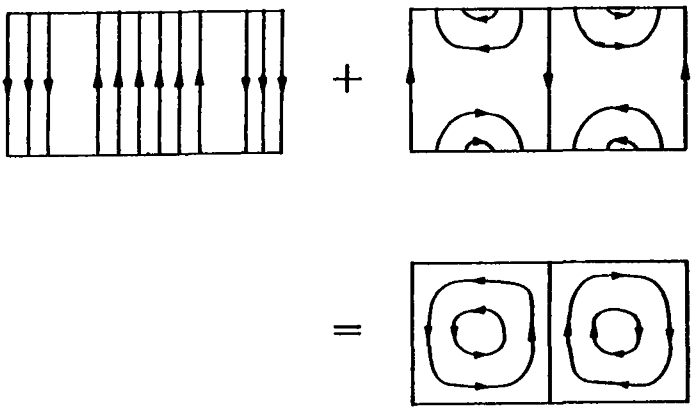


Figure S8.6.6

(c) A sketch of the lines of constant vector potential and thus **H** for the particular, homogeneous and total solution (the sum of these) is shown in Fig. S8.6.6. It is perhaps easiest to envision the sum by picturing the addition of contour maps of the two parts, the axes out of the paper being the height A_z of the respective surfaces.

8.6.7 (a) This is a problem involving a particular and a homogeneous solution of the vector Poisson equation. The particular solution is due to uniform current density $J_o = n_o i$

$$\mathbf{A}_p = -\mu_o n_o i \frac{x^2 - a^2}{2}\mathbf{i}_z$$

Alternatively, we may find the homogeneous solution by comparison with Prob. 8.6.6. In that problem the wire density was sinusoidal. Now it is uniform. A_z was antisymmetric, now it is symmetric. We can expand the symmetric wire distribution as a square wave.

$$J_z(x,y) = n_o i = \sum_{\substack{n \\ n-\text{odd}}} \frac{4 n_o i}{n\pi}\cos\frac{n\pi}{2a}x$$

The particular solution of the vector potential is thus

$$\mathbf{A}_p = -\mathbf{i}_z \mu_o n_o i \sum_{\substack{n \\ n-\text{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)$$

The complete solution is

$$\mathbf{A} = \mathbf{i}_z \mu_o n_o i \sum_{\substack{n \\ \text{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)\left[\frac{\cosh \frac{n\pi}{2a}(y-\frac{b}{2})}{\cosh \frac{n\pi}{4a}b} - 1\right]$$

(b) The flux linkage of a wire at x, y is

$$\lambda = lA_z(x,y)$$

and thus

$$v = \frac{d\lambda}{dt} = \mu_o n_o l \sum_{\substack{n \\ \text{odd}}} \frac{4}{n\pi}\Big(\frac{2a}{n\pi}\Big)^2 \cos\Big(\frac{n\pi}{2a}x\Big)\left[\frac{\cosh \frac{n\pi}{2a}(y-\frac{b}{2})}{\cosh \frac{n\pi}{4a}b} - 1\right]\frac{di}{dt}$$

8.6.8 (a) Here we have a solution very much like that of Prob. 8.6.6, except that the particular solution

$$\mathbf{A}_p = -\mathbf{i}_z \mu_o i n_o (a/\pi)^2 \sin\Big(\frac{\pi x}{a}\Big)$$

has to be replaced by an infinite sum whose second derivative reproduces the square wave of magnitude in_o. Thus

$$\mathbf{A}_b = -\mathbf{i}_z \mu_o i n_o \sum_{n-\text{odd}} \frac{4}{n\pi}\Big(\frac{a}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{a}\Big)$$

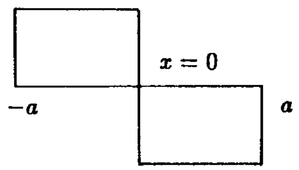


Figure S8.6.8

The complete solution is (compare Prob. 8.6.6)

$$\mathbf{A} = \mathbf{i}_z \mu_o i n_o \sum_{n-\text{odd}} \frac{4}{n\pi}\Big(\frac{a}{n\pi}\Big)^2 \sin\Big(\frac{n\pi x}{a}\Big)\left[\frac{\cosh(n\pi/a)(y-\frac{b}{2})}{\cosh\left(\frac{n\pi b}{2a}\right)} - 1\right]$$

(b) The inductance is computed from

$$Li = -\int_0^a \int_0^b n_o dx' dy' 2lA_z(x', y')$$

where $2lA_z$ is the flux linkage of one turn $n_o dx' dy'$ is the wire density. Thus integrating one typical term:

$$\int_0^u dx' \sin\left(\frac{n\pi x'}{a}\right) \int_0^b \left[\frac{\cosh \frac{n\pi}{a}\left(y - \frac{b}{2}\right)}{\cosh \frac{n\pi b}{2a}} - 1\right] dy' = 2\left(\frac{a}{n\pi}\right)\left[2\frac{a}{n\pi} \tanh \frac{n\pi b}{2a} - b\right]$$

and the inductance is

$$L = \mu_o n_o^2 l \sum_{n-\text{odd}} \frac{16}{n\pi}\left(\frac{a}{n\pi}\right)^4 \left[\frac{n\pi b}{2a} - \tanh\left(\frac{n\pi b}{2a}\right)\right]$$

SOLUTIONS TO CHAPTER 9

9.1 MAGNETIZATION DENSITY

9.2 LAWS AND CONTINUITY CONDITIONS WITH MAGNETIZATION

9.2.1

$$\mathbf{M} = M_o \cos\beta x(\mathbf{i_x} + \mathbf{i_y})$$

The volume charge density

$$\rho_m = -\nabla \cdot \mu_o \mathbf{M} = \mu_o M_o \sin\beta x$$

$$\sigma_m = \mathbf{n} \cdot \mu_o(\mathbf{M}^a - \mathbf{M}^b)$$

and thus there is positive surface charge density on top

$$\sigma_m = \mu_o M_o \cos\beta x \qquad y = d$$

and a charge density of opposite sign at the bottom, $y = -d$.

9.2.2 (a) The magnetization is uniform, with the orientation shown in Fig. P9.2.1. Thus, it is solenoidal and the right hand side of (9.2.2) is zero and therefore equal to the left hand side, which is zero because $\mathbf{H} = 0$. Certainly a zero H field is irrotational, so Ampère's law is also satisfied. Associated with $\mathbf{M}$ inside is a magnetic surface charge density. However, this is cancelled by a surface charge density of opposite sign induced in the infinitely permeable wall so as to prevent there being an $\mathbf{H}$ outside the cylinder.

(b) In view of the direction defined as positive for the wire, the flux linked by the coil is

$$\lambda = \mathbf{B} \cdot \mathbf{i_y} 2Rd = \mu_o M_y 2Rd = \mu_o 2RdM_o \cos\gamma \tag{1}$$

Thus, with the terminus of the right wire defined as the + terminal and $\gamma = \Omega t$, the voltage is

$$v = \frac{d\lambda}{dt} = -\mu_o 2RdM_o\Omega \sin\Omega t \tag{2}$$

9.2.3 (a) From Ampère's law

$$\oint_C \mathbf{H}\cdot d\mathbf{s} = \int_S \mathbf{J}\cdot d\mathbf{a}$$

we find

$$\oint \mathbf{H}\cdot d\mathbf{s} = 0$$

because there is no $\mathbf{J}$ present. This means that $\mathbf{H} = -\nabla\Psi$ and Ψ is a scalar potential that satisfies Laplace's equations, since $\mathbf{H}$ is divergence-free. The only possible solution to this problem, subject to $\Psi = \text{const}$ at $y = 0$ and $y = a$, is $\Psi = \text{const}$; and hence $\mathbf{H} = 0$.

(b) Since

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \tag{1}$$

we have

$$\mathbf{B} = \mathbf{i_y}\mu_o M_o \cos\beta(x - Ut) \tag{2}$$

The flux linked by the turn is

$$\begin{aligned}\lambda &= \mu_o l\int_{x=-d}^{x=d} M_o\cos\beta(x-Ut)dx\\ &= \mu_o l d M_o\left\{\frac{\sin(\beta d - \beta U t)}{\beta d} + \frac{\sin(\beta d + \beta U t)}{\beta d}\right\}\\ &= \mu_o l d M_o\left\{\frac{\sin\beta d\cos\beta U t - \cos\beta d\sin\beta U t}{\beta d}\right.\\ &\left.+\frac{\sin\beta d\cos\beta U t + \cos\beta d\sin\beta U t}{\beta d}\right\}\\ &= 2\mu_o l d M_o\frac{\sin\beta d}{\beta d}\cos\beta U t\end{aligned}$$

The voltage is

$$v = \frac{d\lambda}{dt} = -2\beta U\mu_o l d M_o\frac{\sin\beta d}{\beta d}\sin\beta U t$$

9.3 PERMANENT MAGNETIZATION

9.3.1 The given answer is the result of using (4.5.24) twice. First, the result is written with the identification of variables

$$\frac{\sigma_o}{\epsilon_o} \rightarrow \frac{\mu_o M_o}{\mu_o};\quad x_1 = a, x_2 = -a, y \rightarrow y - b \tag{1}$$

representing the upper magnetic surface charge. Second, representing the potential of the lower magnetic surface charge,

$$\frac{\sigma_o}{\mu_o} \rightarrow -M_o; \quad x_1 = a, x_2 = -a, y \rightarrow y + b \tag{2}$$

The sum of these two results is the given answer.

9.3.2 In the upper half-space, where there is the given magnetization density, the magnetic charge density is

$$\rho_m = -\nabla \cdot \mu_o \mathbf{M} = \mu_o M_o \alpha \cos \beta x e^{-\alpha y} \tag{1}$$

while at the interface there is the surface magnetic charge density

$$\sigma_m = -\mu_o M_x(y = 0) = -\mu_o M_o \cos \beta x \tag{2}$$

In the upper region, a particular solution is needed to balance the source term, (1) introduced into the magnetic potential Poisson's equation

$$\nabla^2 \Psi_p = -M_o \alpha \cos \beta x e^{-\alpha y} \tag{3}$$

given the constant coefficient nature of the Laplacian on the left, it is natural to look for a product solution having the same x and y dependence as what is on the right. Thus, if

$$\Psi_p = F \cos \beta x e^{-\alpha y} \tag{4}$$

then (3) requires that

$$F[-\beta^2 + \alpha^2] = -M_o \alpha \Rightarrow F = M_o \alpha / (\beta^2 - \alpha^2) \tag{5}$$

Thus, to satisfy the boundary conditions at $y = 0$

$$\Psi^a = \Psi^b; \quad -\mu_o \frac{\partial \Psi^a}{\partial y} + \mu_o \frac{\partial \Psi^b}{\partial y} = -\mu_o M_o \cos \beta x \tag{6}$$

we take the solution in the upper region to be a superposition of (5) and a suitable solution to Laplace's equation that goes to zero at $y \rightarrow \infty$ and has the same x dependence.

$$\Psi^a = \left[A e^{-\beta y} + \frac{M_o \alpha}{(\beta^2 - \alpha^2)} e^{-\alpha y} \right] \cos \beta x \tag{7}$$

Similarly, in the lower region where there is no source,

$$\Psi^b = C e^{\beta y} \cos \beta x \tag{8}$$

Substitution of these solutions into the two boundary conditions of (6) gives

$$A = \frac{M_o}{2(\alpha - \beta)} \tag{9}$$

$$C = -\frac{M_o}{2(\alpha + \beta)} \tag{10}$$

and hence the given solution.

9.3.3 We have

$$\nabla \cdot \mu_o \mathbf{H} = -\mu_o \nabla^2 \Psi = -\nabla \cdot \mu_o \mathbf{M} = -\mu_o \beta M_o \cos \beta x \exp \alpha y$$

This is Poisson's equation for Ψ with the particular solution:

$$\Psi_p = \frac{\beta M_o}{\alpha^2 - \beta^2} \cos \beta x \exp \alpha y$$

The homogeneous solution has to take care of the fact that at $y = 0$ the magnetic charge density stops. We have the following solutions of Laplace's equation

$$\Psi_h = \begin{cases} A \cos \beta x e^{-\beta y} & y > 0 \\ B \cos \beta x e^{\beta y} & y < 0 \end{cases}$$

There is no magnetic surface charge density. At the boundary, Ψ and $\partial\Psi/\partial y$ must be continuous

$$-\frac{\beta M_o}{\alpha^2 - \beta^2} + B = A$$

and

$$\frac{\alpha \beta M_o}{\alpha^2 - \beta^2} + \beta B = -\beta A$$

Solving, we find

$$B = -\frac{M_o}{2(\alpha - \beta)}\left(1 + \frac{\alpha}{\beta}\right)$$

and

$$A = -\frac{M_o}{2(\alpha + \beta)}\left(1 - \frac{\alpha}{\beta}\right)$$

9.3.4 The magnetic volume charge density is

$$\begin{aligned}
\rho_m &= -\nabla \cdot \mu_o \mathbf{M} = -\mu_o \frac{1}{r}\frac{\partial}{\partial r}(r M_r) - \mu_o \frac{1}{r}\frac{\partial}{\partial \phi} M_\phi \\
&= -\mu_o \frac{M_o}{r} p (r/R)^{p-1} \cos p(\phi - \gamma) + \mu_o \frac{M_o}{r} p (r/R)^{p-1} \cos p(\phi - \gamma) \\
&= 0
\end{aligned}$$

There is no magnetic volume charge density. All the charge density is on the surface

$$\sigma_m = \mu_o M_r \big|_{r=R} = \mu_o M_o \cos p(\phi - \gamma)$$

This magnetic surface charge density produces $\mu_o \mathbf{H}$ just like σ_s produces $\epsilon_o \mathbf{E}$ (EQS). We set

$$\Psi = \begin{cases} A (R/r)^p \cos p(\phi - \gamma) & r > R \\ B (r/R)^p \cos p(\phi - \gamma) & r < R \end{cases}$$

Because there is no current present, Ψ is continuous at $r = R$ and thus

$$A = B$$

On the surface

$$-\mu_o \frac{\partial \Psi}{\partial r}\Big|_{r=R_+} + \mu_o \frac{\partial \Psi}{\partial r}\Big|_{r=R_-} = \sigma_m = \mu_o M_o \cos p(\phi - \gamma)$$

We find

$$2p\frac{A}{R} = M_o \qquad A = \frac{R}{2p} M_o$$

(b) The radial field at $r = d + R$ is

$$\mu_o H_r(r = d + R) = \mu_o \frac{M_o}{2} \cos p(\phi - \gamma)\left(\frac{R}{R+d}\right)^{p+1}$$

The flux linkage is

$$\lambda = \mu_o N^2 H_r al = \frac{\mu_o N^2 M_o}{2} al\left(\frac{R}{R+d}\right)^{p+1} \cos p\left(\frac{\pi}{2} - \Omega t\right)$$

The voltage is

$$\frac{d\lambda}{dt} = \frac{p\Omega \mu_o N^2 M_o al}{2}\left(\frac{R}{R+d}\right)^{p+1} \cos p\Omega t$$

(c) If p is high, then

$$\left(\frac{R}{R+d}\right)^{p+1} \ll 1$$

unless d is made very small.

9.4 MAGNETIZATION CONSTITUTIVE LAWS

9.4.1 (a) With the understanding that **B** and **H** are collinear, the magnitude of **B** is related to that of **H** by the constitutive law

$$B = \mu_o[H + M_o \tanh(\alpha H)] \tag{1}$$

For small argument, the tanh function is approximately its argument. Thus, like the saturation law of Fig. 9.4.4, in the neighborhood of the origin, for $\alpha H \ll 1$, the curve is a straight line with slope $\mu_o(1 + \alpha M_o)$. In the range of $\alpha H \approx 1$ the curve makes a transition to a lesser slope μ_o.

(b) It follows from (9.4.1) and (1) that

$$B = \mu_o\left[\frac{N_1 i}{2\pi R} + M_o \tanh\left(\frac{\alpha N_1 i}{2\pi R}\right)\right] \tag{2}$$

and in turn from (9.4.2) that

$$\lambda_2 = \frac{\pi w^2 N_2 \mu_o}{4}\left[\frac{N_1 i}{2\pi R} + M_o \tanh\left(\frac{\alpha N_1 i}{2\pi R}\right)\right] \tag{3}$$

Thus, the voltage is $v = d\lambda_2/dt$, the given expression.

9.4.2 The flux linkage is according to (9.4.2)

$$\lambda_2 = \frac{\pi w^2}{4} N_2 B \tag{1}$$

The field intensity is according to (9.4.1)

$$H_\phi = \frac{N_1 i}{2\pi R}$$

Therefore

$$\frac{d\lambda_2}{dt} = \frac{\pi w^2}{4} N_2 \frac{dB}{dt}$$

where we need the dispersion diagram to relate H_ϕ (i.e. i) to B (see Fig. S9.4.2).

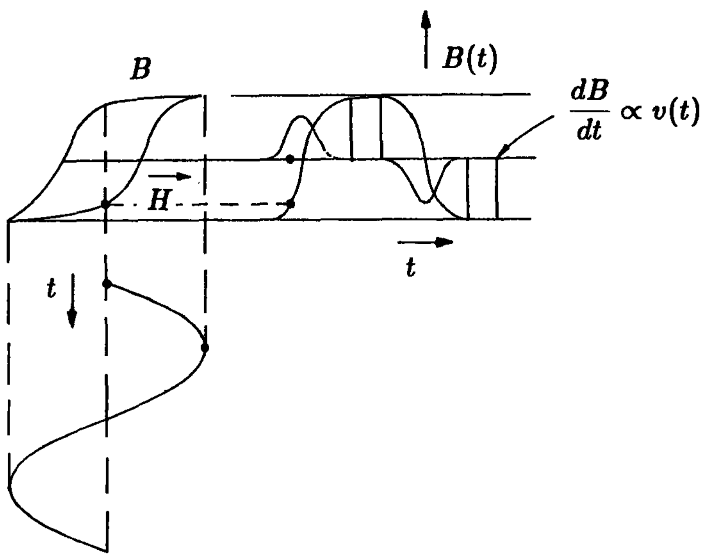


Figure S9.4.2

9.5 FIELDS IN THE PRESENCE OF MAGNETICALLY LINEAR INSULATING MATERIALS

9.5.1 The postulated uniform H field satisfies (9.5.1) and (9.5.2) everywhere inside the regions of uniform permeability. It also satisfies the continuity conditions, (9.5.3) and (9.5.4). Finally, with no H outside the conductors, (9.5.3) is satisfied. The only way in which the permeable materials can alter the uniform field that exists in their absence is by having a component collinear with the permeability gradient. As shown by (9.5.21), only then is there induced the magnetic charge necessary to altering the distribution of H. Here, such a component would be perpendicular to the interface between permeable materials, where it would produce a surface magnetic charge in accordance with (9.5.22). Because H is simply i/w throughout, the total flux linking the one turn circuit is simply

$$\lambda = \int_{A_a} \mu_a H da + \int_{A_b} \mu_b H da = (\mu_a A_a + \mu_b A_b) H = (\mu_a A_a + \mu_b A_b) i/w \qquad (1)$$

and hence, because $\lambda = Li$, the inductance is as given.

9.5.2 From Ampère's law applied to a circular contour around the inner cylinder, anywhere within the region $b < r < a$, one finds

$$H_\phi = \frac{i}{2\pi r}$$

where $\mathbf{i}_\phi$ points in the clock-wise direction, and z along the axis of the cylinder. The flux densities are

$$B_\phi = \frac{\mu_a i}{2\pi r} \quad \text{and} \quad B_\phi = \frac{\mu_b i}{2\pi r}$$

in the two media. The flux linkage is

$$\begin{aligned}\lambda &= l\left\{\int_b^R \frac{\mu_b i}{2\pi r} dr + \int_R^a \frac{\mu_a i}{2\pi r} dr\right\} \\ &= \frac{l}{2\pi}[\mu_b ln(R/b) + \mu_a ln(a/R)]i\end{aligned}$$

The inductance is

$$L = \frac{\lambda}{i} = \frac{l}{2\pi}[\mu_b ln(R/b) + \mu_a ln(a/R)]$$

9.5.3 For the reasons given in the solution to Prob. 9.5.1, the H field is simply $(i/w)\mathbf{i}_z$. Thus, the magnetic flux density is

$$B = \mu H = -\left(\frac{\mu_m x}{l}\right)\frac{i}{w} \qquad (1)$$

and the total flux linked by the one turn is

$$\lambda = \int_S B_z dy dx = d\int_{-l}^{0} \left(\frac{-\mu_m x}{l}\right)\frac{i}{w} dx = \frac{\mu_m l d}{2w} i \qquad (2)$$

By definition, $\lambda = Li$, so it follows that L is as given.

9.5.4 The magnetic field does not change from that of Prob. 9.5.2. The flux linkage is

$$\lambda = l\int_b^a \mu_m(r/b)\frac{i}{2\pi r}dr = \mu_m l\left(\frac{a-b}{b}\right)i$$

The inductance is

$$L = \mu_m l\frac{a-b}{b}$$

9.5.5 (a) The postulated fields have the r dependence of the H produced by a line current i on the z axis, as can be seen using Ampère's integral law (Fig. 1.4.4). Direct substitution into (9.5.1) and (9.5.2) written in polar coordinates also shows that fields in this form satisfy Ampère's law and the continuity condition everywhere in the regions of uniform permeability.

(b) Using the postulated fields, (9.5.4) requires that

$$\frac{\mu_a A}{r} = \frac{\mu_b C}{r} \Rightarrow C = \frac{\mu_a}{\mu_b}A \tag{1}$$

(c) For a contour that encloses the interior conductor, which carries the total current i, Ampère's integral law requires that $(\beta \equiv 2\pi - \alpha)$

$$\oint_C H_\phi r dr = i = \alpha r\frac{A}{r} + \beta r\frac{C}{r} = \alpha A + \beta C \tag{2}$$

Thus, from (1),

$$i = A\left(\alpha + \beta\frac{\mu_a}{\mu_b}\right) \Rightarrow A = \frac{i}{\alpha + \beta\frac{\mu_a}{\mu_b}};$$

$$C = \frac{(\mu_a/\mu_b)i}{\alpha + \beta\frac{\mu_a}{\mu_b}} \tag{3}$$

(d) The inductance follows by integrating the flux density over the gap. Note that the same answer must be obtained from integrating over the gap region occupied by either of the permeable materials. Integration over a surface in region a gives

$$\lambda = l\int_b^a \frac{\mu_a A}{r}dr = l\mu_a A ln(a/b) = \frac{l\mu_a ln(a/b)i}{\alpha + (2\pi - \alpha)(\mu_a/\mu_b)} \tag{4}$$

Because $\lambda = Li$, it follows that the inductance of the shorted coaxial section is as given.

(e) Since the field inside the volume of the inner conductor is zero, it follows from Ampère's continuity condition, (9.5.3), that

$$K_z = H_\phi \Rightarrow K_z = \begin{cases} A/b = i/b\left[\alpha + \beta\frac{\mu_a}{\mu_b}\right]; & \text{region (a)} \\ C/b = i(\mu_a/\mu_b)/b\left(\alpha + \beta\frac{\mu_a}{\mu_b}\right); & \text{region (b)} \end{cases} \tag{5}$$

Note that these surface current densities are not equal, but are consistent with having the total current in the inner conductor equal to i.

$$i = \frac{A}{b}(\alpha b) + \frac{C}{b}\beta b = \frac{i\alpha}{\alpha + \beta(\frac{\mu_a}{\mu_b})} + \frac{i(\mu_a/\mu_b)\beta}{\alpha + \beta\frac{\mu_a}{\mu_b}} \tag{6}$$

9.5.6 The H-field changes as one proceeds from medium μ_a to the medium μ_b. For the contour shown, Ampère's law gives (see Fig. S9.5.6):

$$H_z^a a + H_z^b(w - a) = i$$

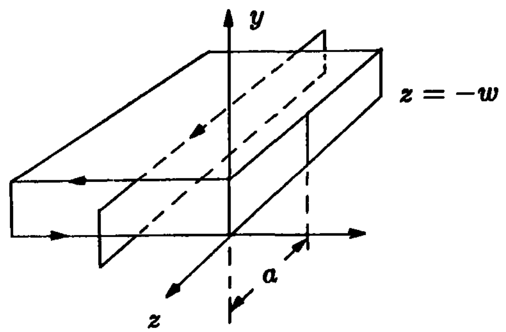


Figure S9.5.6

The flux continuity gives

$$\mu_a H_z^a = \mu_b H_z^b$$

Therefore

$$H_z^a\left[a + \frac{\mu_a}{\mu_b}(w - a)\right] = i$$

and the flux linkage is

$$\lambda = dl\mu_a H_z^a = \frac{\mu_a dl i}{a + \frac{\mu_a}{\mu_b}(w - a)}$$

and the inductance is

$$L = \frac{\lambda}{i} = \frac{dl}{\frac{a}{\mu_a} + \frac{w-a}{\mu_b}}$$

9.6 FIELDS IN PIECE-WISE UNIFORM MAGNETICALLY LINEAR MATERIALS

9.6.1 (a) At the interface, Ampère's law and flux continuity require the boundary conditions

$$H_z^a - H_z^b = -\frac{\partial \Psi^a}{\partial z} + \frac{\partial \Psi^b}{\partial z} = K_o \cos\beta z \tag{1}$$

$$\mu_o H_y^a - \mu H_y^b = -\mu_o \frac{\partial \Psi^a}{\partial y} + \mu \frac{\partial \Psi^b}{\partial y} = 0 \tag{2}$$

The z dependence of the surface current density in (1) suggests that the magnetic potential be taken as the solutions to Laplace's equation

$$\Psi = \begin{cases} Ae^{-\beta y}\sin\beta z \\ Ce^{\beta y}\sin\beta z \end{cases} \tag{3}$$

Substitution of these relations into (1) and (2) gives

$$\begin{bmatrix} -\beta & \beta \\ \mu_o\beta & \mu\beta \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} K_o \\ 0 \end{bmatrix} \tag{4}$$

and hence

$$A = -\frac{\mu}{\mu_o}\frac{K_o}{\beta\left[1+\frac{\mu}{\mu_o}\right]}; \quad C = -\frac{\mu_o}{\mu}A \tag{5}$$

Thus, the magnetic potential is as given.

(b) In the limit where the lower region is infinitely permeable, the boundary condition at $y = 0$ for the upper region becomes

$$H_z^a(y=0) = -\frac{\partial \Psi^a}{\partial z}(y=0) = K_o\cos\beta z \tag{6}$$

This suggests a solution in the form of (3a). Substitution gives

$$A = -K_o/\beta \tag{7}$$

which is the same as the limit $\mu/\mu_o \to \infty$ of (5a).

(c) Given the solution in the upper region, flux continuity determines the field in the lower region. In the lower region, the condition at $y = 0$ is

$$\frac{\partial \Psi^b}{\partial y}(y=0) = \frac{\mu_o}{\mu}\frac{\partial \Psi^a}{\partial y}(y=0) = \frac{\mu_o}{\mu}K_o\sin\beta z \tag{8}$$

and it follows that

$$\beta C\sin\beta z = \frac{\mu_o}{\mu}K_o\sin\beta z \Rightarrow C = \frac{\mu_o}{\mu}K_o/\beta \tag{9}$$

which agrees with (5) in the limit where $\mu/\mu_o \gg 1$.

9.6.2 (a) The H-field is the gradient of a Laplacian potential to the left and right of the current sheet. Because $\mathbf{n} \times \mathbf{H} = 0$ at $y = \pm d$, $\Psi =$ const.

(b) At the sheet

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \tag{1}$$

and thus

$$-\frac{\partial \Psi^a}{\partial y} + \frac{\partial \Psi^b}{\partial y} = K_o \sin\left(\frac{\pi y}{2d}\right) \tag{2}$$

From flux density continuity we obtain

$$\mu_o \frac{\partial \Psi^a}{\partial x} = \mu_o \frac{\partial \Psi^b}{\partial x} \tag{3}$$

From (2) we see that Ψ^a and $\Psi^b \propto \cos(\pi y/2d)$ and thus

$$\Psi^a = A \cos\left(\frac{\pi y}{2d}\right) e^{-\pi x/2d} \tag{4a}$$

$$\Psi^b = B \cos\left(\frac{\pi y}{2d}\right) e^{\pi x/2d} \tag{4b}$$

This satisfies $\Psi =$ const at $y = \pm d$. We have from (3)

$$-\frac{\pi}{2d} A = \frac{\pi}{2d} B$$

and from (2)

$$\frac{\pi}{2d} A - \frac{\pi}{2d} B = K_o$$

giving

$$A = -B = \frac{K_o}{(\pi/d)}$$

Therefore

$$\Psi^{a}_{b} = \pm \frac{K_o}{(\pi/d)} \cos\left(\frac{\pi y}{2d}\right) e^{\mp \pi x/2d}$$

9.6.3 (a) Boundary conditions at $r = R$ are

$$H_\phi^a - H_\phi^b = -\frac{1}{R}\frac{\partial \Psi^a}{\partial \phi} + \frac{1}{R}\frac{\partial \Psi^b}{\partial \phi} = \frac{Ni}{2R} \sin\phi \tag{1}$$

$$B_r^a - B_r^b = -\mu \frac{\partial \Psi^a}{\partial r} + \mu_o \frac{\partial \Psi^b}{\partial r} = 0 \tag{2}$$

To satisfy these, it is appropriate to choose as solutions to Laplace's equation outside and inside the winding

$$\Psi = \begin{cases} (A/r)\cos\phi; & R < r \\ Cr\cos\phi; & r < R \end{cases} \tag{3}$$

Substitution of these relations in (1) and (2) shows that the coefficients are

$$A = \frac{NiR}{2[1+(\mu/\mu_o)]}; \quad C = -\frac{\mu}{\mu_o}\frac{A}{R^2} \tag{4}$$

and substitution of these into (3) results in the given expressions for the magnetic potential.

(b) The magnetic field intensity inside is uniform and x directed. Thus, the integration over the area of the loop amounts to a multiplication by the area. The component normal to the loop is $H_x \cos\alpha, H_x = -C$. Therefore,

$$\lambda = n\mu_o H_x \cos\alpha(2al) = -n\mu_o C \cos\alpha(2al) \tag{5}$$

With no current in the rotating loop, the flux linkage-current relation reduces to $\lambda = L_m i$, so the desired mutual inductance multiplies i in (5).

9.6.4 (a) It is best to find the H-field first, then determine the vector potential. The vector potential can then be used to find the flux according to 8.6.5. Look at stator field first ($r = a$). The scalar potential of the stator that vanishes at $r = b$ is

$$\Psi^s = A\cos\phi\left(\frac{r}{b} - \frac{b}{r}\right) \tag{1}$$

On surface of stator

$$\mathbf{n} \times \mathbf{H}^s = \mathbf{K} \tag{2}$$

where $\mathbf{n} = -\mathbf{i}_r$.

$$\mathbf{K} = \mathbf{i}_z i_1 N_s \sin\phi \tag{3}$$

where the stator wire density N_s is

$$N_s = \frac{N_1}{2a}$$

with N_1 the total number of turns. Since

$$\mathbf{n} \times \mathbf{H}^s = \frac{1}{r}\frac{\partial\Psi}{\partial\phi}\Big|_{r=a}\mathbf{i}_z = -\frac{1}{a}A\sin\phi\left(\frac{a}{b} - \frac{b}{a}\right)\mathbf{i}_z$$

We find

$$A = -\frac{N_1}{2}i_1\frac{ab}{a^2-b^2} \tag{5}$$

The $\mathbf{H}$ field due to stator windings is:

$$\mathbf{H}^s = \frac{N_1 i_1}{2}\frac{a}{a^2-b^2}\left[\left(1+\frac{b^2}{r^2}\right)\cos\phi\mathbf{i}_r - \left(1-\frac{b^2}{r^2}\right)\sin\phi\mathbf{i}_\phi\right] \tag{6}$$

The rotor potential is

$$\Psi^r = B\cos(\phi-\theta)\left(\frac{r}{a} - \frac{a}{r}\right) \tag{7}$$

We find similarly,

$$B = \frac{N_2}{2} i_2 \frac{ab}{b^2 - a^2} \tag{8}$$

The H-field is

$$\mathbf{H}^r = \frac{N_2 i_2}{2} \frac{b}{a^2 - b^2} \left[\left(1 + \frac{a^2}{r^2}\right) \cos(\phi - \theta) \mathbf{i}_r - \left(1 - \frac{a^2}{r^2}\right) \sin(\phi - \theta) \mathbf{i}_\phi\right] \tag{9}$$

Fluxes linking the windings can be obtained by evaluating $\int_S \mathbf{B} \cdot d\mathbf{a}$ or by use of the vector potential A_z. Here we use A_z. The vector potential is z-directed and is related to the $\mathbf{B}$ field by

$$\nabla \times \mathbf{A} = \mathbf{B} = \mu_o \mathbf{H} = \frac{1}{r} \frac{\partial A_z}{\partial \phi} \mathbf{i}_r - \frac{\partial A_z}{\partial r} \mathbf{i}_\phi \tag{10}$$

From the r-components of $\mathbf{H}$ we find by inspection

$$\begin{aligned} A_z = \mu_o & \frac{N_1 i_1}{2} \frac{ab}{a^2 - b^2} \left(\frac{r}{b} + \frac{b}{r}\right) \sin \phi \\ + \mu_o & \frac{N_2 i_2}{2} \frac{ba}{a^2 - b^2} \left(\frac{r}{a} + \frac{a}{r}\right) \sin(\phi - \theta) \end{aligned} \tag{11}$$

Of course, the ϕ component gives the same result.

(b) The inductances follow from evaluation of the flux linkages. The flux of one stator turn, extending from $\phi = -\phi_o$ to $\phi = \pi - \phi_o$ is

$$\begin{aligned} \Phi_\lambda^{ss}(\phi_o) &= l[A_z^s(\pi - \phi_o) - A_z^s(-\phi_o)]_{r=a} \\ &= \mu_o l \frac{N_1 i_1}{2} \frac{ab}{a^2 - b^2} \left(\frac{a}{b} + \frac{b}{a}\right) 2 \sin \phi_o \end{aligned} \tag{12}$$

The inductance is obtained by computing the flux linkage

$$\lambda_{11} = \int_{\phi_o = 0}^{\pi} \frac{N_1 \Phi_\lambda^{ss}(\phi_o)}{2a} a d\phi_o = \mu_o l N_1^2 i_1 \frac{a^2 + b^2}{a^2 - b^2} \tag{13}$$

The inductance is

$$L_{11} = \frac{\lambda_{11}}{i_1} = \mu_o l N_1^2 \frac{a^2 + b^2}{a^2 - b^2} \tag{14}$$

In a similar way we find

$$L_{22} = \frac{\lambda_{22}}{i_2} = \mu_o l N_2^2 \frac{a^2 + b^2}{a^2 - b^2} \tag{15}$$

The mutual inductance is evaluated from Φ_λ^{rs}, the flux due to the field produced by the stator, passing a turn of the rotor extending from $-\phi_o + \theta$ to $\pi - \phi_o + \theta$

$$\begin{aligned} \Phi_\lambda^{rs} &= l[A_z^s(\pi - \phi_o + \theta) - A_z^s(-\phi_o + \theta)]_{r=b} \\ &= \mu_o l N_1 i_1 \frac{2ab}{a^2 - b^2} \sin(\phi_o - \theta) \end{aligned} \tag{16}$$

The mutual flux linkage is

$$\lambda_{21} = \int_{\phi_o=0}^{\pi} \frac{N_2}{2b}\Phi_\lambda^{rs} b\,d\phi_s = \mu_o l N_1 N_2 i_1 \frac{2ab}{a^2-b^2}\cos\theta \tag{17}$$

$$L_{21} = \mu_o l N_1 N_2 \frac{2ab}{a^2-b^2}\cos\theta$$

A similar analysis gives L_{12} which is found equal to L_{21}. From energy arguments presented in Chap. 11, it can be proven that $L_{12} = L_{21}$ is a necessity. Note that

$$L_{21}^2 = L_{12}L_{21} \le L_{11}L_{22}$$

9.6.5 (a) The vector potential of the wire carrying a current I is

$$A_z = -\frac{\mu_o I}{2\pi} ln(\frac{r_1}{a}) \tag{1}$$

where

$$r_1 = \sqrt{(y-h)^2 + x^2}$$

and a is a reference radius. If we mount an image of magnitude i_b at the position $x=0, y=-h$, we have

$$A_z = -\frac{\mu_o i_b}{2\pi} ln(\frac{r_2}{a}) \tag{2}$$

where

$$r_2 = \sqrt{(y+h)^2 + x^2}$$

The field in the μ-material is represented by the vector potential

$$A_z = -\frac{\mu_o i_a}{2\pi} ln(\frac{r_1}{a}) \tag{3}$$

where i_a is to be determined. We find for the $\mathbf{B} = \mu\mathbf{H}$ field

$$\begin{aligned}\mu_o\mathbf{H} = \nabla\times\mathbf{A} &= \mathbf{i_x}\frac{\partial A_z}{\partial y} - \mathbf{i_y}\frac{\partial A_z}{\partial x} \\ &= -\frac{\mu_o}{2\pi}\left\{\mathbf{i_x}\left(I\frac{y-h}{\sqrt{(y-h)^2+x^2}^3} + i_b\frac{y+h}{\sqrt{(y+h)^2+x^2}^3}\right)\right. \\ &\left. -\mathbf{i_y}\left(I\frac{x}{\sqrt{(y-h)^2+x^2}^3} + i_b\frac{x}{\sqrt{(y+h)^2+x^2}^3}\right)\right\}; \qquad y>0\end{aligned} \tag{4a}$$

$$\mu\mathbf{H} = -\frac{\mu_o i_a}{2\pi}\frac{1}{\sqrt{(y-h)^2+x^2}^3}\{\mathbf{i_x}(y-h) - \mathbf{i_y}x\}; \qquad y<0 \tag{4b}$$

At $y = 0$ we match H_x and μH_y obtaining

$$I - i_b = \frac{\mu_o i_a}{\mu} \tag{5}$$

$$I + i_b = i_a \tag{6}$$

By adding the two equations we obtain:

$$i_a = \frac{2I}{1 + \frac{\mu_o}{\mu}} \tag{7}$$

and thus

$$i_b = \frac{1 - \frac{\mu_o}{\mu}}{1 + \frac{\mu_o}{\mu}} \tag{8}$$

(b) When $\mu \gg \mu_o$, then $\mathbf{H}_{\tan} \simeq 0$ on the interface. We need an image that cancels the tangential magnetic field, i.e.

$$i_b = I$$

(c) We have a normal flux as found in (4a) for $i_b = I$

$$\mu_o H_y = \frac{\mu_o}{2\pi} 2I \frac{x}{\sqrt{h^2 + x^2}^3}$$

This normal flux must be continuous. It can be produced by a fictitious source at $y = h$ of magnitude $i_a = 2I$. The field is (compare (4b))

$$H = -\frac{\mu_o}{\mu}\frac{I}{\pi}\frac{1}{\sqrt{(y-h)^2 + x^2}^3}\{\mathbf{i_x}(y-h) - \mathbf{i_y}x\}$$

(d) When $\mu \gg \mu_o$, we find from (2) and (8)

$$i_a \simeq 2I$$

$$i_b \simeq I$$

in concordance with the above!

9.6.6 The field in the upper region can be taken as the sum of the field due to the wire, a particular solution, and the field of an image current at the position $y = -h, x = 0$, a homogeneous solution. The polarity of this latter current is determined by which of the two physical situations is of interest.

(a) If the material is perfectly conducting, there is no flux density normal to its surface in the upper region. In this case, the image current must be in the $-z$ direction so that its y directed field is in the opposite direction to that of the actual current in the plane $y = 0$. The field at $y = h, x = 0$ due to this image current is

$$\mu_o \mathbf{H} = \frac{\mu_o i}{(2\pi)(2h)} \mathbf{i}_x \tag{1}$$

and therefore the force per unit length is as given. The wire is repelled by a perfectly conducting wall.

(b) In this case, there is no tangential magnetic field intensity at the interface, so the image current is in the same direction as the actual current. As a result, the field intensity of the image current, evaluated at the position of the actual current, is the negative of that given by (1). The resulting force is also the negative of that for the perfect conductor, as given. The wire is attracted by a permeable wall.

9.6.7 (a) In this version of an "inside-outside" problem, the "inside" region is the highly permeable one. The field intensity must be $H_o \mathbf{i}_z$ in that region and have no tangential component in the plane $z = 0$. The latter condition is satisfied by taking the configuration as being that of a spherical cavity centered at the origin with the surrounding highly permeable material extending to infinity in the $\pm z$ directions. At the surface where $r = a$, the normal flux density in the highly permeable material tends to be zero. Thus, the approximate field takes the form

$$\Psi^a = -H_o r \cos\theta + A \frac{\cos\theta}{r^2} \tag{1}$$

where the coefficient A is adjusted to make

$$\mathbf{n} \cdot \mathbf{B}\big|_{r=a} = 0 \Rightarrow \frac{\partial \Psi^a}{\partial r}(r = a) = 0 \tag{2}$$

Substitution of (1) into (2) gives $A = -a^3 H_o/2$ and hence the given magnetic potential.

(b) Because there is no surface current density at $r = A$, the magnetic potential (the tangential field intensity) is continuous there. Thus, for the field inside

$$\Psi^b(r = a) = \Psi^a(r = a) = -3H_o a/2 \tag{3}$$

To satisfy this condition, the interior magnetic scalar potential is taken to have the form

$$\Psi^b = Cr\cos\theta = Cz \tag{4}$$

Substitution of this expression into (3) to evaluate $C = -3H_o/2$ results in the given expression.

9.6.8 The perfectly permeable walls force the boundary condition $\Psi = 0$ on the surfaces. The bottom magnetic surface charge density is neutralized by the image charges in the wall (see Fig. S9.6.8). The top magnetic surface charge density produces a magnetic potential Ψ that is

$$\Psi = A\sinh\beta(y-a)\cos\beta x \qquad y > d/2 \tag{1a}$$

and

$$\Psi = B\sinh\beta\Big(y+\frac{d}{2}\Big)\cos\beta x \qquad y < d/2 \tag{1b}$$

At the interface at $y = d/2, \Psi$ is continuous

$$A\sinh\beta\Big(\frac{d}{2}-a\Big) = B\sinh\beta d \tag{2}$$

and thus

$$B = -A\frac{\sinh\beta(a-\frac{d}{2})}{\sinh\beta d} \tag{3}$$

The magnetic surface charge density at $y = d/2$ is

$$\sigma_m = \mu_o M_o \cos\beta x \tag{4}$$

It forces a jump of $\partial\Psi/\partial y$ at $y = d/2$:

$$-\frac{\partial\Psi}{\partial y}\bigg|_{y=d/2_+} + \frac{\partial\Psi}{\partial y}\bigg|_{y=d/2_-} = M_o\cos\beta x \tag{5}$$

and we find

$$-A\cosh\beta\Big(\frac{d}{2}-a\Big) + B\cosh\beta d = \frac{M_o}{\beta} \tag{6}$$

Using (3) we obtain

$$\begin{aligned} A &= -\frac{M_o}{\beta}\frac{\sinh\beta d}{\cosh\beta(\frac{d}{2}-a)\sinh\beta d - \cosh\beta d\sinh\beta(\frac{d}{2}-a)} \\ &= -\frac{M_o}{\beta}\frac{\sinh\beta d}{\sinh\beta(\frac{d}{2}+a)} \end{aligned} \tag{7}$$

The vertical component of B, B_y, above the tape, for $y > d/2$, is

$$B_y = -\mu_o\frac{\partial\Psi}{\partial y} = \mu_o M_o\frac{\sinh\beta d}{\sinh\beta(\frac{d}{2}+a)}\cosh\beta(y-a)\cos\beta x \tag{8}$$

Note that in the limit $a \to d/2$, the flux is simply $\mu_o M_o$ as expected. If the tape moves, $\cos\beta x$ has to be expressed as $\cos\beta(x'-Ut)$. The flux is

$$\lambda = wN\mu_o M_o\frac{\sinh\beta d}{\sinh\beta(\frac{d}{2}+a)}\cosh\beta\Big(h+\frac{d}{2}-a\Big)\times\int_{-l/2}^{l/2}\cos\beta(x'-Ut)dx' \tag{9}$$

The integral evalues to

$$\frac{1}{\beta}\Big[\sin\beta\Big(\frac{l}{2}-Ut\Big)+\sin\beta\Big(\frac{l}{2}+Ut\Big)\Big] = \frac{2}{\beta}\sin\beta\frac{l}{2}\cos\beta Ut \tag{10}$$

and from here on one proceeds as in the Example 9.3.2.

$$v_o = \frac{d\lambda}{dt}$$

9.6.9 In terms of the magnetic scalar potential, boundary conditions are

$$\Psi(x,b) = 0; \quad \Psi(x,0) = 0 \tag{1}$$

$$H_y = -\frac{\partial \Psi}{\partial y}(0,y) = -K_o \cos\frac{\pi y}{a}; \quad \frac{\partial \Psi}{\partial y}(b,y) = K_o \cos\frac{\pi y}{a} \tag{2}$$

To satisfy the first pair of these while matching the y dependence of the second pair, the potential is taken as having the y dependence $\sin(\pi y/a)$. In terms of Ψ, the conditions at the surfaces $x = 0$ and $x = b$ are even with respect to $x = b/2$. Thus, the combination of $\exp(\pm\pi x/a)$ chosen to complete the solution to Laplace's equation is even with respect to $x = b/2$.

$$\Psi = A \cosh\left[\frac{\pi}{a}\left(x - \frac{b}{2}\right)\right] \sin\left(\frac{\pi y}{a}\right) \tag{3}$$

Thus, both of the relations (2) are satisfied by making the coefficient A equal to

$$A = \frac{aK_o}{\pi \cosh(\pi b/2a)} \tag{4}$$

9.6.10 The solution can be divided into a particular part due to the current density in the wire and a homogeneous part associated with the field that is uniformly applied at infinity. Because of the axial symmetry in the absence of the applied field, the particular part can be found using Ampère's integral law. Thus, from an integration at a constant radius r, it follows that

$$\begin{aligned} H_{\phi p} 2\pi r &= \pi r^2 J_o; \quad r < R \\ H_{\phi p} 2\pi r &= \pi R^2 J_o; \quad R < r \end{aligned} \tag{1}$$

so that the particular field intensity is

$$H_{\phi p} = \begin{cases} rJ_o/2; & r < R \\ R^2 J_o/2r; & R < r \end{cases} \tag{2}$$

in polar coordinates

$$\mathbf{H} = \frac{1}{\mu}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi}\mathbf{i}_r - \frac{\partial A_z}{\partial r}\mathbf{i}_\phi\right) \tag{3}$$

and it follows from (2), integrated in accordance with (3), that

$$A_{zp} = \begin{cases} -\mu_b r^2 J_o/4; & r < R \\ -\frac{1}{2}\mu_a J_o R^2 ln(r/R) - \frac{1}{4}\mu_b R^2 J_o; & R < r \end{cases} \tag{4}$$

In view of the applied field, the homogeneous solution is assumed to take the form

$$A_{zh} = \begin{cases} Dr \sin\phi; & r < R \\ -\mu_a H_o r \sin\phi + C\frac{\sin\phi}{r}; & R < r \end{cases} \tag{5}$$

The coefficients C and D are adjusted to satisfy the boundary conditions at $r = R$,

$$A_z^a - A_z^b = 0 \tag{6}$$

$$-\frac{1}{\mu_a}\frac{\partial A_z^a}{\partial r} + \frac{1}{\mu_b}\frac{\partial A_z^b}{\partial r} = 0 \tag{7}$$

The first of these guarantees that the flux density normal to the surface is continuous at $r = R$ while the second requires continuity of the tangential magnetic field intensity. Substitution of (5) into these relations gives a pair of equations that can be solved for the coefficients C and D.

$$\begin{bmatrix} 1/R & -R \\ 1/\mu_a R^2 & 1/\mu_b \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \mu_a H_o R \\ -H_o \end{bmatrix} \tag{8}$$

The coefficients which follow are substituted into (5) and those expressions respectively added to (4) provide the given expressions.

✓

9.6.11 (a) Given the magnetization, the associated H is found by first finding the distribution of magnetic charge. There is none in the volume, where $\mathbf{M}$ is uniform. The surface magnetization charge density at the surface, say at $r = R$, is

$$\sigma_{sm} = -\mu_o \mathbf{n} \cdot (\mathbf{M}^a - \mathbf{M}^b) = \mu_o M \mathbf{n} \cdot \mathbf{i_r} = \mu_o M \cos\theta \tag{1}$$

Thus, boundary conditions to be satisfied at $r = R$ by the scalar magnetic potential are

$$\Psi^a - \Psi^b = 0 \tag{2}$$

$$-\mu_o \frac{\partial \Psi^a}{\partial r} + \mu_o \frac{\partial \Psi^b}{\partial r} = \mu_o M \cos\theta \tag{3}$$

From the θ dependence in (3), it is reasonable to assume that the fields outside and inside the sphere take the form

$$\Psi = \begin{cases} -H_o r \cos\theta + A\frac{\cos\theta}{r^2} \\ -H r \cos\theta \end{cases} \tag{4}$$

Substitution of these expressions into (2) and (3) gives

$$H = H_o - \frac{1}{3}M \Rightarrow M = 3(H_o - H) \tag{5}$$

Thus, it follows that

$$B \equiv \mu_o(H + M) = \mu_o(-2H + 3H_o) \tag{6}$$

(b) This relation between B and H is linear and therefore a straight line in the $B - H$ plane. Where $B = 0$ in (6), $H = 3H_o/2$ and where $H = 0, B = 3\mu_o H_o$. Thus, the load line is as shown in Fig. S9.6.11.

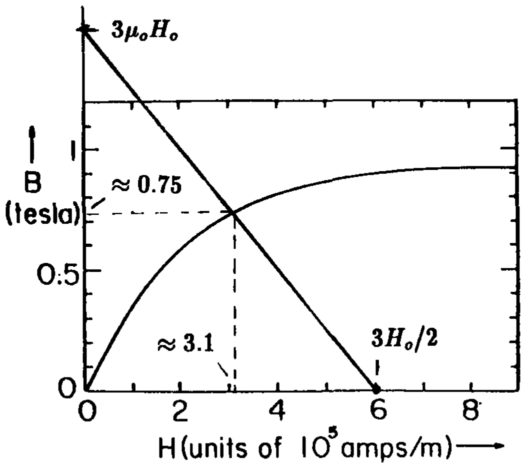


Figure S9.6.11

(c) The values of B and H within the sphere are given by the intersection of the load line with the saturation curve representing the constitutive law for the magnetization of the sphere.

(d) For the specific values given, the load line is as shown in Fig. S9.6.11. The values of B and H deduced from the intersection are also indicated in the figure.

9.6.12 We assume that the field is uniform inside the cylinder and then confirm the correctness of the assumption. The scalar potentials inside and outside the cylinder are

$$\Psi = \begin{cases} -H_o R \cos\phi(r/R) + A\cos\phi(R/r) & r > R \\ C\cos\phi(r/R) & r < R \end{cases} \tag{1}$$

Because Ψ is continuous at $r = R$

$$-H_o R + A = C \tag{2}$$

If there is an internal uniform magnetization $\mathbf{M} = M\mathbf{i}_x$, then

$$\mathbf{n}\cdot\mathbf{M} = M\cos\phi \tag{3}$$

The boundary condition for the normal component of $\mu_o\mathbf{H}$ at $r = R$ gives

$$\left(\mu_o H_o + \mu_o\frac{A}{R}\right) + \mu_o\frac{C}{R} = \mu_o M \tag{4}$$

Therefore, from (2) and (4)

$$\frac{C}{R} = -H_o + \frac{M}{2} \tag{5}$$

and the internal ($r < R$)$\mathbf{H}$ field is (we use no subscripts to denote the field internal to cylinder):

$$\mathbf{H} = (H_o - \frac{M}{2})\mathbf{i_x} \tag{6}$$

The magnetization causes a "demagnetization" field of magnitude $M/2$. We can construct "load line" to find internal $\mathbf{B}$ graphically. Since

$$B = \mu_o(H + M) \tag{7}$$

we find from (6) for the magnitude of the internal H field

$$H = \left(H_o - \frac{M+H}{2} + \frac{H}{2}\right) = H_o - \frac{B}{2\mu_o} + \frac{H}{2} \tag{8}$$

or

$$H = 2H_o - \frac{B}{\mu_o} \tag{9}$$

The two intersection points are (see Fig. S9.6.12)

$$H = 2H_o \quad for \quad B = 0$$

and

$$B = 2\mu_o H_o \quad \text{for} \quad H = 0$$

We read off the graph: $B = 0.67$ tesla, $H = 2.5 \times 10^5$ amps/m.

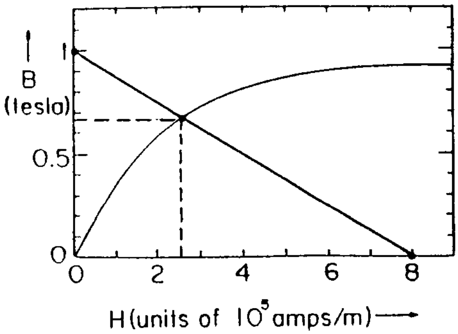


Figure S9.6.12

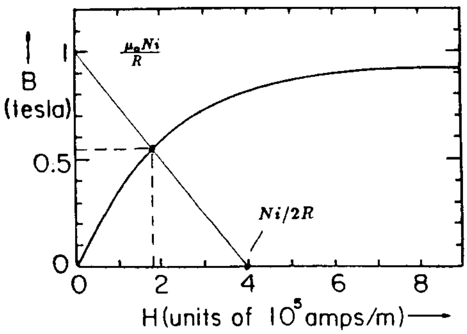


Figure S9.6.13

9.6.13 The relation between the current in the winding and H and M in the sphere are given by (9.6.15).

$$M = 3(\frac{Ni}{3R} - H) \tag{1}$$

From this, the load line follows as

$$B \equiv \mu_o(H + M) = \mu_o(\frac{Ni}{R} - 2H) \tag{2}$$

The intercepts that can be used to plot this straight line are shown in Fig. S9.6.13. The line shown is for the given specific numbers. Thus, within the sphere, $B \approx 0.54$ and $H \approx 1.8$.

9.7 MAGNETIC CIRCUITS

9.7.1 (a) Because of the high core permeability, the fields are approximated by taking an "inside-outside" approach. First, the field inside the core is approximately subject to the condition that

$$\mathbf{n}\cdot\mathbf{B} = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b \tag{1}$$

which is satisfied because the given field distribution has no radial component. Further, Ampère's integral law requires that

$$\int_0^{2\pi} H_\phi r d\phi = Ni = \int_0^{2\pi} \frac{Ni}{2\pi r} r d\phi = Ni \tag{2}$$

In terms of the magnetic scalar potential, with the integration constant adjusted to define the potential as zero at $\phi = \pi$,

$$\begin{aligned} -\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = \frac{Ni}{2\pi r} \Rightarrow \Psi &= -\frac{Ni}{2\pi}\phi + \text{const} \\ &= \frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right) \end{aligned} \tag{3}$$

This potential satisfies Laplace's equation, has no radial derivative on the inside and outside walls, suffers a discontinuity at $\phi = 0$ that is Ni and has a continuous derivative normal to the plane of the wires at $\phi = 0$ (as required by flux continuity). Thus, the proposed solution meets the required conditions and is uniquely specified.

(b) In the interior region, the potential given by (3), evaluated at $r = b$, provides a boundary condition on the field. This potential (and actually any other potential condition at $r = b$) can be represented by a Fourier series, so we represent the solution for $r < b$ by solutions to Laplace's equation taking the form

$$\Psi = \sum_{m=1}^{\infty} \psi_m \sin m\phi \left(\frac{r}{b}\right)^m \tag{4}$$

Because the region includes the origin, solutions r^{-m} are omitted. Thus, at the boundary, we require that

$$\frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right) = \sum_{m=1}^{\infty} \psi_m \sin m\phi \tag{5}$$

Multiplication by $\sin n\phi$ and integration gives

$$\begin{aligned} \int_0^{2\pi} \frac{Ni}{2}\left(1 - \frac{\phi}{\pi}\right)\sin(n\phi)d\phi &= \int_0^{2\pi} \sum_{m=1}^{\infty} \psi_m \sin m\phi \sin n\phi d\phi \\ &= \psi_n \pi \end{aligned} \tag{6}$$

Thus,

$$\psi_m = \frac{Ni}{2\pi}\int_0^{2\pi} \left(1 - \frac{\phi}{\pi}\right)\sin m\phi d\phi = \frac{Ni}{m\pi} \tag{7}$$

Substitution of this coefficient into (4) results in the given solution.

9.7.2 The approximate magnetic potential on the outer surface is

$$\Psi = \sum_{m=1}^{\infty} \frac{Ni}{m\pi} \sin m\phi \tag{1}$$

according to (b) of Prob. 9.7.1. The outside potential is a solution to Laplace's equation that must match (1) and decays to zero as $r \rightarrow \infty$. This is clearly

$$\Psi = \sum_{m=1}^{\infty} \frac{Ni}{m\pi} (a/r)^m \sin m\phi \tag{2}$$

9.7.3 Using contours C_1 and C_2 respectively, as defined in Fig. S9.7.3, Ampère's integral law gives

$$H_a a = Ni \Rightarrow H_a = Ni/a \tag{1}$$

$$H_b b = Ni \Rightarrow H_b = Ni/b \tag{2}$$

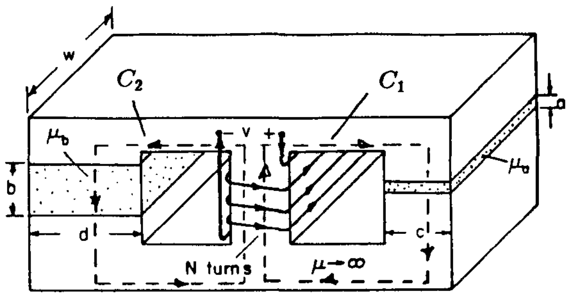


Figure S9.7.3

From the integral form of flux continuity, for a closed surface S that intersects the middle leg and passes through the gaps to right and left, we know that the flux through the middle leg is equal to the sum of those through the gaps. This flux is linked N times, so

$$\lambda = N(cw\mu_a H_a + dw\mu_b H_b) \tag{3}$$

Substitution of (1) and (2) into this expression gives

$$\lambda = N^2 w\Big(\frac{c\mu_a}{a} + \frac{d\mu_b}{b}\Big)i \tag{4}$$

where the coefficient of i is the given inductance.

9.7.4 The field in the gap due to the coil of N turns is approximately uniform because the hemisphere is small. From Ampère's law

$$Hh = Ni \tag{1}$$

where $\mathbf{H}$ directed downward is defined positive. This field is distorted by the sphere. The scalar magnetic potential around the sphere is

$$\Psi = R\frac{Ni}{h}\cos\theta[(r/R) - (R/r)^2] \tag{2}$$

where θ is the angle measured from the vertical axis. The field is

$$\mathbf{H} = -\frac{Ni}{h}\{\mathbf{i}_r \cos\theta[1 + 2(R/r)^2] - \mathbf{i}_\theta \sin\theta[1 - (R/r)^2]\} \tag{3}$$

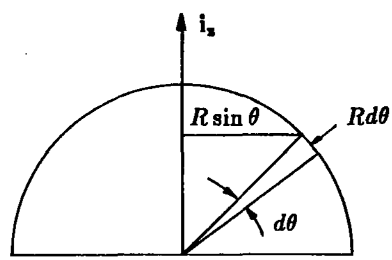


Figure S9.7.4

The flux linked by one turn at angle α is (see Fig. S9.7.4)

$$\begin{aligned}\Phi_\lambda &= \int_0^\alpha \mu_o H_r 2\pi R^2 \sin\theta d\theta \\ &= -3\mu_o \frac{Ni}{h} 2\pi R^2 \int_0^\alpha \sin\theta\cos\theta d\theta \\ &= -\frac{3\mu_o}{2}\frac{Ni}{h}\pi R^2(1 - \cos 2\alpha)\end{aligned} \tag{4}$$

But $1 - \cos 2\alpha = 2\sin^2\alpha$ which will be used below. The flux linkage is λ_{21} where 1 stands for the coil on the $\pi/2$ leg of the "circuit", 2 for the hemispherical coil

$$\begin{aligned}\lambda_{21} &= \int_0^{\pi/2} \Phi_\lambda \frac{n}{R}\sin\alpha R d\alpha \\ &= -\frac{3}{4}\mu_o\frac{Nn}{h}i\pi R^2 \int_0^{\pi/2} \sin^3\alpha d\alpha \\ &= -\mu_o\frac{Nn}{2h}\pi R^2 i\end{aligned} \tag{5}$$

The mutual inductance is

$$L_{21} = \frac{\lambda_{21}}{i} = -\mu_o\frac{Nn}{2h}\pi R^2 \tag{6}$$

9.7.5 In terms of the air-gap magnetic field intensities defined in Fig. S9.7.5, Ampère's integral law for a contour passing around the magnetic circuit through the two windings and across the two air-gaps, requires that

$$N_1 i_1 + N_2 i_2 = H_a x + H_b x \tag{1}$$

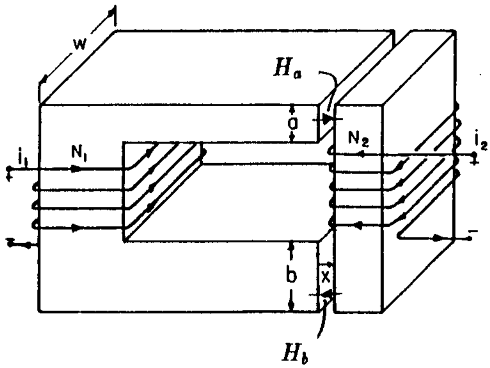


Figure S9.7.5

In terms of these same field intensities, flux continuity for a surface S that encloses the movable member requires that

$$aw\mu_o H_a = bw\mu_o H_b \Rightarrow H_b = \frac{a}{b} H_a \tag{2}$$

From these relations, it follows that

$$H_a = (N_1 i_1 + N_2 i_2)/x\left(1 + \frac{a}{b}\right) \tag{3}$$

The flux linking the first winding is that through either of the gaps, say the upper one, multiplied by N_1

$$\lambda_1 = N_1 aw\mu_o H_a = L_o(N_1^2 i_1 + N_1 N_2 i_2) \tag{4}$$

The second equation has been written using (3). Similarly, the flux linking the second coil is that crossing the upper gap multiplied by N_2.

$$\lambda_2 = N_2 aw\mu_o H_a = L_o(N_2 N_1 i_1 + N_2^2 i_2) \tag{5}$$

Identification of the coefficients of the respective currents in these two relations results in the given self and mutual inductances.

9.7.6 Denoting the H field in the gap of width x by H_x and that in the gap g by H_g, Ampère's integral law gives

$$\oint \mathbf{H} \cdot d\mathbf{s} = xH_x + gH_g = Ni \tag{1}$$

where flux continuity requires

$$\mu_o H_x \pi a^2 = \mu_o H_g 2\pi a d \tag{2}$$

Thus

$$xH_x + \frac{\pi a^2}{2\pi a d} gH_x = Ni \tag{3}$$

The flux is

$$\Phi_\lambda = \mu_o \pi a^2 H_x = \frac{\mu_o Ni}{\frac{x}{\pi a^2} + \frac{g}{2\pi a d}}$$

The inductance is

$$L = \frac{N\Phi_\lambda}{i} = \frac{\mu_o N^2}{\frac{x}{\pi a^2} + \frac{g}{2\pi a d}}$$

9.7.7 We pick two contours (Fig. S9.7.7) to find the H field which is indicated in the three gaps as H_a, H_b and H_c. The fields are defined positive if they point radially outward. From contour C_1:

$$(-H_a + H_b)g = N_1 i_1 \tag{1}$$

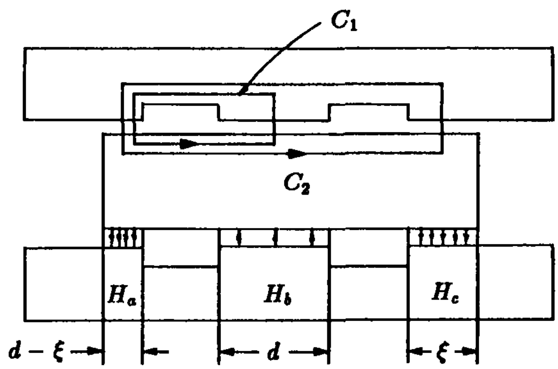


Figure S9.7.7

From contour C_2

$$(-H_a + H_c)g = N_1 i_1 + N_2 i_2 \tag{2}$$

The flux must be continuous so that

$$2\pi a[(d-\xi)\mu_o H_a + d\mu_o H_b + \xi \mu_o H_c] = 0 \tag{3}$$

We find from these three equations

$$H_a = -\frac{d+\xi}{2d}\frac{N_1 i_1}{g} - \frac{\xi}{2d}\frac{N_2 i_2}{g} \tag{4}$$

$$H_b = \frac{d-\xi}{2d}\frac{N_1 i_1}{g} - \frac{\xi}{2d}\frac{N_2 i_s}{g} \tag{5}$$

$$H_c = \frac{d-\xi}{2d}\frac{N_1 i_1}{g} + \frac{2d-\xi}{2d}\frac{N_2 i_2}{g} \tag{6}$$

The flux linkage of coil (1) is:

$$\begin{aligned}\lambda_1 &= -N_1 2\pi a(d-\xi)\mu_o H_a \\ &= \mu_o \pi a\left[\frac{d^2-\xi^2}{d}\frac{N_1^2 i_1}{g} + \frac{\xi(d-\xi)}{d}\frac{N_1 N_2 i_2}{g}\right]\end{aligned}$$

The flux linkage of coil (2) is:

$$\begin{aligned}\lambda_2 &= N_2 2\pi a\xi\mu_o H_c \\ &= \mu_o \pi a\left[\frac{\xi(2d-\xi)}{d}\frac{N_2^2 i_2}{g} + \frac{\xi(d-\xi)}{d}\frac{N_1 N_2 i_1}{g}\right]\end{aligned}$$

The inductance matrix is, by inspection

$$\begin{aligned}L_{11} &= \mu_o \pi a\frac{d^2-\xi^2}{dg}N_1^2 \\ L_{22} &= \mu_o \pi a\frac{\xi(2d-\xi)}{dg}N_2^2 \\ L_{12} = L_{21} &= \mu_o \pi a\frac{\xi(d-\xi)}{dg}N_1 N_2\end{aligned}$$

9.7.8 (a) Ψ must be constant over the surfaces of the central leg at $x = \mp l/2$ where we have perfectly permeable surfaces. In solving for the field internal to the central leg we assume that $\partial\Psi/\partial n = 0$ on the interfaces with μ_o.

(b) If we assume an essentially uniform field H_μ in the central leg, Ampère's integral law applied to a contour following the central leg and closing around the upper part of the magnetic circuit gives

$$H_\mu l = N_1 i_1 + N_2 i_2 \tag{1}$$

Therefore

$$\Psi(x=-l/2) = \frac{N_1 i_1 + N_2 i_2}{2} \tag{2}$$

$$\Psi(x = l/2) = -\frac{N_1 i_1 + N_2 i_2}{2} \tag{3}$$

(c) In region a, at $y = 0$, Ψ must decrease linearly from the value (2) to the value (1)

$$\Psi = -(N_1 i_1 + N_2 i_2)\frac{x}{l} \tag{4}$$

At

$$y = a, \quad \Psi = 0 \tag{5}$$

At $x = \pm l/2, 0 < y < a, \Psi$ must change linearly from (2) and (3) respectively, to zero

$$\Psi(x = -\frac{l}{2}, y) = \frac{N_1 i_1 + N_2 i_2}{2}\frac{(a - y)}{a} \tag{6}$$

$$\Psi(x = \frac{l}{2}, y) = -\frac{N_1 i_1 + N_2 i_2}{2}\frac{(a - y)}{a} \tag{7}$$

(d) Ψ must obey Laplace's equation and match boundary conditions that vary linearly with x and y. An obvious solution is

$$\Psi = Axy + Bx + Cy$$

We have, at $y = 0$

$$Bx = -(N_1 i_1 + N_2 i_2)\frac{x}{l}$$

and thus

$$B = -\frac{N_1 i_1 + N_2 i_2}{l}$$

In a similar way we find at $y = a$

$$Aax + Bx + Ca = 0$$

and thus

$$C = 0, \quad Aa = -B$$

which gives

$$\Psi = \frac{(N_1 i_1 + N_2 i_2)}{la}[xy - ax]$$

9.7.9 From Ampère's integral law we find for the H fields

$$l_1 H_1 + l_2 H_2 = Ni + Kl_1 \tag{1}$$

where K is the ("surface-") current in the thin sheet. This surface current is driven by the electric field induced by Faraday's law

$$\begin{aligned} 2\frac{K}{\sigma\Delta}(3a + w) &= \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int \mu_o \mathbf{H} \cdot d\mathbf{a} \\ &= -\mu a w \frac{dH_1}{dt} \end{aligned} \tag{2}$$

Finally, the flux is continuous so that

$$\mu H_1 3aw = \mu H_2 aw \tag{3}$$

and

$$H_2 = 3H_1 \tag{4}$$

When we introduce complex notation and use (4) in (1) we find

$$\hat{H}_1(l_1 + 3l_2) = N\hat{i}_o + \hat{K}l_1 \tag{5}$$

and

$$\hat{K} = -\frac{j\omega\mu aw}{2(3a+w)}\sigma\Delta\hat{H}_1 \tag{6}$$

Introducing (6) into (5) yields

$$\hat{H}_1 = \frac{N\hat{i}_o}{(l_1 + 3l_2)}\frac{1}{1 + j\omega\tau_m} \tag{7}$$

where

$$\tau_m = \mu\sigma\Delta\frac{awl_1}{(l_1 + 3l_2)(6a + 2w)}$$

9.7.10 The cross-sectional areas of the legs to either side are half of that through the center leg. Thus, the flux density, B, tends to be the same over the cross-sections of all parts of the magnetic circuit. For this reason, we can expect that each point within the core will tend to be at the same operating point on the given magnetization characteristic. Thus, with H_g defined as the air-gap field intensity and H defined as the field intensity at each point in the core, Ampère's integral law requires that

$$2Ni = (l_1 + l_2)H + dH_g \tag{1}$$

In the gap, the flux density is $\mu_o H_g$ and that must be equal to the flux density just inside the adjacent pole faces.

$$\mu_o H_g = B \tag{2}$$

The given load-line is obtained by combining these relations. Evaluation of the intercepts of this line gives the line shown in Fig. S9.7.10. Thus, in the core, $B \approx 0.75$ Tesla and $H \approx 0.3 \times 10^4$ A/m.

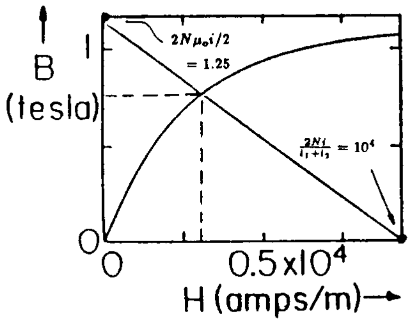


Figure S9.7.10

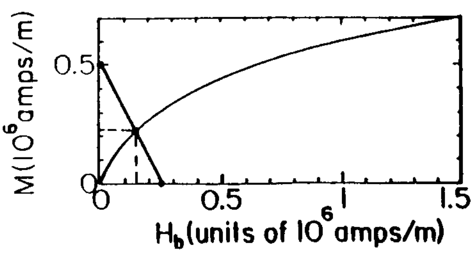


Figure S9.7.11

9.7.11 (a) From Ampère's integral law we obtain for the field H_b in the μ material and H_a in the air gap:

$$bH_b + aH_a = Ni \tag{1}$$

Further, from flux continuity

$$AB_b = A\mu_o H_a \tag{2}$$

and thus

$$H_b = \frac{Ni}{b} - \frac{a}{b}\frac{B_b}{\mu_o} \tag{3}$$

Now $B_b = \mu_o(H_b + M)$ and thus

$$H_b = \frac{Ni}{b} - \frac{a}{b}(H_b + M) \tag{4}$$

or

$$H_b = \frac{Ni}{a+b} - \frac{b}{a+b}M \tag{5}$$

This is the load line.

(b) The intercepts are at $M = 0$

$$H_b = \frac{Ni}{a+b} = \frac{Ni}{2a} = 0.25 \times 10^6$$

and at $H_b = 0$

$$M = \frac{Ni}{b} = 0.5 \times 10^6$$

We find

$$M = 0.22 \times 10^6 \text{ A/m}$$

$$H_b = 0.13 \times 10^6 \text{ A/m}$$

The B field is

$$\mu_o(H_b + M) = 4\pi \times 10^{-7}(0.13 + 0.22) \times 10^6 = 0.44\,\text{tesla}$$

SOLUTIONS TO CHAPTER 10

10.0 INTRODUCTION

10.0.1 (a) The line integral of the electric field along C_1 is from Faraday's law:

$$\oint_{C_1} \mathbf{E} \cdot dl = 0 \qquad (1)$$

because no flux is linked (see Fig. S10.0.1a). Therefore

$$-v + iR = 0$$

because the voltage drop across the resistor is iR. Hence

$$v = iR \qquad (2)$$

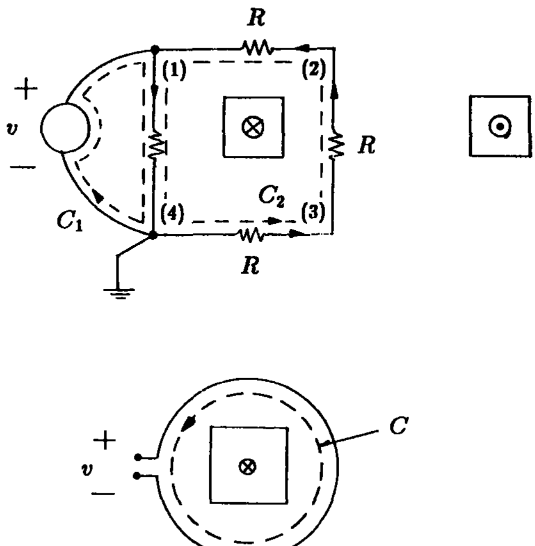


Figure S10.0.1a,b

The line integral along C_2 is

$$4iR = \frac{d\Phi_\lambda}{dt} \qquad (3)$$

which leads to

$$iR = \left(\frac{d\Phi_\lambda}{dt}\right)/4 \qquad (4)$$

Therefore, we find for the voltage across the voltmeter

$$v = \frac{1}{4}\frac{d\Phi_\lambda}{dt} \tag{5}$$

(b) With the voltmeter connected to 2, (1) becomes

$$v = 2iR$$

Using (2),

$$v = 2\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big]$$

and similarly for the other modes

$$v(3) = 3[iR] = 3\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big]$$

$$v(4) = 4iR = 4\Big[\frac{1}{4}\frac{d\Phi_\lambda}{dt}\Big] = \frac{d\Phi_\lambda}{dt}$$

For a transformer with a one turn secondary (see Fig. S10.0.1b),

$$v = \oint_C \mathbf{E}\cdot d\mathbf{l} = \frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{a} = \frac{d}{dt}\Phi_\lambda$$

10.0.2 Given the following one-turn inductor (Figs. S10.0.2a and S10.0.2b), we want to find (a) v_2 and (b) v_1. The current per unit length (surface current) flowing along the sheet is $K = i/d$. The tangential component of the magnetic field has to have the discontinuity K. A magnetic field (the gradient of a Laplacian potential)

$$\begin{aligned} H_z &= \frac{i}{d} \quad \text{inside} \\ &= 0 \quad \text{outside} \end{aligned} \tag{1}$$

has the proper discontinuity. This is the field in a single turn "coil" of infinite width d and finite $K = i/d$. It serves here as an approximation.

(a) v_2 can be found by applying Faraday's law to the contour C_2,

$$\oint_{C_2} \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_2} \mathbf{B}\cdot d\mathbf{a}$$

Using (1), and the constitutive relation $\mathbf{B} = \mu_o\mathbf{H}$,

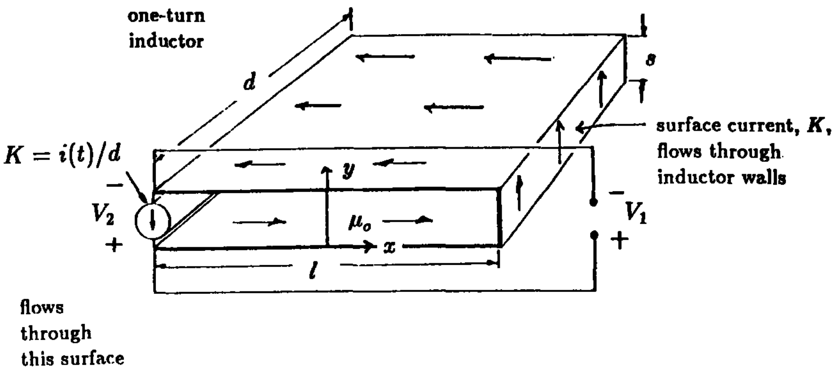
$$\int_{(A)C_2}^{(B)} \mathbf{E}\cdot d\mathbf{s} + \int_{(B)C_2}^{(A)} \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_2} \mu_o\frac{i(t)}{d}dxdy \tag{2}$$

Since the inductor walls are perfectly conducting, $\mathbf{E} = 0$ for the second integral on the left in (2). Therefore,

$$-v_2 = -\frac{d}{dt}\left(sl\mu_o \frac{i(t)}{d}\right)$$

or,

$$\Rightarrow v_2 = \frac{sl\mu_o}{d}\frac{di(t)}{dt}$$



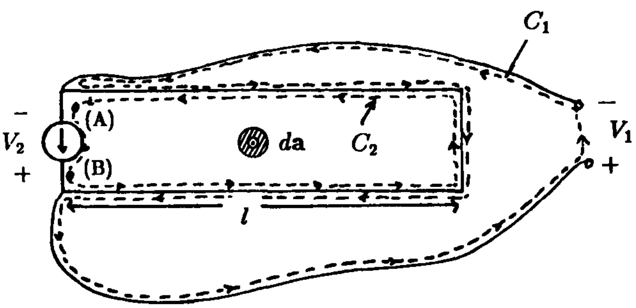


Figure S10.0.2

(b) Now, v_1 can be found by a similar method. Writing Faraday's law on C_1,

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_{S_1} \mathbf{B} \cdot d\mathbf{a} \tag{3}$$

Since C_1 does not link any flux, (3) can be written

$$-v_1 = -\frac{d}{dt}(0) = 0$$

10.1 MAGNETOQUASISTATIC ELECTRIC FIELDS IN SYSTEMS OF PERFECT CONDUCTORS

10.1.1 The magnetic field intensity from Problem 8.4.1 is

$$\mathbf{H} = \frac{i\pi R^2}{4\pi}\left[2\cos\theta\left(\frac{1}{r^3} - \frac{1}{b^3}\right)\mathbf{i_r} + \sin\theta\left(\frac{1}{r^3} + \frac{2}{b^3}\right)\mathbf{i}_\theta\right]$$

The E-field induced by Faraday's law has lines that link the dipole field and uniform field. By symmetry they are ϕ-directed. Using the integral law of Faraday's law using a spherical cap bounded by the contour $r =$ constant, $\theta =$ constant, we have

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{s} = 2\pi r\sin\theta E_\phi &= -\frac{d}{dt}\int_0^\theta \mu_o \mathbf{H}_r 2\pi r\sin\theta r d\theta \\ &= -\mu_o\frac{di}{dt}\frac{\pi R^2}{4\pi}\int_0^\theta 2\pi r^2 2\sin\theta\cos\theta d\theta\left(\frac{1}{r^3} - \frac{1}{b^3}\right) \\ &= -\mu_o\frac{di}{dt}\frac{\pi R^2}{4\pi}\pi r^2\left(\frac{1}{r^3} - \frac{1}{b^3}\right)2\sin^2\theta\end{aligned}$$

Thus:

$$E_\phi = \mu_o\frac{R^2}{4b^2}\left(\frac{r}{b} - \frac{b^2}{r^2}\right)\sin\theta\frac{di}{dt}$$

10.1.2 (a) The H-field is similar to that of Prob. 10.0.2 with K specified. It is z-directed and uniform

$$H_z = \begin{cases} K & \text{inside} \\ 0 & \text{outside}\end{cases} \tag{1}$$

Indeed, it is the gradient of a Laplacian potential and has the proper discontinuity at the sheet.

(b) The particular solution does not need to satisfy all the boundary conditions. Suppose we look for one that satisfies the boundary conditions at $y = 0, x = 0$, and $y = a$. If we set

$$\mathbf{E}_p = \mathbf{i_x}E_{xp}(y,t) \tag{2}$$

with $E_{xp}(0,t) = 0$ we have satisfied all three boundary conditions. Now, from Faraday's law,

$$\frac{\partial E_{xp}}{\partial y} = \mu_o\frac{\partial H_z}{\partial t} = \mu_o\frac{dK}{dt} \tag{3}$$

Integration gives

$$E_{xp} = y\mu_o\frac{dK}{dt} \tag{4}$$

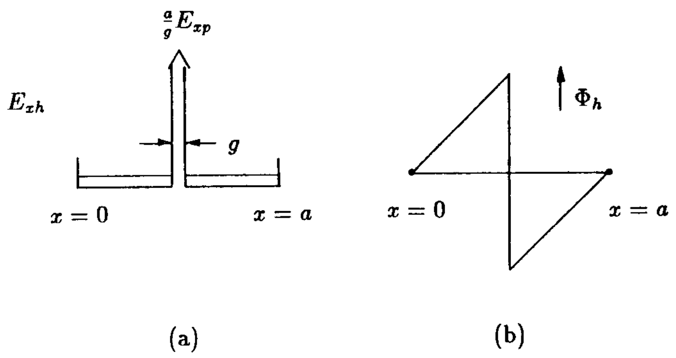


Figure S10.1.2a,b

The total field has to satisfy the boundary condition at $y = -l$. There, the field has to vanish for almost all $0 \leq x \leq a$, except for the short gap at the center of the interval. Thus the E_x-field must consist of a large field $\frac{a}{g}E_{xp}$, over the gap g, and zero field elsewhere. The homogeneous solution must have an E_x-field that looks as shown in Fig. S10.1.2a, or a potential that looks as shown in Fig. S10.1.2b. The homogeneous solution is derivable from a Laplacian potential Φ_h

$$\Phi_h = \sum A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right) \tag{5}$$

which obeys all the boundary conditions, except at $y = -l$. Denote the potential Φ_h at $y = -l$ by

$$\Phi_h(y = -l) = aE_{xp}f(x) \tag{6}$$

so that the jump of $f(x)$ at $x = a/2$ is normalized to unity. Using the orthogonality properties of the sine function, we have

$$-\sinh\left(\frac{m\pi}{a}l\right)\frac{a}{2}A_m = aE_{xp}\int_{x=0}^{a} f(x)\sin\left(\frac{m\pi}{a}x\right)dx \tag{7}$$

It is clear that all odd orders integrate to zero, only even order terms remain. For an even order, except $m = 0$,

$$\begin{aligned}
\int_{x=0}^{a} f(x)\sin\left(\frac{m\pi}{a}x\right) &= 2\int_{x=0}^{a/2}\frac{x}{a}\sin\left(\frac{m\pi}{a}x\right)dx \\
&= \frac{2a}{(m\pi)^2}\int_{u=0}^{m\pi/2} u\sin u\,du \\
&= \frac{2a}{(m\pi)^2}\left[-u\cos u\Big|_0^{m\pi/2} + \int_0^{m\pi/2}\cos u\,du\right] \\
&= \frac{a}{m\pi}(-1)^{\frac{m}{2}+1}
\end{aligned} \tag{8}$$

Therefore

$$A_m = \begin{cases} \dfrac{2aE_{xp}}{m\pi\sinh\left(\frac{m\pi}{a}l\right)}(-1)^{m/2} & m\text{-even} \\ 0 & m\text{-odd} \end{cases} \tag{9}$$

The total field is

$$\mathbf{E} = \mu_o \frac{dK}{dt}\left\{\mathbf{i}_x\left[y - l\sum_{\substack{m\\ \text{even}}} 2(-1)^{m/2}\frac{\sinh\frac{m\pi}{a}y}{\sinh\frac{m\pi}{a}l}\cos\left(\frac{m\pi}{a}x\right)\right]\right.$$
$$\left. - \mathbf{i}_y l\sum_{\substack{m\\ \text{even}}} 2(-1)^{m/2}\frac{\cosh\frac{m\pi}{a}y}{\sinh\frac{m\pi}{2}l}\sin\left(\frac{m\pi}{a}x\right)\right]\right\} \tag{10}$$

10.1.3 (a) The magnetic field is uniform and z-directed

$$\mathbf{H} = \mathbf{i}_z K(t)$$

(b) The electric field is best analyzed in terms of a particular solution that satisfies the boundary conditions at $\phi = 0$ and $\phi = \alpha$, and a homogeneous solution that obeys the last boundary condition at $r = a$. The particular solution is ϕ-directed and is identical with the field encircling an axially symmetric uniform H-field

$$2\pi r E_{\phi p} = -\pi r^2 \mu_o \frac{dH_z}{dt} \tag{1}$$

and thus

$$E_{\phi p} = -\frac{r}{2}\mu_o \frac{dK}{dt} \tag{2}$$

The homogeneous solution is composed of the gradients of solutions to Laplace's equation

$$\Phi_h = \sum_n A_n (r/a)^{n\pi/\alpha}\sin\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

At $r = a$, these solutions must cancel the field along the boundary, except at and around $\phi = \alpha/2$. Because $\delta \ll \alpha$, we approximate the field $E_{\phi h}$ at $r = a$ as composed of a unit impulse function at $\phi = \alpha/2$ of content

$$\alpha E_{\phi p} = -\frac{a}{2}\alpha\mu_o \frac{dK}{dt} \tag{4}$$

and a constant field

$$E_{\phi h} = \frac{a}{2}\mu_o \frac{dK}{dt}$$

over the rest of the interval as shown in Fig. S10.1.3. From (3)

$$E_{\phi h}\Big|_{r=a} = -\frac{1}{a}\frac{\partial \Phi_h}{\partial \phi} = -\frac{1}{a}\sum_n (n\pi/\alpha)A_n \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{5}$$

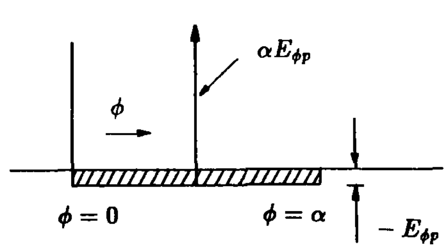


Figure S10.1.3

Here we take an alternative approach to that of 10.1.2. We do not have to worry about the part of the field over $0 < \phi < \alpha$, excluding the unit impulse function, because the line integral of $E_{\phi h}$ from $\phi = 0$ to $\phi = \alpha$ is assured to be zero (conservative field). Thus we need solely to expand the unit impulse at $\phi = \alpha/2$ in a series of $\cos\left(\frac{m\pi}{\alpha}\phi\right)$. By integrating

$$-\frac{1}{a}(m\pi/\alpha)A_m\frac{\alpha}{2} = \cos(m\pi/2)\alpha E_{\phi p} \tag{6}$$

where the right hand side is the integral through the unit impulse function. Thus,

$$A_m = \begin{cases} -\frac{2a}{m\pi}(-1)^{m/2}\alpha E_{\phi p} & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd} \end{cases} \tag{7}$$

Therefore

$$\Phi_h = \sum_{m-\text{even}} \frac{a^2\alpha}{m\pi}(-1)^{m/2}\mu_o\frac{dK}{dt}(r/a)^{m\pi/\alpha}\sin\left(\frac{m\pi}{\alpha}\phi\right) \tag{8}$$

and

$$\begin{aligned}\mathbf{E} = -\mu_o\frac{dK}{dt}\frac{a}{2}\Big\{\frac{r}{a} + \sum_{\substack{m=2\\ m-\text{even}}}^{\infty} 2(-1)^{m/2}(r/a)^{\frac{m\pi}{\alpha}-1} \\ \Big[\mathbf{i}_r\sin\left(\frac{m\pi\phi}{\alpha}\right) + \mathbf{i}_\phi\cos\left(\frac{m\pi\phi}{\alpha}\right)\Big]\Big\}\end{aligned} \tag{9}$$

10.1.4 (a) The coil current produces an equivalent surface current $K = Ni/d$ and hence, because the coil is long

$$\mathbf{B} \simeq \mathbf{i}_z\mu_o\frac{Ni}{d} \tag{1}$$

(b) The (semi-) conductor is cylindrical and uniform. Thus $\mathbf{E}$ must be axisymmetric and, by symmetry, ϕ-directed. From Faraday's law applied to a circular contour of radius r inside the coil

$$2\pi r E_\phi = -\frac{dB_z}{dt}\pi r^2$$

and

$$E_\phi = -\frac{r}{2}\mu_o\frac{N}{d}\frac{di}{dt}$$

(c) The induced H-field is due to the circulating current density:

$$J_\phi = \sigma E_\phi = \omega\frac{\sigma r}{2}\mu_o\frac{N}{d}I\sin\omega t$$

where we have set

$$i(t) = I\cos\omega t$$

The H field will be axial, z- and ϕ-independent, by symmetry. (The z-"independence" follows from the fact that $d \gg b$.) From Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J}$$

we have

$$-\frac{dH_z}{dr} = J_\phi$$

and thus

$$H_{z \text{ induced}} = -\omega\sigma\frac{r^2}{4}\mu_o\frac{N}{d}I\sin\omega t$$

For $H_{z \text{ induced}} \ll H_{z \text{ imposed}}$ for $r \le b$

$$\frac{\omega\mu_o\sigma b^2}{4} \ll 1$$

10.1.5 (a) From Faraday's law

$$\nabla \times \mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{B} \tag{1}$$

and thus

$$\frac{\partial E_{yp}}{\partial x} = -\mu_o\frac{N}{d}\frac{di}{dt} \tag{2}$$

Therefore,

$$E_{yp} = -\mu_o\frac{N}{d}\left(x - \frac{b}{2}\right)\frac{di}{dt} \tag{3}$$

(b) We must maintain $\mathbf{E}\cdot\mathbf{n} = 0$ inside the material. Thus, adding the homogeneous solution, a gradient of a scalar potential Φ, we must leave $E_x = 0$ at $x = 0$ and $x = b$. Further, we must eliminate E_y at $y = 0$ and $y = a$. We need an infinite series

$$\Phi_h = \sum_n A_n \cos\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right) \tag{4}$$

with the electric field

$$\mathbf{E}_h = \sum_n A_n\left(\frac{n\pi}{b}\right)\left[\sin\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right)\mathbf{i}_x - \cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}y\right)\mathbf{i}_y\right] \tag{5}$$

At $y = \pm a/2$

$$\begin{aligned} E_{yh} &= -\sum_n A_n\left(\frac{n\pi}{b}\right)\cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}\frac{a}{2}\right) \\ &= -E_{yp} = \mu_o\frac{N}{d}\left(x - \frac{b}{2}\right)\frac{di}{dt} \end{aligned} \tag{6}$$

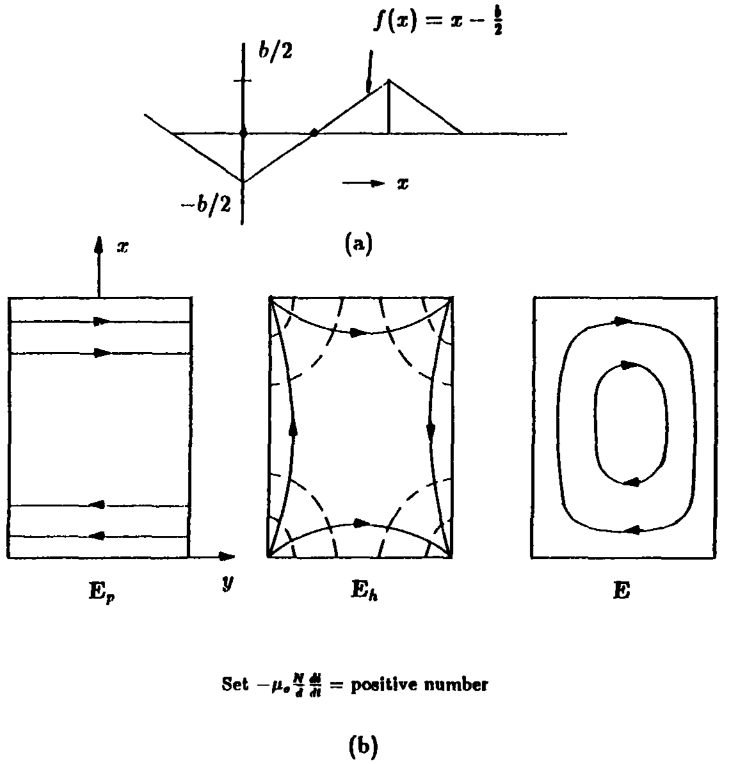


Figure S10.1.5

We must expand the function shown in Fig. S10.1.5a into a cosine series. Thus, multiplying (6) by $\cos\frac{m\pi}{b}x$ and integrating from $x = 0$ to $x = b$, we obtain

$$-\frac{m\pi}{b}\frac{b}{2}A_m\cosh\left(\frac{m\pi}{2b}a\right) = \mu_o\frac{N}{d}\frac{di}{dt}\int_0^b\left(x-\frac{b}{2}\right)\cos\frac{m\pi}{b}x\,dx$$
$$= \begin{cases} -\mu_o\frac{N}{d}\frac{di}{dt}2\left(\frac{b}{m\pi}\right)^2 & m\text{ - odd} \\ 0 & m\text{ - even}\end{cases} \tag{7}$$

Solving for A_m

$$A_m = \begin{cases} \frac{4b^2/(m\pi)^3}{\cosh(m\pi a/2b)}\mu_o\frac{N}{d}\frac{di}{dt} & m\text{ - even} \\ 0 & m\text{ - odd}\end{cases} \tag{8}$$

The $\mathbf{E}$-field is

$$\mathbf{E} = -\mu_o\frac{N}{d}\frac{di}{dt}\left\{\left(x-\frac{b}{2}\right)\mathbf{i}_y - \sum_{n-\text{odd}}\frac{4b/(m\pi)^2}{\cosh(m\pi a/2b)}\right.$$
$$\left[\sin\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}y\right)\mathbf{i}_x\right. \tag{9}$$
$$\left.\left.-\cos\left(\frac{n\pi}{b}x\right)\cosh\left(\frac{n\pi}{b}y\right)\mathbf{i}_y\right]\right\}$$

(c) See Fig. S10.1.5b.

10.2 NATURE OF FIELDS INDUCED IN FINITE CONDUCTORS

10.2.1 The approximate resistance of the disk is

$$R = \frac{1}{\sigma}\frac{2\pi a}{2}\frac{1}{a\Delta}$$

where we have taken half of the circumference as the length. The flux through the disk is [compare (10.2.15)]

$$\lambda = \int \mu_o \mathbf{H}\cdot d\mathbf{a} = \mu_o \frac{i_2}{2\pi a}\pi a^2$$

$$\lambda = \frac{\mu_o i_2 a}{2}$$

This is caused by the current i_2 so the inductance of the disk L_{22} is (using $N = 1$):

$$L_{22} = \frac{\mu_o a}{2}$$

The time constant is

$$\tau_m = \frac{L_{22}}{R} = \frac{\mu_o a}{2}\frac{\sigma\Delta}{\pi} = \frac{\mu_o a\Delta\sigma}{2\pi}$$

This is roughly the same as (10.2.17).

10.2.2 Live bone is fairly "wet" and hence conducting like the surrounding flesh. Current lines have to close on themselves. Thus, if one mounts a coil with its axis perpendicular to the arm and centered with the arm as shown in Fig. S10.2.2, circulating currents are set up. If perfect symmetry prevailed and the bone were precisely at center, then no current would flow along its axis. However, such symmetry does not exist and thus longitudinal currents are set up with the bone off center.

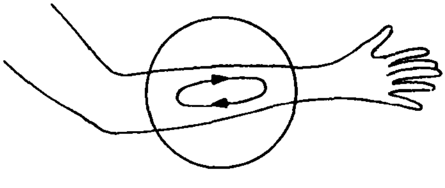


Figure S10.2.2

10.2.3 The field of coil (1) is, according to (10.2.8)

$$H_1 = \frac{N_1 i_1}{2\pi a} \tag{1}$$

The net field is

$$H_1 + H_{ind}$$

with $H_{ind} = K_\phi$ where K_ϕ is the ϕ directed current in the shell. The E-field is from Faraday's law, using symmetry

$$2\pi r E_\phi = -\mu_o \frac{d}{dt}(H_o + H_{ind})\pi r^2 \tag{2}$$

But

$$E_\phi\Big|_{r=a} = \frac{K_\phi}{\sigma\Delta} = \frac{H_{ind}}{\sigma\Delta} \tag{3}$$

and thus, for $r = a$

$$\frac{2H_{ind}}{\mu_o\sigma\Delta a} + \frac{d}{dt}H_{ind} = -\frac{d}{dt}H_o \tag{4}$$

In the sinusoidal steady state, using complex notation

$$H_o = \text{Re}\,\hat{H}_o e^{j\omega t} \quad \text{etc.} \tag{5}$$

and

$$\hat{H}_{ind} = -\frac{j\omega\tau_m}{j\omega\tau_m + 1}\hat{H}_o \tag{6}$$

where

$$\tau_m = \frac{\mu_o\sigma\Delta a}{2}$$

At small values of $\omega\tau_m$

$$|\hat{H}_{ind}| = \omega\tau_m|\hat{H}_o| \tag{7}$$

10.3 DIFFUSION OF AXIAL MAGNETIC FIELDS THROUGH THIN CONDUCTORS

√

10.3.1 The circulating current $K(t)$ produces an approximately uniform axial field

$$H_z = K(t) \tag{1}$$

As the field varies with time, there is an induced E-field obeying Faraday's law

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_S \mu_o \mathbf{H} \cdot d\mathbf{a} \tag{2}$$

The E-field drives the surface current

$$K = \Delta\sigma E \tag{3}$$

that must be constant along the circumference. Hence E must be constant. From (1), (2), and (3)

$$4aE = 4a\frac{K}{\Delta\sigma} = -\frac{d}{dt}\mu_o K a^2 \tag{4}$$

and thus

$$\frac{d}{dt}K + \frac{4}{\mu_o\sigma\Delta a}K = 0 \tag{5}$$

Thus

$$K(t) = K_o e^{-t/\tau_m} \tag{6}$$

with

$$\tau_m = \frac{\mu_o\sigma\Delta a}{4} \tag{7}$$

10.3.2 (a) This problem is completely analogous to 10.3.1. One has

$$H_z = K(t) \tag{1}$$

and, because $K = \Delta\sigma E$ must be constant along the surface, so that E must be constant

$$(2d + \sqrt{2}d)E = -\frac{d}{dt}\mu_o K(t)\frac{d^2}{2} \tag{2}$$

Therefore

$$(2+\sqrt{2})\frac{K}{\Delta\sigma} = -\frac{d}{dt}(\mu_o K)\frac{d}{2} \tag{3}$$

or

$$\frac{dK}{dt} + \frac{K}{\tau_m} = 0 \tag{4}$$

with

$$\tau_m = \frac{\mu_o\sigma\Delta d}{2(2+\sqrt{2})} \tag{5}$$

The solution for $J = K/\Delta$ is

$$J = J_o e^{-t/\tau_m} \tag{6}$$

(b) Since

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = 0 \tag{7}$$

and the line integral along the surface is $\sqrt{2}dE$, we have

$$v + \sqrt{2}dE = 0 \tag{8}$$

$$v = -\sqrt{2}\frac{dK}{\Delta\sigma} = -\sqrt{2}\frac{dJ_o}{\sigma}e^{-t/\tau_m} \tag{9}$$

(c) Again from Faraday's law

$$\begin{aligned}\oint_{C_2} \mathbf{E} \cdot d\mathbf{s} = -v &= -\frac{d}{dt}\mu_o H_z \frac{d^2}{2} = -\mu_o \frac{d^2}{2}\frac{dK}{dt} \\ &= \mu_o \frac{1}{\tau_m}\frac{d^2}{2}\Delta J_o e^{-t/\tau_m} = (2+\sqrt{2})\frac{J_o d}{\sigma}e^{-t/\tau_m}\end{aligned} \tag{10}$$

10.3.3 (a) We set up the boundary conditions for the three uniform axial fields, in the regions $r < b, b < r < a, r > a$ (see Fig. S10.3.3).

$$H_o(t) - H_1(t) = -K_{\text{out}}(t) = -J_{\text{out}}\Delta = -\sigma E_{\text{out}}\Delta \tag{1}$$

$$H_1(t) - H_2(t) = -K_{\text{in}}(t) = -J_{\text{in}}\Delta = -\sigma E_{\text{in}}\Delta \tag{2}$$

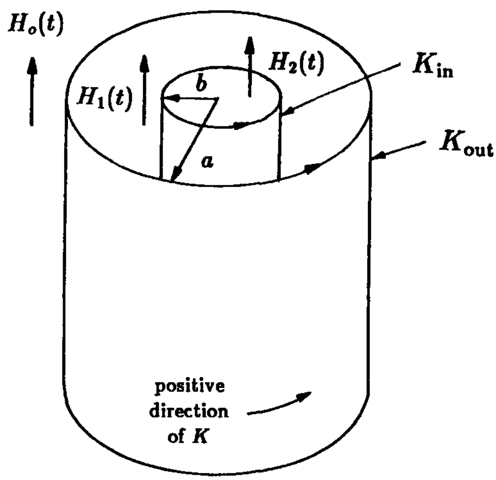


Figure S10.3.3

From the integral form of Faraday's law:

$$2\pi a E_{\text{out}} = -\mu_o \frac{d}{dt}\left[H_1(t))\pi(a^2 - b^2) + H_2(t)\pi b^2\right] \tag{3}$$

$$2\pi b E_{\text{in}} = -\mu_o \frac{d}{dt}\left[H_2(t)\pi b^2\right] \tag{4}$$

We can solve for E_{out} and E_{in} and substitute into (1) and (2)

$$H_o(t) - H_1(t) = \mu_o \frac{\sigma\Delta}{2}\left[\frac{a^2 - b^2}{a}\frac{dH_1(t)}{dt} + \frac{b^2}{a}\frac{dH_2(t)}{dt}\right] \tag{5}$$

$$H_1(t) - H_2(t) = \mu_o \frac{\sigma\Delta b}{2}\frac{dH_2(t)}{dt} \tag{6}$$

We obtain from (6)

$$\tau_m \frac{dH_2(t)}{dt} + H_2(t) - H_1(t) = 0 \tag{7}$$

where

$$\tau_m \equiv \frac{\mu_o \sigma \Delta b}{2}$$

From (5), after some rearrangement, we obtain:

$$\Rightarrow \tau_m \frac{b}{a}\frac{dH_2}{dt} + \tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\frac{dH_1(t)}{dt} + H_1(t) = H_o(t) \tag{8}$$

(b) We introduce complex notation

$$H_o = H_m \cos\omega t = \text{Re}\left\{H_m e^{j\omega t}\right\} \tag{9}$$

Similarly H_1 and H_2 are replaced by $H_{1,2} = \text{Re}\left[\hat{H}_{1,2}e^{j\omega t}\right]$. We obtain two equations for the two unknowns $\hat{H}_1$ and $\hat{H}_2$:

$$-\hat{H}_1 + (1 + j\omega\tau_m)\hat{H}_2 = 0$$

$$\left[1 + j\omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\right]\hat{H}_1 + \frac{b}{a}j\omega\tau_m\hat{H}_2 = H_m$$

They can be solved in the usual way

$$\hat{H}_1 = \frac{\begin{vmatrix} 0 & 1 + j\omega\tau_m \\ H_m & \frac{b}{a}j\omega\tau_m \end{vmatrix}}{Det} = -\frac{(1 + j\omega\tau_m)H_m}{Det}$$

$$\hat{H}_2 = \frac{\begin{vmatrix} -1 & 0 \\ 1 + \omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right) & H_m \end{vmatrix}}{Det} = -\frac{H_m}{Det}$$

where Det is the determinant.

$$Det \equiv -\left\{\left[1 + j\omega\tau_m\left(\frac{a}{b} - \frac{b}{a}\right)\right](1 + j\omega\tau_m) + j\omega\tau_m\frac{b}{a}\right\}$$

10.3.4 (a) To the left of the sheet (see Fig. S10.3.4),

$$\mathbf{H} = K_o \mathbf{i}_z \tag{1}$$

To the right of the sheet

$$\mathbf{H} = K \mathbf{i}_z \tag{2}$$

Along the contour C_1, use Faraday's law

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a} \tag{3}$$

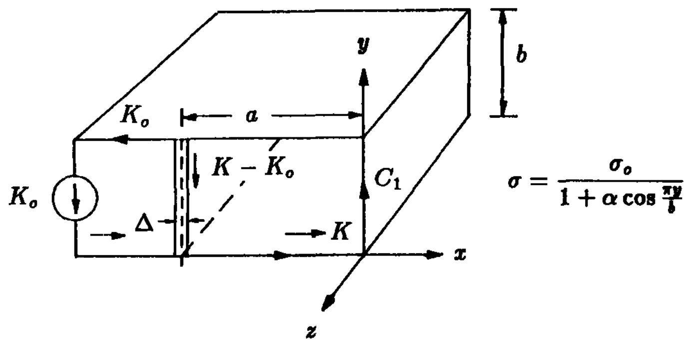


Figure S10.3.4

Along the three perfectly conducting sides of the conductor $\mathbf{E} = 0$. In the sheet the current $K - K_o$ is constant so that

$$\nabla \cdot J = 0 \Rightarrow \nabla \cdot (\sigma E) = 0 \tag{4}$$

$$\oint_{C_1} \mathbf{E} \cdot d\mathbf{s} = \int_{y=0}^{b} \left(\frac{K - K_o}{\Delta \sigma} \right) dy = -\mu_o a b \frac{dK}{dt} \tag{5}$$

$$\frac{K - K_o}{\Delta \sigma_o} \int_{y=0}^{b} \left(1 + \alpha \cos \frac{\pi y}{b} \right) dy = -\mu_o a b \frac{dK}{dt} \tag{6}$$

The integral yields b and thus

$$\frac{dK}{dt} + \frac{K}{\mu_o a \Delta \sigma_o} = \frac{K_o}{\mu_o a \Delta \sigma_o} \tag{7}$$

From (7) we can find K as a function of time for a given $K_o(t)$.

(b) The y-component of the electric field at $x = -a$ has a uniform part and a y-dependent part according to (5). The y-dependent part integrates to zero and hence is part of a conservative field. The uniform part is

$$E_{yp} b = -\frac{K - K_o}{\Delta \sigma_o} b = \mu_o a b \frac{dK}{dt} \tag{8}$$

This is the particular solution of Faraday's law

$$\frac{\partial E_{yp}}{\partial x} = -\mu_o \frac{\partial H_z}{\partial t} = -\mu_o \frac{dK}{dt} \tag{9}$$

with the integral

$$E_{yp} = -\mu_o x \frac{dK}{dt} \tag{10}$$

and indeed, at $x = -a$, we obtain (8). There remains

$$E_{yh} = -\frac{K - K_o}{\Delta\sigma_o} \alpha \cos\left(\frac{\pi y}{b}\right) \tag{11}$$

It is clear that this field can be found from the gradient of the Laplacian potential

$$\Phi = A \sin\left(\frac{\pi y}{b}\right) \sinh\left(\frac{\pi x}{b}\right) \tag{12}$$

that satisfies the boundary conditions on the perfect conductors. At $x = -a$

$$-\frac{\partial \Phi}{\partial y}\Big|_{x=-a} = \frac{\pi}{b} A \cos\frac{\pi y}{b} \sinh\left(\frac{\pi a}{b}\right) = -\frac{K - K_o}{\Delta\sigma_o} \alpha \cos\frac{\pi y}{b} \tag{13}$$

and thus

$$A = -\frac{\alpha b}{\pi \Delta\sigma_o} \frac{(K - K_o)}{\sinh(\pi a/b)} \tag{14}$$

10.4 DIFFUSION OF TRANSVERSE MAGNETIC FIELDS THROUGH THIN CONDUCTORS

10.4.1 (a) Let us consider an expanded view of the conductor (Fig. S10.4.1). At $y = \Delta$, the boundary condition on the normal component of $\mathbf{B}$ gives

$$\mathbf{i}_y \cdot [\mathbf{B}^a - \mathbf{B}^c] = 0 \tag{1}$$

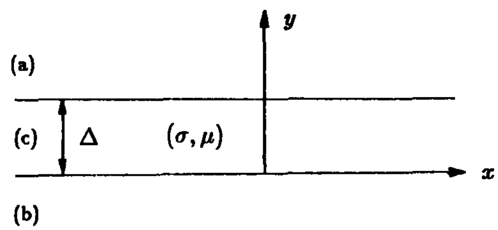


Figure S10.4.1

Therefore

$$B_y^a\Big|_{y=\Delta} - B_y^c\Big|_{y=\Delta} = 0 \tag{2}$$

At $y = 0$

$$B_y^c\Big|_{y=0} - B_y^b\Big|_{y=0} = 0 \tag{3}$$

Since the thickness, Δ, of the sheet is very small, we can assume that B is uniform across the sheet so that,

$$B_y^c\Big|_{y=\Delta} = B_y^c\Big|_{y=0} \tag{4}$$

Using (3) and (4) in (2),

$$B_y^a - B_y^b = 0 \tag{5}$$

From the continuity condition associated with Ampère's law

$$\mathbf{n} \times [\mathbf{H}^a - \mathbf{H}^b] = \mathbf{K}$$

Since

$$\mathbf{K} = K_z \mathbf{i}_z, \quad \mathbf{n} = \mathbf{i}_y,$$

$$-H_x^a + H_x^b = K_z \tag{6}$$

The current density $\mathbf{J}$ in the sheet is

$$J_z = \frac{K_z}{\Delta} \tag{7}$$

And so, from Ohm's law

$$E_z = \frac{K_z}{\Delta\sigma} \tag{8}$$

Finally from Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{9}$$

Since only B_y matters (only time rate of change of flux normal to the sheet will induce circulating $\mathbf{E}$-fields) and $\mathbf{E}$ only has a z-component,

$$-\frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

From (8) therefore,

$$\frac{\partial}{\partial x}\left[\frac{K_z}{\Delta\sigma}\right] = \frac{\partial B_y}{\partial t}$$

and finally, from (6),

$$\frac{\partial}{\partial x}[H_x^a - H_x^b] = -\Delta\sigma \frac{\partial B_y}{\partial t} \tag{10}$$

(b) At $t = 0$ we are given $\mathbf{K} = \mathbf{i}_z K_o \sin \beta x$. Everywhere except within the current sheet, we have $\mathbf{J} = 0$

$$\Rightarrow \mathbf{H} = -\nabla\Psi$$

So from $\nabla \cdot \mu_o \mathbf{H} = 0$, we have

$$\nabla^2 \Psi = 0$$

Boundary conditions are given by (5) and (10) and by the requirement that the potential mut decay as $y \to \pm\infty$. Since H_x will match the $\sin \beta x$ dependence of the current, pick solutions with $\cos \beta x$ dependence

$$\Psi^{(a)} = A(t) \cos \beta x e^{-\beta y} \tag{11a}$$

$$\Psi^{(b)} = C(t) \cos \beta x e^{\beta y} \tag{11b}$$

$$\mathbf{H}^{(a)} = \beta A(t) \sin \beta x e^{-\beta y} \mathbf{i_x} + \beta A(t) \cos \beta x e^{-\beta y} \mathbf{i_y} \tag{12a}$$

$$\mathbf{H}^{(b)} = \beta C(t) \sin \beta x e^{\beta y} \mathbf{i_x} - \beta C(t) \cos \beta x e^{\beta y} \mathbf{i_y} \tag{12b}$$

From (5),

$$\mu_o \beta A(t) \cos \beta x e^{-\beta y}\Big|_{y=0} + \mu_o \beta C(t) \cos \beta x e^{\beta y}\Big|_{y=0} = 0$$

Therefore,

$$A(t) = -C(t) \tag{13}$$

From (10),

$$\frac{\partial}{\partial x}\left[\beta A(t) \sin \beta x e^{-\beta y}\Big|_{y=0} - \beta C(t) \sin \beta x e^{\beta y}\Big|_{y=0}\right]$$
$$= -\Delta\sigma\mu_o \beta \cos \beta x e^{-\beta y}\Big|_{y=0} \frac{dA(t)}{dt}$$

Using (13)

$$2\beta^2 A(t) \cos \beta x = -\Delta\sigma\mu_o \beta \cos \beta x \frac{dA(t)}{dt}$$

The cosines cancel and

$$\frac{dA(t)}{dt} + \frac{2\beta}{\Delta\sigma\mu_o} A(t) = 0 \tag{14}$$

The solution is

$$A(t) = A(0) e^{-t/\tau} \qquad \tau = \frac{\mu_o \Delta\sigma}{2\beta} \tag{15}$$

So the surface current, proportional to H_x according to (6), decays similarly as

$$\mathbf{K} = \mathbf{i_z} K_o \sin \beta x e^{-t/\tau} \qquad \tau = \frac{\mu_o \Delta\sigma}{2\beta}$$

10.4.2 (a) If the sheet acts like a perfect conductor (see Fig. S10.4.2), the component of $\mathbf{H}$ perpendicular to the sheet must be zero.

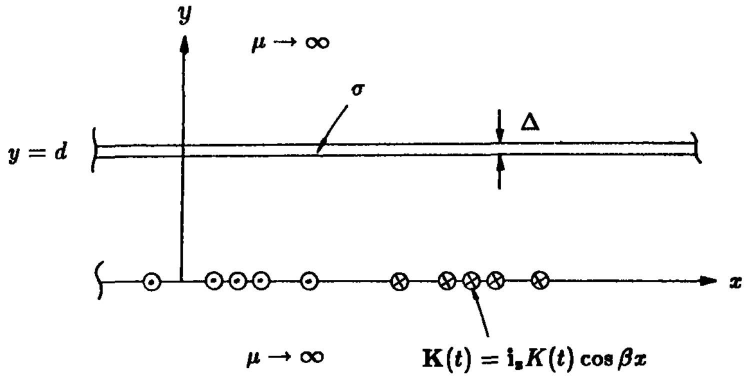


Figure S10.4.2

At $y = 0$ the magnetic field experiences a jump of the tangential component

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K} \tag{1}$$

with $\mathbf{n} \parallel \mathbf{i_y}$ and $\mathbf{H}_2 = 0$,

$$H_x = -K(t) \cos \beta x \tag{2}$$

The field in the space $0 < y < d$ is the gradient of a Laplacian potential

$$\Psi = A \sin \beta x \cosh \beta (y - d) \tag{3}$$

The cosh is chosen so that H_y is zero at $y = d$:

$$\mathbf{H} = -A\beta[\cos \beta x \cosh \beta(y-d)\mathbf{i_x} + \sin \beta x \sinh \beta(y-d)\mathbf{i_y}] \tag{4}$$

Satisfying the boundary condition at $y = 0$

$$-A\beta \cos \beta x \cosh \beta d = -K(t) \cos \beta x \tag{5}$$

Therefore

$$A = \frac{K(t)}{\beta \cosh \beta d} \tag{6}$$

$$\Psi = \frac{K(t) \sin \beta x \cosh \beta(y-d)}{\beta \cosh \beta d} \tag{7}$$

(b) For $K(t)$ slowly varying, the magnetic field diffuses straight through so the sheet acts as if it were not there. The field "sees" $\mu \to \infty$ material and, therefore, has no tangential H

$$\Psi = A \sin \beta x \sinh \beta (y - d) \tag{8}$$

which satisfies the condition $H_x = 0$ at $y = d$. Indeed,

$$\mathbf{H} = -A\beta[\cos\beta x \sinh\beta(y-d)\mathbf{i_x} + \sin\beta x \cosh\beta(y-d)\mathbf{i_y}]$$

Matching the boundary condition at $y = 0$, we obtain

$$A = -\frac{K(t)}{\beta\sinh\beta d} \tag{9}$$

$$\Psi = -\frac{K(t)\sin\beta x \sinh\beta(y-d)}{\beta\sinh\beta d} \tag{10}$$

(c) Now solving for the general time dependence, we can use the previous results as a clue. Initially, the sheet acts like a perfect conductor and the solution (7) must apply. As $t \to \infty$, the sheet does not conduct, and the solution (10) must apply. In between, we must have a transition between these two solutions. Thus, postulate that the current $\mathbf{i_z}K_s(t)\cos\beta x$ is flowing in the top sheet. We have

$$K_s(t)\cos\beta x = \sigma\Delta E_z \tag{11}$$

Postulate the potential

$$\Psi = C(t)\frac{\sin\beta x\cosh\beta(y-d)}{\beta\cosh\beta d} - D(t)\frac{\sin\beta x\sinh\beta(y-d)}{\beta\sinh\beta d} \tag{12}$$

The boundary condition at $y = 0$ is

$$\begin{aligned}-\frac{\partial\Psi}{\partial x}\Big|_{y=0} = H_x\Big|_{y=0} &= -K(t)\cos\beta x \\ &= -C(t)\cos\beta x - D(t)\cos\beta x\end{aligned} \tag{13}$$

Therefore

$$C + D = K \tag{14}$$

At $y = d$

$$-\frac{\partial\Psi}{\partial x}\Big|_{y=d} = H_x\Big|_{y=d} = K_s(t)\cos\beta x = -C(t)\frac{\cos\beta x}{\cosh\beta d} \tag{15}$$

The current in the sheet is driven by the E-field induced by Faraday's law and is z-directed by symmetry

$$\begin{aligned}\frac{\partial E_z}{\partial y} = -\frac{\partial}{\partial t}\mu_o H_x = \mu_o &\frac{\cos\beta x\cosh\beta(y-d)}{\cosh\beta d}\frac{dC}{dt} \\ - \mu_o &\frac{\cos\beta x\sinh\beta(y-d)}{\sinh\beta d}\frac{dD}{dt}\end{aligned} \tag{16}$$

Therefore,

$$E_z = \frac{\mu_o\cos\beta x\sinh\beta(y-d)}{\beta\cosh\beta d}\frac{dC}{dt} - \mu_o\frac{\cos\beta x\cosh\beta(y-d)}{\beta\sinh\beta d}\frac{dD}{dt} \tag{17}$$

At $y = d$

$$E_z = -\mu_o \frac{1}{\beta \sinh \beta d} \cos \beta x \frac{dD}{dt} = \frac{K_s \cos \beta x}{\sigma \Delta} \tag{18}$$

Hence, combining (14), (15), and (18)

$$\cosh \beta d K_s = -C(t) = -K + D = -\frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \frac{dD}{dt} \tag{19}$$

resulting in the differential equation

$$\frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \frac{dD}{dt} + D = K \tag{20}$$

With K a step function

$$D = K_o\left[1 - e^{-t/\tau_m}\right] \tag{21}$$

where

$$\tau_m = \frac{\mu_o \sigma \Delta}{\beta} \coth \beta d \tag{22}$$

and

$$C = K_o e^{-t/\tau_m}$$

At $t = 0$, $D = 0$ and at $t = \infty$, $C = 0$. This checks with the previously obtained solutions.

10.4.3 (a) If the shell (Fig. S10.4.3) is thin enough it acts as a surface of discontinuity at which the usual boundary conditions are obeyed. From the continuity of the normal component of $\mathbf{B}$,

$$B_r^a - B_r^b = 0 \tag{1}$$

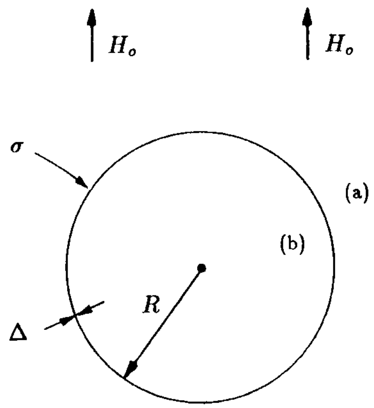


Figure S10.4.3

the continuity condition associated with Ampère's law

$$\mathbf{n} \times [\mathbf{H}^a - \mathbf{H}^b] = \mathbf{K} \tag{2}$$

use of Ohm's law

$$E = \frac{J}{\sigma} = \frac{K}{\Delta\sigma} \tag{3}$$

results in

$$H_\theta^a - H_\theta^b = K_\phi = \Delta\sigma E_\phi \tag{4}$$

The electric field obeys Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

Only flux normal to the shell induces E in the sheet. By symmetry, $\mathbf{E}$ is ϕ-directed

$$(\nabla \times \mathbf{E})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi \sin\theta) = -\frac{\partial B_r}{\partial t} \tag{6}$$

And thus, at the boundary

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta[H_\theta^a - H_\theta^b] = -\mu_o \Delta\sigma \frac{\partial H_r}{dt} \tag{7}$$

(b) Set

$$H_o(t) = \mathrm{Re}\left\{H_o e^{j\omega t}\right\}[\cos\theta \mathbf{i}_r - \sin\theta \mathbf{i}_\theta] \tag{8}$$

The H-field outside and inside the shell must be the gradient of a scalar potential

$$\hat{\Psi}_a = -H_o r\cos\theta + \frac{\hat{A}\cos\theta}{r^2} \tag{9}$$

$$\hat{\Psi}_b = \hat{C} r\cos\theta \tag{10}$$

$$\hat{H}_\theta^a = -H_o \sin\theta + \frac{\hat{A}}{r^3}\sin\theta \tag{11}$$

$$\hat{H}_\theta^b = \hat{C}\sin\theta \tag{12}$$

$$\hat{H}_r^a = H_o\cos\theta + \frac{2\hat{A}}{r^3}\cos\theta \tag{13}$$

$$\hat{H}_r^b = -\hat{C}\cos\theta \tag{14}$$

From (1)

$$B_r^a = B_r^b \Rightarrow H_o + \frac{2\hat{A}}{R^3} = -\hat{C} \tag{15}$$

Introducing (11), (12), and (13) into (7) we find

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin^2\theta\left(-H_o + \frac{\hat{A}}{R^3} - \hat{C}\right)\right\} = -j\omega\mu_o\Delta\sigma\left\{H_o\cos\theta + \frac{2\hat{A}\cos\theta}{R^3}\right\} \tag{16}$$

from which we find A, using (15) to eliminate $\hat{C}$.

$$\hat{A} = -\frac{j\omega\mu_o\Delta\sigma R^4 H_o}{2(j\omega\mu_o\Delta\sigma R + 3)} \tag{17}$$

$\hat{A}$ provides the dipole term

$$\frac{\hat{m}}{4\pi} = \hat{A} = \frac{-j\omega\mu_o\Delta\sigma R^4 H_o}{2(j\omega\mu_o\Delta\sigma R + 3)} \tag{18}$$

and thus

$$\hat{m} = -\frac{j\omega\tau(2\pi R^3)\hat{H}_o)}{(1 + j\omega\tau)} \tag{19}$$

with

$$\tau = \frac{\mu_o\sigma\Delta R}{3}$$

(c) In the limit $\omega\tau \to \infty$, we find

$$\hat{m} \to -2\pi H_o R^3$$

as in Example 8.4.4.

10.4.4 (a) The field is that of a dipole of dipole moment $m = ia$

$$\Psi = \frac{ia}{4\pi r^2}\cos\theta \tag{1}$$

(b) The normal component has to vanish on the shell. We add a uniform field

$$\Psi = Ar\cos\theta + \frac{ia}{4\pi r^2}\cos\theta \tag{2}$$

The normal component of H at $r = R$ is

$$-\frac{\partial\Psi}{\partial r}\Big|_{r=R} = 0 = -\big(A - 2\frac{ia}{4\pi R^3}\big)\cos\theta$$

and thus

$$A = \frac{2ia}{4\pi R^3} \tag{3}$$

and

$$\Psi = \frac{ia}{4\pi R^2}\cos\theta\big(2\frac{r}{R} + \frac{R^2}{r^2}\big)$$

(see Fig. S10.4.4).

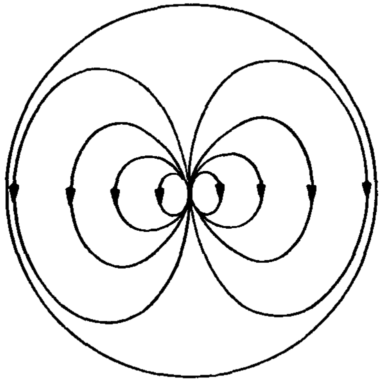


Figure S10.4.4

(c) There is now also an outside field. For $r < R$

$$\Psi = \frac{ia}{4\pi r^2}\cos\theta + A(t)r\cos\theta \tag{5}$$

For $r > R$,

$$\Psi = C(t)\frac{\cos\theta}{r^2} \tag{6}$$

The θ-components of $\mathbf{H}$ are

$$H_\theta = \frac{ia}{4\pi r^3}\sin\theta + A\sin\theta; \quad r < R \tag{7a}$$

and

$$H_\theta = \frac{C}{r^3}\sin\theta; \quad r > R \tag{7b}$$

The normal component at $r = R$ is

$$H_r = \left(\frac{2ia}{4\pi R^3} - A\right)\cos\theta \tag{8a}$$

and

$$H_r = \frac{2C}{R^3}\cos\theta \tag{8b}$$

With the boundary condition (7) of Prob. 10.4.3, we have

$$\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\left(\frac{C}{R^3} - \frac{ia}{4\pi R^3} - A\right)\right] = -\frac{2\mu_o\Delta\sigma}{R^3}\cos\theta\frac{dC}{dt} \tag{9}$$

From the continuity of the normal component of B, we find

$$\frac{2ia}{4\pi R^3} - A = 2\frac{C}{R^3} \tag{10}$$

The equation for C becomes

$$\frac{1}{R^4 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\Big(C - \frac{ia}{4\pi} + 2C - \frac{2ia}{4\pi}\Big)\right] = -\frac{2\mu_o\Delta\sigma}{R^3}\cos\theta\frac{dC}{dt} \tag{11}$$

or

$$\tau_m\frac{dC}{dt} + C = \frac{ia}{4\pi} \tag{12}$$

with $\tau_m = \mu_o\sigma\Delta R/3$. If we consider the steady state, then

$$C = \mathrm{Re}\,[\hat{C}e^{j\omega t}] \tag{13}$$

$$\hat{C} = \frac{1}{(1 + j\omega\tau_m)}\frac{ia}{4\pi} \tag{14}$$

$$\hat{A} = \frac{2ia}{4\pi R^3} - \frac{2C}{R^3} = \frac{2ia}{4\pi R^3}\frac{j\omega\tau_m}{1 + j\omega\tau_m} \tag{15}$$

Jointly with (5) and (6), this determines Ψ.

(d) When $\omega\tau_m \rightarrow \infty$, we have $\hat{C} \rightarrow 0$, no outside field and $\hat{A} = 2ia/4\pi R^3$ which checks with (3). When $\omega\tau_m \rightarrow 0$, we have no shield and $\hat{A} \rightarrow 0$. The shell behaves as if it were infinitely conducting in the limit $\omega\tau_m \rightarrow \infty$.

10.4.5 (a) If the current density varies so rapidly that the sheet is a perfect conductor, then it imposes the boundary condition (see Fig. S10.4.5),

$$\mathbf{n}\cdot\mu_o\mathbf{H} = 0 \quad \text{at} \quad r = b$$

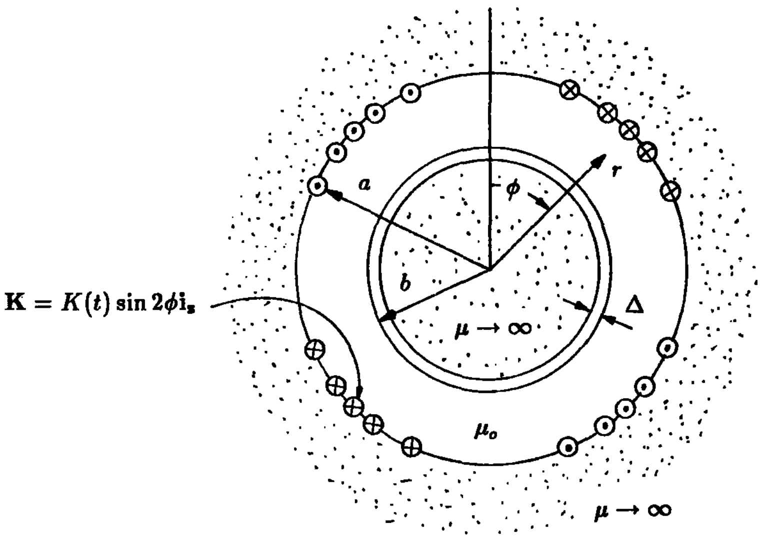


Figure S10.4.5

Inside the high μ material $\mathbf{H} = 0$ to keep $\mathbf{B}$ finite. So at $r = a$,

$$\mathbf{n} \times \mathbf{H} = \mathbf{K}$$

Therefore

$$-\mathbf{i}_z H_\phi = K(t) \sin 2\phi \mathbf{i}_z$$

Thus, the potential has to obey the boundary conditions

$$\frac{\partial \Psi}{\partial r} = 0 \quad \text{at} \quad r = b \tag{1}$$

$$-\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = -K(t) \sin 2\phi \quad \text{at} \quad r = a \tag{2}$$

In order to satisfy (2), we must pick a $\cos 2\phi$ dependence for Ψ. To satisfy (1), one picks a $[(r/b)^2 + (b/r)^2] \cos 2\phi$ type solution. Guess

$$\Psi = A[(r/b)^2 + (b/r)^2] \cos 2\phi$$

Indeed,

$$\frac{\partial \Psi}{\partial r} = A\left[\frac{2r}{b^2} - \frac{2b^2}{r^3}\right] \cos 2\phi = 0 \quad \text{at} \quad r = b$$

$$\frac{\partial \Psi}{\partial \phi} = -A[(r/b)^2 + (b/r)^2] 2 \sin 2\phi$$

From (2),

$$\frac{A}{a}[(a/b)^2 + (b/a)^2] 2 \sin 2\phi = -K(t) \sin 2\phi$$

Therefore,

$$\Psi = -\frac{K(t)a}{2} \frac{[(r/b)^2 + (b/r)^2]}{[(a/b)^2 + (b/a)^2]} \cos 2\phi \tag{3}$$

(b) Now the current induced in the sheet is negligible, so all the field diffuses straight through. The sheet behaves as if it were not there at all. But at $r = b$ we have $\mu \to \infty$ material, so $H = 0$ inside. Also, since now there is no $\mathbf{K}$ at $r = b$, we must have

$$H_\phi = 0 \quad \text{at} \quad r = b$$

It is clear that the following potential obeys the boundary condition at $r = b$

$$\Psi = A[(r/b)^2 - (b/r)^2] \cos 2\phi$$

$$H_\phi = -\frac{1}{r}\frac{\partial \Psi}{\partial \phi} = \frac{A}{r}[(r/b)^2 - (b/r)^2] 2 \sin 2\phi = 0 \quad \text{at} \quad r = b$$

Again, applying (2)

$$\frac{A}{a}[(a/b)^2 - (b/a)^2] 2 \sin 2\phi = -K(t) \sin 2\phi$$

Thus,

$$\Psi = -\frac{K(t)a}{2}\frac{[(r/b)^2 - (b/r)^2]}{[(a/b)^2 - (b/a)^2]}\cos 2\phi \tag{4}$$

(c) At the sheet, the normal B is continuous assuming that Δ is small. Also, from Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \tag{5}$$

Since only a time varying field normal to the sheet will induce currents, we are only interested in $(\nabla \times \mathbf{E})_r$

$$\left(\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) = -\frac{dB_r}{dt}$$

By symmetry there is only a z-component of E

$$\frac{1}{r}\frac{\partial}{\partial \phi}E_z = -\frac{\partial B_r}{\partial t} \tag{6}$$

One should note, however, that there are some subtleties involve in the determination of the E-field. We do not attempt to match the boundary conditions on the coil surface. Such matching would require the addition of the gradient of a solution of Laplace's equation to $\mathbf{E}_p = \mathbf{i}_z E_z$. Such a field would induce surface charges in the conducting sheet, but otherwise not affect its current distribution. Remember that in MQS $\epsilon_o \frac{\partial \mathbf{E}}{\partial t}$ is ignored which means that the charging currents responsible for the build-up of charge are negligible compared to the MQS currents flowing in the systems.

From Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$. But, $\mathbf{J} = \mathbf{K}/\Delta$.

$$\frac{1}{r}\frac{\partial}{\partial \phi}\frac{K_z}{\Delta\sigma} = -\frac{\partial B_r}{\partial t} \tag{7}$$

Applying the boundary conditions from Ampère's law,

$$\mathbf{n} \times \left[\mathbf{H}_{\text{gap}}\big|_{r=b} - \mathbf{H}_{\mu\to\infty}\right] = K_z \mathbf{i}_z$$

$$H_\phi\big|_{r=b} = K_z$$

So at $r = b$

$$\frac{1}{b}\frac{\partial}{\partial \phi}\frac{H_\phi}{\Delta\sigma} = -\mu_o \frac{\partial H_r}{\partial t} \tag{8}$$

Now guess a solution for Ψ in the gap. Since we have two current sources (the windings at $r = a$ and the sheet at $r = b$) and we do not necessarily know that they are in phase, we need to use superposition. This involves setting up the field due to each of the two sources individually

$$\Psi = \left\{A(t)\left[\underbrace{(r/a)^2 - (a/r)^2}\right] + C(t)\left[\underbrace{(r/b)^2 - (b/r)^2}\right]\right\}\cos 2\phi \tag{9}$$

Here, A represents the field due to the current at $r = b$, and C is produced by the current at $r = a$. Apply the boundary condition (2), at $r = a$. We find from the tangential H-field

$$\frac{2C(t)}{a}[(a/b)^2 - (b/a)^2] = -K(t)$$

Thus,

$$C(t) = \frac{-aK(t)}{2[(a/b)^2 - (b/a)^2]} \tag{10}$$

The normal and tangential components of $\mathbf{H}$ at $r = b$ are

$$H_r = -\{A(t)[\frac{2b}{a^2} + \frac{2a^2}{b^3}] + C(t)\frac{4}{b}\}\cos 2\phi \tag{11}$$

$$H_\phi = \{\frac{A(t)}{b}[(b/a)^2 - (a/b)^2]\}2\sin 2\phi \tag{12}$$

From (8)

$$\frac{1}{\mu_o \Delta\sigma b}\frac{\partial}{\partial\phi}[\frac{A(t)}{b}[(b/a)^2 - (a/b)^2]2\sin 2\phi] = \left\{(\frac{2b}{a^2} + \frac{2a^2}{b^3})\frac{dA(t)}{dt} + \frac{4}{b}\frac{dC}{dt}\right\}\cos 2\phi$$

Using (10),

$$\frac{dA(t)}{dt} + A(t)\frac{2}{\mu_o b\Delta\sigma}\frac{[(a/b)^2 - (b/a)^2]}{[(a/b)^2 + (b/a)^2]}$$
$$= \frac{a}{[(a/b)^2 + (b/a)^2][(a/b)^2 - (b/a)^2]}\frac{dK(t)}{dt}$$

Simplifying,

$$\frac{dA(t)}{dt} + \frac{A(t)}{\tau} = D\frac{dK(t)}{dt} \tag{13}$$

$$\tau = \frac{\mu_o b\Delta\sigma}{2}\frac{[(a/b)^2 + (b/a)^2]}{[(a/b)^2 - (b/a)^2]} \tag{14}$$

$$D = \frac{a}{[(a/b)^2 + (b/a)^2][(a/b)^2 - (b/a)^2]} \tag{15}$$

dK/dt is a unit impulse function in time. The homogeneous solution for A is

$$A(t) \propto e^{-t/\tau} \tag{16}$$

and the solution that has the proper discontinuity at $t = 0$ is

$$A = DK_o \tag{17}$$

Using (10) and (17) in (9), we obtain,

$$\Psi = \frac{aK_o}{[(a/b)^2 - (b/a)^2]}\left\{\frac{[(r/a)^2 - (a/r)^2]}{[(a/b)^2 + (b/a)^2]}e^{-t/\tau} - \frac{(r/b)^2 - (b/r)^2}{2}\right\}\cos 2\phi$$

First consider the early time $t \rightarrow 0^+$

$$\Psi = \frac{aK_o}{[(a/b)^2 - (b/a)^2]}\left\{\frac{[(r/a)^2 - (a/r)^2]}{[(a/b)^2 + (b/a)^2]} - \frac{(r/b)^2 - (b/r)^2}{2}\right\}\cos 2\phi$$

Therefore

$$\Psi = \frac{-aK_o}{2}\left[\frac{(r/b)^2 + (b/r)^2}{(a/b)^2 + (b/a)^2}\right]\cos 2\phi$$

It is the same as if the surface currents spontaneously arose to buck out the field. At $t \rightarrow \infty$, $e^{-t/\tau} \rightarrow 0$

$$\Psi = \frac{-aK_o}{2}\left[\frac{(r/b)^2 - (b/r)^2}{(a/b)^2 - (b/a)^2}\right]\cos 2\phi$$

This is when the field has enough time to diffuse through the shell so it is as if no surface currents were present.

10.4.6 (a) When ω is very high, the sheet behaves as a perfect conductor, and (see Fig. S10.4.6)

$$\Psi = bK\frac{[(r/a) + (a/r)]}{\left[\frac{b}{a} + \frac{a}{b}\right]}\cos\phi \tag{1}$$

Then, indeed, $\partial\Psi/\partial r = 0$ at $r = a$, and $-\frac{1}{b}\frac{\partial\Psi}{\partial\phi}$ accounts for the surface current K.

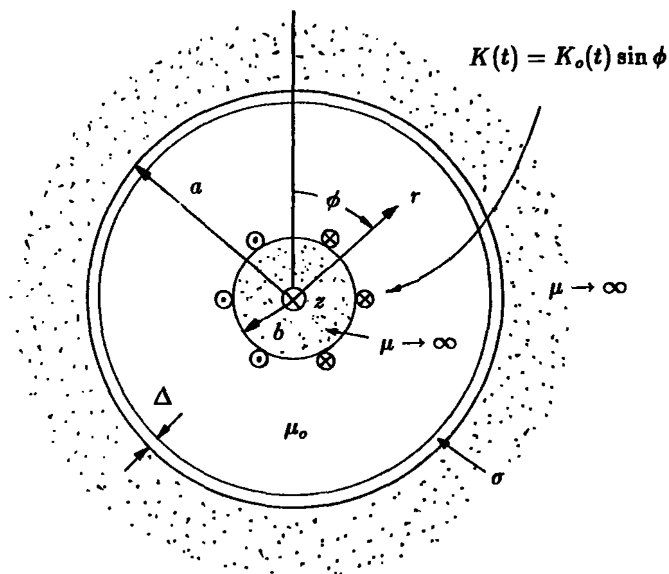


Figure S10.4.6

(b) When ω is very low, then $\partial\Psi/\partial\phi = 0$ at $r = a$ and

$$\Psi = bK\frac{[(r/a) - (a/r)]}{[\frac{b}{a} - \frac{a}{b}]}\cos\phi \tag{2}$$

(c) As before in Prob. 10.4.5, we superimpose the field caused by the two current distributions

$$\Psi = \left\{A(t)[\frac{r}{a} - \frac{a}{r}] + C(t)[\frac{r}{b} - \frac{b}{r}]\right\}\cos\phi \tag{3}$$

The r- and ϕ-components of the field are:

$$H_r = -\left\{A(t)[\frac{1}{a} + \frac{a}{r^2}] + C(t)[\frac{1}{b} + \frac{b}{r^2}]\right\}\cos\phi \tag{4}$$

$$H_\phi = \left\{\frac{A(t)}{r}[\frac{r}{a} - \frac{a}{r}] + \frac{C(t)}{r}[\frac{r}{b} - \frac{b}{r}]\right\}\sin\phi \tag{5}$$

At $r = b$,

$$H_\phi = K_o(t)\sin\phi \tag{6}$$

and thus

$$A(t) = \frac{K_o(t)b}{\frac{b}{a} - \frac{a}{b}} \tag{7}$$

At $r = a$,

$$-H_\phi\Big|_{r=a} = K_s$$

where K_s is the current in the sheet. From (7) of the preceding problem solution, we have at $r = a$

$$-\frac{1}{a}\frac{\partial}{\partial\phi}\frac{H_\phi}{\Delta\sigma} = -\mu_o\frac{\partial H_r}{\partial t} \tag{8}$$

Thus, using (4) and (5) in (8):

$$\frac{C(t)}{a}[\frac{a}{b} - \frac{b}{a}] = -\mu_o\Delta\sigma a\left\{\frac{2}{a}\frac{dA(t)}{dt} + \frac{dC(t)}{dt}[\frac{1}{b} + \frac{b}{a^2}]\right\} \tag{9}$$

Replacing A through (7) we obtain

$$\frac{dC}{dt} + \frac{[\frac{a}{b} - \frac{b}{a}]C(t)}{\mu_o\Delta\sigma a[\frac{a}{b} + \frac{b}{a}]} = \frac{2b}{(a/b)^2 - (b/a)^2}\frac{dK_o(t)}{dt} \tag{10}$$

Thus

$$\frac{dC}{dt} + \frac{C(t)}{\tau} = D\frac{dK}{dt} \tag{11}$$

with

$$\tau = \mu_o a\Delta\sigma\frac{[\frac{a}{b} + \frac{b}{a}]}{[\frac{a}{b} - \frac{b}{a}]} \tag{12}$$

$$D = \frac{2b}{(a/b)^2 - (b/a)^2}$$

The solution for a step of $K_o(t)$ is

$$C = DK_o e^{-t/\tau} \tag{13}$$

$$C(t) = DK_o e^{-t/\tau} = \frac{2bK_o}{(a/b)^2 - (b/a)^2} e^{-t/\tau}$$

Combining all the expressions gives the final answer:

$$\Psi = \frac{K_o b}{\frac{b}{a} - \frac{a}{b}} \left\{ \left[\frac{r}{a} - \frac{a}{r}\right] - 2\frac{\left[\frac{r}{b} - \frac{b}{r}\right]}{\left[\frac{a}{b} + \frac{b}{a}\right]} e^{-t/\tau} \right\} \cos\phi$$

For very short times $t/\tau \ll 1$, one has

$$\Psi = \frac{K_o b}{\frac{b}{a} - \frac{a}{b}} \left[\frac{r}{a} - \frac{a}{r} - 2\frac{\frac{r}{b} - \frac{b}{r}}{\frac{a}{b} + \frac{b}{a}}\right] \cos\phi = \frac{K_o b}{\frac{b}{a} + \frac{a}{b}} \left(\frac{r}{a} + \frac{a}{r}\right) \cos\phi$$

which is the same as (1). For very long times $\exp -t/\tau = 0$ and one obtains (2).

10.5 MAGNETIC DIFFUSION LAWS

10.5.1 (a) We first list the five equations (10.5.1)-(10.5.5)

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{10.5.1}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{10.5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu \mathbf{H} \tag{10.5.3}$$

$$\nabla \cdot \mu \mathbf{H} = 0 \tag{10.5.4}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{10.5.5}$$

Take the curl of (10.5.3) and use the identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla\nabla \cdot \mathbf{F} - \nabla^2 \mathbf{F} \tag{1}$$

also note that

$$\nabla \cdot \mathbf{J} = \nabla \cdot \sigma \mathbf{E} = \sigma \nabla \cdot \mathbf{E} = 0 \tag{2}$$

because σ is uniform. Therefore,

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mu \mathbf{H} \tag{3}$$

or

$$-\nabla^2(\mathbf{J}/\sigma) = -\mu\frac{\partial}{\partial t}\mathbf{J} \tag{4}$$

(b) Since $\mathbf{J} = \mathbf{i}_z J_z$, equation (b) follows immediately from (4). We now use (10.5.3)

$$\nabla\times(\mathbf{J}/\sigma) = -\frac{\partial}{\partial t}\mu\mathbf{H}$$

But

$$\nabla\times(\mathbf{J}/\sigma) = \frac{1}{\sigma}\nabla\times(\mathbf{i}_z J_z(x,y)) = \frac{1}{\sigma}\Big(\mathbf{i}_x\frac{\partial}{\partial y}J_z - \mathbf{i}_y\frac{\partial}{\partial x}J_z\Big)$$

and thus

$$\frac{\partial\mathbf{H}}{\partial t} = -\frac{\partial}{\partial y}\Big(\frac{J_z}{\sigma\mu}\Big)\mathbf{i}_x + \frac{\partial}{\partial x}\Big(\frac{J_z}{\sigma\mu}\Big)\mathbf{i}_y$$

10.6 MAGNETIC DIFFUSION TRANSIENT RESPONSE

10.6.1 The expressions for H_z and J_y obey the diffusion equation, no matter what signs are assigned to the coefficients. The summations cancel the field $-K_p x/b$ and current density K_p/b respectively, at $t = 0$ and eventually decay. If one turns off a drive from a steady state, the current density is initially uniform, equal to K_p/b and the field is equal to $-K_p x/b$ and then decays. But, the symmations with reversed signs have precisely that behavior.

10.6.2 (a) The magnetic field is

$$\mathbf{H} = \mathbf{i}_z H_z = K_p \tag{1}$$

and there is no E-field, nor J within the block.

(b) When the current-source is suddenly turned off, the H-field cannot disappear instantaneously; the current returns through the conducting block, but still circulates in the perfect conductor around the block. For this boundary value problem we must change the eigenfunctions. At $x = 0$, the field remains finite, because there is a circulation current terminating it. Thus we have, instead of (10.6.15),

$$H_z = \sum_{n-\text{odd}}^{\infty} C_n \cos\Big(\frac{n\pi}{2b}x\Big)e^{-t/\tau_n} \tag{2}$$

with the decay times

$$\tau_n = \frac{4\mu\sigma b^2}{(n\pi)^2} \tag{3}$$

Initially, H_z is uniform, and thus, using orthogonality

$$\int_{-b}^{0} H_z \cos\frac{m\pi}{2b}x\,dx = K_p\frac{2b}{m\pi}\sin\frac{m\pi}{2} = \frac{b}{2}C_m \tag{4}$$

and thus

$$C_m = (-1)^{\frac{m-1}{2}}\Big(\frac{4}{m\pi}\Big)K_p; \quad m \text{ odd} \tag{5}$$

$$H_z = \sum_{n-\mathrm{odd}} (-1)^{\frac{n-1}{2}} \frac{4}{n\pi} K_p \cos\Big(\frac{n\pi}{2b}x\Big)e^{-t/\tau_n}$$

The current density is

$$J_y = -\frac{\partial H_z}{\partial x} = \frac{2}{b}\Sigma(-1)^{\frac{n-1}{2}} K_p \sin\Big(\frac{n\pi}{2b}x\Big)e^{-t/\tau_n}$$

If we pick a new origin at $x' = x + b$, then

$$\begin{aligned}\sin\Big(\frac{n\pi}{2b}x\Big) &= \sin\Big(\frac{n\pi}{2b}x' - \frac{n\pi}{2}\Big) = -\cos\frac{n\pi}{2b}x' \sin\Big(\frac{n\pi}{2}\Big)\\ &= -(-1)^{\frac{n-1}{2}}\cos\Big(\frac{n\pi}{2b}x'\Big) \quad \text{for } n \text{ odd}\end{aligned}$$

Interestingly, we find

$$J_y = -\frac{2K_p}{b}\sum_{n-\mathrm{odd}} \cos\Big(\frac{n\pi}{2b}x'\Big)e^{-t/\tau_n}$$

At $t = 0$ this is the expansion of a unit impulse function at $x' = 0$ of content $-2K_p$. All the current now flows through a thin sheet at the end of the block. The factor of 2 comes in because the problem has been solved as a symmetric problem at $x' = 0$, and thus half of the current "flows" in the "imagined" other half.

10.7 SKIN EFFECT

10.7.1 (a) In order to find the impedance, we need to know the voltage v, the complex current being $\hat{K}_s$. The voltage is (see Fig. 10.7.2)

$$v = aE_y \tag{1}$$

and, from Faraday's law

$$\frac{\partial \hat{E}_y}{\partial x} = -j\omega\mu\hat{H}_z \tag{2}$$

From (2) and (10.7.10)

$$E_y = \frac{j\omega\mu\delta}{(1+j)}\frac{\big(e^{(1+j)\frac{x}{\delta}} + e^{-(1+j)\frac{x}{\delta}}\big)}{\big(e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}\big)}\hat{K}_s \tag{3}$$

and thus the impedance is at $x = -b$

$$Z = \frac{a\hat{E}_y}{d\hat{K}_s} = \frac{ja\omega\mu\delta}{d(1+j)} \frac{e^{(1+j)\frac{b}{\delta}} + e^{-(1+j)\frac{b}{\delta}}}{e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}} \tag{4}$$

But the factor in front is

$$\frac{ja\omega\mu\delta}{d(1+j)} = \frac{a(1+j)}{d\sigma\delta} \tag{5}$$

(b) When $b \ll \delta$, we can expand the exponentials and obtain

$$\begin{aligned} Z &= \frac{a(1+j)}{d\sigma\delta} \frac{1 + (1+j)\frac{b}{\delta} + 1 - (1+j)\frac{b}{\delta}}{1 + (1+j)\frac{b}{\delta} - 1 + (1+j)\frac{b}{\delta}} \\ &= \frac{a(1+j)}{d\sigma\delta} \frac{1}{(1+j)\frac{b}{\delta}} = \frac{a}{d\sigma b} \end{aligned} \tag{6}$$

(c) When $b \gg \delta$, then we need retain only the exponential $\exp[(1+j)b/\delta]$ with the result:

$$Z = \frac{a(1+j)}{d\sigma\delta} \tag{7}$$

so that

$$\mathrm{Re}(Z) = \frac{a}{d\sigma\delta}$$

This looks like (6) with b replaced by δ.

10.7.2 (a) When the block is shorted, we have to add the two solutions $\exp \pm (1+j)\frac{x}{\delta}$ so that they add at the termination. Indeed, if we set

$$\hat{H}_z = A[e^{-(1+j)\frac{x}{\delta}} + e^{(1+j)\frac{x}{\delta}}] \tag{1}$$

then the E-field is, from

$$\frac{\partial \hat{E}_y}{\partial x} = -j\omega\mu\hat{H}_z \tag{2}$$

and thus through integration

$$\hat{E}_y = \frac{j\omega\mu\delta}{(1+j)}\hat{A}[e^{-(1+j)\frac{x}{\delta}} - e^{(1+j)\frac{x}{\delta}}] \tag{3}$$

and is indeed zero at $x = 0$. In order to obtain $\hat{H}_z = \hat{K}_s$ at $x = -b$ we adjust $\hat{A}$ so that

$$\hat{H}_z = \hat{K}_s \frac{e^{(1+j)\frac{x}{\delta}} + e^{-(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}} + e^{-(1+j)\frac{b}{\delta}}} \tag{4}$$

(b) The high frequency distribution is governed by the $\exp -(1+j)\frac{x}{\delta}(x<0)$ and thus

$$\hat{H}_z \simeq \hat{K}_s \frac{e^{-(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}}} = \hat{K}_s e^{-(1+j)\frac{x-b}{\delta}} \tag{5}$$

This is the same expression as the one obtained from (10.7.10) by neglecting $\exp -(1+j)\frac{x}{\delta}$ and $\exp(1+j)b/\delta$.

(c) The impedance is obtained from (3) and (4)

$$\frac{a\hat{E}_y}{dK_s}\Big|_{x=-b} = \frac{a(1+j)}{d\sigma\delta}\frac{e^{(1+j)b/\delta} - e^{-(1+j)b/\delta}}{e^{(1+j)b/\delta} + e^{-(1+j)b/\delta}}$$

SOLUTIONS TO CHAPTER 11

11.0 INTRODUCTION

11.0.1 The Kirchhoff voltage law gives

$$v = v_c + L\frac{di}{dt} + Ri \tag{1}$$

where

$$i = C\frac{dv_c}{dt} \tag{2}$$

Multiplying (1) by i we get the power flowing into circuit

$$vi = v_c i + \frac{d}{dt}\Big(\frac{1}{2}Li^2\Big) + Ri^2 \tag{3}$$

But

$$v_c i = C\frac{dv_c}{dt}v_c = \frac{d}{dt}\Big(\frac{1}{2}Cv_c^2\Big) \tag{4}$$

and thus we have shown

$$vi = \frac{d}{dt}w + i^2R \tag{5}$$

where

$$w = \Big(\frac{1}{2}Cv_c^2 + \frac{1}{2}Li^2\Big) \tag{6}$$

Since w is under a total time derivative it integrates to zero, when the excitation i starts from zero and ends at zero. This indicates storage, since the energy supplied by the excitation is extracted after deexcitation. The term i^2R is positive definite and indicates power consumption.

11.1 INTEGRAL AND DIFFERENTIAL CONSERVATION STATEMENTS

11.1.1 (a) If $S = S_x\mathbf{i_x}$, then there is no power flow through surfaces with normals perpendicular to x. The surface integral

$$\oint_S \mathbf{S}\cdot d\mathbf{a}$$

gives $(x_1 > x_2)$

$$[S_x(x_1) - S_x(x_2)]A$$

because S_x is independent of y and z.

(b) Because W and P_d are also independent of y and z, the integrations transverse to the x-axis are simply multiplications by A. Hence from (11.1.1)

$$-A[S_x(x_1) - S_x(x_2)] = A\frac{d}{dt}\int W dx + A\int P_d dx$$

When $x_1 - x_2 = \Delta x$,

$$S_x(x_1) = S_x(x_2) + \frac{\partial S_x}{\partial x}\Big|_{x_2}\Delta x$$

$\int W dx = W\Delta x$, $\int P_d dx = P_d \Delta x$ and we get

$$-\frac{\partial S_x}{\partial x} = \frac{\partial W}{\partial t} + P_d$$

We have to use partial time derivatives, because W is also a function of x.

(c) The time rate of change of energy and the power dissipated must be equal to the net power flow, which is equal to the difference of the power flowing in and the power flowing out.

11.2 POYNTING'S THEOREM

11.2.1 (a) The power flow is

$$\mathbf{E} \times \mathbf{H} = -E_x H_z \mathbf{i}_y \tag{1}$$

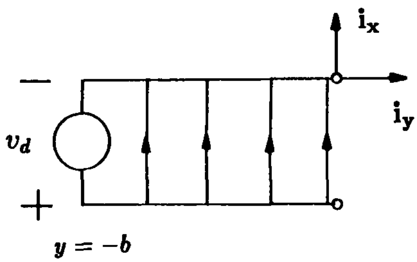


Figure S11.2.1

The EQS field is

$$E_x = \frac{V_d}{a} \tag{2}$$

$$\frac{\partial H_z}{\partial y} = \epsilon\frac{\partial E_x}{\partial t} \tag{3}$$

and thus

$$H_z = y\epsilon_o \frac{\partial E_x}{\partial t} \tag{4}$$

since $H_z = 0$ at $y = 0$. From (1), (2), and (4)

$$\mathbf{E} \times \mathbf{H} = -\mathbf{i}_y y\epsilon_o \frac{V_d}{a} \frac{d}{dt}\left(\frac{V_d}{a}\right) = -\mathbf{i}_y \frac{y\epsilon_o}{a^2} V_d \frac{dV_d}{dt} \tag{5}$$

(b) The power input is:

$$-\int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a}$$

over the cross-section at $y = -b$ where $d\mathbf{a} = -\mathbf{i}_y$ and therefore,

$$-\int \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = \frac{b\epsilon_o}{a^2} awV_d \frac{dV_d}{dt} = \frac{d}{dt}\left(\frac{1}{2}CV_d^2\right) \tag{6}$$

with

$$C = \frac{\epsilon_o bw}{a}$$

(c) The time rate of change of the electric energy is

$$\begin{aligned} \frac{d}{dt}\int W_e dv &= \frac{d}{dt}\int \frac{1}{2}\epsilon_o \mathbf{E}^2 dv = \frac{d}{dt}\left[\frac{1}{2}\epsilon_o\left(\frac{V_d}{a}\right)^2 abw\right] \\ &= \frac{d}{dt}\left(\frac{1}{2}\frac{\epsilon_o bw}{a}V_d^2\right) = \frac{d}{dt}\left(\frac{1}{2}CV_d^2\right) \qquad \text{QED} \end{aligned} \tag{7}$$

(d) The magnetic energy is

$$\begin{aligned} W_m &= \int \frac{1}{2}\mu_o \mathbf{H}^2 dv = \frac{1}{2}\mu_o aw \int_{-b}^{0} H_z^2 dy \\ &= \frac{1}{2}\mu_o aw \frac{b^3}{3}\left[\epsilon_o \frac{d}{dt}\frac{V_d}{a}\right]^2 \end{aligned} \tag{8}$$

Now

$$\frac{d}{dt}V_d \sim \frac{V_d}{\tau}$$

where τ is the time of interest. Therefore,

$$W_m = \frac{1}{6}\frac{\mu_o\epsilon_o b^2}{\tau^2}\epsilon_o \frac{bw}{a}V_d^2 \ll \frac{1}{2}\epsilon_o \frac{bw}{a}V_d^2$$

if

$$\frac{1}{3}\frac{\mu_o\epsilon_o b^2}{\tau^2} = \frac{1}{3}\frac{b^2}{c^2\tau^2} \ll 1$$

11.2.2 (a)

$$H_z = -\frac{I_d}{w} \tag{1}$$

From Faraday's law

$$-\frac{\partial E_x}{\partial y} = -\mu_o \frac{\partial H_z}{\partial t} \tag{2}$$

and therefore

$$E_x = -\mu_o y \frac{d}{dt}\left(\frac{I_d}{w}\right) \tag{3}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = -E_x H_z \mathbf{i}_y = -\mathbf{i}_y \frac{\mu_o y}{w^2} I_d \frac{dI_d}{dt}$$

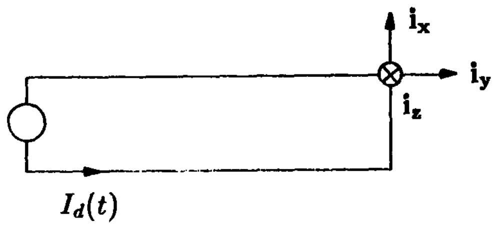


Figure S11.2.2

(b) The input power is $-\int \mathbf{S} \cdot d\mathbf{a}$, integrated over the cross-section at $y = -b$ with $d\mathbf{a} \parallel -\mathbf{i}_y$. The result is

$$-\int \mathbf{S} \cdot d\mathbf{a} = \frac{\mu_o b}{w^2} aw \frac{d}{dt} \frac{1}{2} I_d^2 = \frac{d}{dt} \frac{1}{2} L I_d^2$$

with

$$L = \frac{\mu_o ab}{w}$$

(c) The magnetic energy is

$$\int W_m dv = \int dv \frac{1}{2} \mu_o \mathbf{H}^2 = \frac{1}{2} abw\mu_o \frac{I_d^2}{w^2} = \frac{1}{2} L I_d^2$$

with the same L as defined above. Thus the magnetic energy by itself balances the conservation equation.

(d) The electric energy storage is

$$\begin{aligned}\int W_e dv &= \int \frac{1}{2} \epsilon_o \mathbf{E}^2 dv = \frac{1}{2} \epsilon_o \frac{\mu_o^2}{w^2} \left(\frac{dI_d}{dt}\right)^2 \frac{b^3}{3} aw \\ &= \frac{1}{3} \epsilon_o \mu_o b^2 \frac{1}{2} \frac{\mu_o ba}{w} \frac{I_d^2}{\tau^2} = \frac{1}{3} \frac{\epsilon_o \mu_o b^2}{\tau^2} \int W_m dv\end{aligned}$$

where $dI_d/dt \simeq I_d/\tau$, with τ equal to the characteristic time over which I_d changes appreciably. Thus,

$$\int W_e dv \ll \int W_m dv$$

as long as

$$\frac{1}{3}\frac{\epsilon_o \mu_o b^2}{\tau^2} = \frac{1}{3}\frac{b^2}{c^2\tau^2} \ll 1$$

11.3 OHMIC CONDUCTORS WITH LINEAR POLARIZATION AND MAGNETIZATION

11.3.1 (a) The electric field of a dipole current source is

$$\mathbf{E} = \frac{i_p d}{4\pi\sigma r^3}\left[2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta\right] \tag{1}$$

The H-field is given by Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E} \tag{2}$$

Now, by symmetry it appears that $\mathbf{H}$ must be ϕ directed

$$\mathbf{H} = \mathbf{i}_\phi H_\phi \tag{3}$$

and thus

$$\nabla \times \mathbf{H} = \mathbf{i}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \mathbf{i}_\theta \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \tag{4}$$

By inspection of the θ-component of (4), with the aid of (1) and (2), one finds

$$H_\phi = \frac{i_p d}{4\pi r^2}\sin\theta \tag{5}$$

The same result is obtained by comparing r components. Therefore,

$$\mathbf{E} \times \mathbf{H} = \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma}\frac{1}{r^5}\left[-2\cos\theta\sin\theta \mathbf{i}_\theta + \sin^2\theta \mathbf{i}_r\right] \tag{6}$$

The density of dissipated power is

$$\begin{aligned} P_d = \mathbf{E}\cdot\mathbf{J} = \sigma \mathbf{E}^2 &= \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}\left[4\cos^2\theta + \sin^2\theta\right] \\ &= \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}\left[1 + 3\cos^2\theta\right] \end{aligned} \tag{7}$$

(c) Poynting's theorem requires

$$\nabla \cdot \mathbf{S} + P_d = 0 \tag{8}$$

Now $\nabla \cdot \mathbf{S}$ in spherical coordinate is

$$\nabla \cdot \mathbf{S} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(S_\theta \sin\theta)$$

Now

$$\begin{aligned}\nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}[-3\sin^2\theta - 4\cos^2\theta + 2\sin^2\theta] \\ &= -\left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}[1 + 3\cos^2\theta]\end{aligned} \tag{9}$$

Thus, (8) is indeed satisfied according to (7) and (9).

(d)

$$\Phi = \frac{i_p d}{4\pi\sigma}\frac{\cos\theta}{r^2}$$

$$\begin{aligned}\nabla \cdot (\Phi \mathbf{J}) &= \left(\frac{i_p d}{4\pi}\right)^2 \nabla \cdot \frac{1}{\sigma r^5}[2\cos^2\theta \mathbf{i}_r + \sin\theta\cos\theta \mathbf{i}_\theta] \\ &= -\left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}[6\cos^2\theta - 2\cos^2\theta + \sin^2\theta] \\ &= -\left(\frac{i_p d}{4\pi}\right)^2 \frac{1}{\sigma r^6}[1 + 3\cos^2\theta] = \nabla \cdot (\mathbf{E} \times \mathbf{H})\end{aligned}$$

(e) We need not form the cross-product to obtain flow density. The power flow density is the current density weighted by local potential Φ.

11.3.2 (a) The potential is a solution of Laplace's equation

$$\Phi = -\frac{v}{ln\frac{a}{b}} ln(r/a) \tag{1}$$

$$\mathbf{E} = \frac{v}{ln(a/b)}\frac{\mathbf{i}_r}{r} \tag{2}$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E} = \frac{\sigma v}{ln(a/b)}\frac{\mathbf{i}_r}{r} \tag{3}$$

from Ampère's law. By symmetry

$$\mathbf{H} = \mathbf{i}_\phi H_\phi \tag{4}$$

and

$$-\frac{\partial H_\phi}{\partial z} = \frac{\sigma v}{ln(a/b)}\frac{1}{r} \tag{5}$$

and thus

$$H_\phi = -\frac{\sigma v}{ln(a/b)}\frac{z}{r} \tag{6}$$

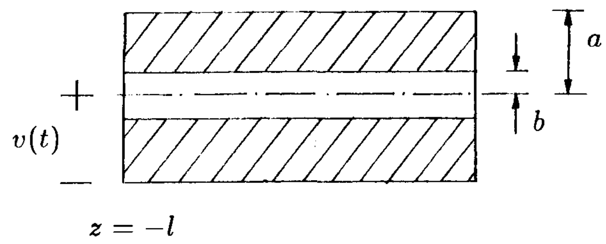


Figure S11.3.2a

(b) The Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = -\mathbf{i}_z\frac{\sigma v^2}{ln^2(a/b)}\frac{z}{r^2} \tag{7}$$

(c) The Poynting flux is

$$\begin{aligned}\oint \mathbf{S}\cdot d\mathbf{a} &= -\int_{r=b}^{r=a} S_z 2\pi r dr\Big|_{z=-l} \\ &= -\frac{2\pi\sigma v^2 l}{ln^2(a/b)}ln(a/b) = -\frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{8}$$

(d) The dissipated power is

$$\begin{aligned}\int dv P_d = \int dv\sigma\mathbf{E}^2 &= \int_{z=-l}^{0}\int_{r=b}^{r=a}\frac{\sigma v^2}{ln^2(a/b)}\frac{2\pi r}{r^2}drdz \\ &= \frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{9}$$

(e) The alternate form for the power flow density is

$$\mathbf{S} = \Phi\mathbf{J} = -\sigma\frac{v^2}{ln^2(a/b)}ln(r/a)\frac{\mathbf{i_r}}{r} \tag{10}$$

$$\begin{aligned}\oint \mathbf{S}\cdot d\mathbf{a} &= -[S_r(r=b) - S_r(r=a)]2\pi b l \\ &= -\frac{2\pi\sigma l}{ln(a/b)}v^2\end{aligned} \tag{11}$$

This is indeed equal to the negative of (9).

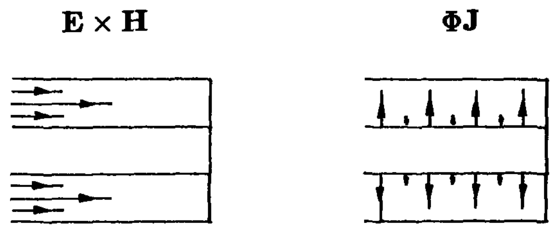


Figure S11.3.2b

(f) See Fig. S11.3.2b.

(g) At $z = -l$,

$$\oint \mathbf{H} \cdot d\mathbf{s} = \frac{2\pi\sigma l v}{ln(a/b)} = i \tag{12}$$

Thus

$$vi = \frac{2\pi\sigma l}{ln(a/b)} v^2 \quad \text{Q.E.D.} \tag{13}$$

11.3.3 (a) The electric field is

$$\mathbf{E} = \frac{v}{d}\mathbf{i}_z \tag{1}$$

From Ampère's law:

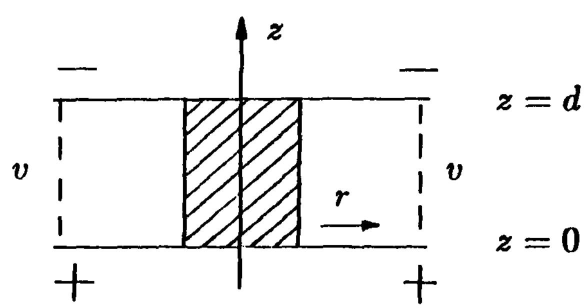


Figure S11.3.3

$$\oint \mathbf{H} \cdot d\mathbf{s} = \int \left(\mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a} \tag{2}$$

$$2\pi r H_\phi = \begin{cases} \pi r^2\left[\frac{\sigma v}{d} + \epsilon\frac{d}{dt}(v/d)\right] & \text{for } r < b \\ \pi b^2\left[\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right] + \pi(r^2 - b^2)\epsilon_o\frac{d}{dt}(v/d) & \text{for } b < r < a \end{cases} \tag{3}$$

and thus

$$H_\phi = \begin{cases} \frac{r}{2}\left[\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right] & \text{for } r < b \\ \frac{1}{2r}\left[\frac{\sigma b^2 v}{d} + \epsilon b^2\frac{d}{dt}(v/d) + (r^2 - b^2)\epsilon_o\frac{d}{dt}(v/d)\right] & \text{for } b < r < a \end{cases} \tag{4}$$

The Poynting flux density

$$\begin{aligned} \mathbf{E}\times\mathbf{H} &= \mathbf{i_z}\times\mathbf{i_\phi} E_z H_\phi \\ &= \begin{cases} -\mathbf{i_r}\frac{r}{2}\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right)\frac{v}{d} & \text{for } r < b \\ -\mathbf{i_r}\frac{1}{2r}\left\{\frac{1}{d}\left[\epsilon b^2 + \epsilon_o(r^2 - b^2)\right]\frac{d}{dt}(v) + \frac{\sigma b^2}{d}v\right\}\frac{v}{d} & \text{for } b < r < a \end{cases} \end{aligned} \tag{5}$$

(b)

$$\begin{aligned} -\int \mathbf{E}\times\mathbf{H}\cdot d\mathbf{a} &= -\int_{z=0}^{d} \mathbf{i_r}\cdot\mathbf{E}\times\mathbf{H}\,dz\,2\pi r \\ &= \begin{cases} \pi r^2\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}(v/d)\right)v & r < b \\ \pi\left\{\frac{1}{d}[\epsilon b^2 + \epsilon_o(r^2 - b^2)]\frac{d}{dt}(v/d) + \frac{\sigma b^2}{d}v\right\}v & b < r < a \end{cases} \end{aligned} \tag{6}$$

For $r < b$,

$$\begin{aligned} \int \frac{dW}{dt}dv + \int P_d dv &= \int_{z=0}^{d}\int_{r=0}^{r} \frac{1}{2}\epsilon\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=0}^{r} \sigma(v/d)^2 2\pi r\,dr\,dz \\ &= \epsilon v\frac{d}{dt}(v/d)\pi r^2 + \sigma\frac{v^2}{d}\pi r^2 \end{aligned} \tag{7a}$$

For $b < r < a$:

$$\begin{aligned} \int \frac{dW}{dt}dv + \int P_d dv &= \int_{z=0}^{d}\int_{r=0}^{b} \frac{1}{2}\epsilon\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=b}^{r} \frac{1}{2}\epsilon_o\frac{d}{dt}(v/d)^2 2\pi r\,dr\,dz \\ &\quad + \int_{z=0}^{d}\int_{r=0}^{b} \sigma(v/d)^2 2\pi r\,dr\,dz \\ &= \pi\left\{\left[\frac{\epsilon b^2}{d}v + \epsilon_o\frac{(r^2 - b^2)}{d}v\right]\frac{d}{dt}(v) + \frac{\sigma b^2}{d}v^2\right\} \quad \text{Q.E.D.} \end{aligned} \tag{7b}$$

(c)

$$\mathbf{S} = \Phi\left(\mathbf{J} + \epsilon\frac{\partial\mathbf{E}}{\partial t}\right) \tag{8}$$

The potential Φ is given by

$$\Phi = -\frac{v}{d}(z-d)$$

and

$$\mathbf{J} + \epsilon\frac{\partial \mathbf{E}}{\partial t} = \begin{cases} \mathbf{i}_z\left(\sigma\frac{v}{d} + \epsilon\frac{d}{dt}\frac{v}{d}\right) & \text{for } r < b \\ \mathbf{i}_z\epsilon_o\frac{d}{dt}\frac{v}{d} & \text{for } b < r < a \end{cases} \tag{9}$$

Therefore,

$$\mathbf{S} = \begin{cases} -\mathbf{i}_z\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)(z-d)\frac{v}{d} & \text{for } r < b \\ -\mathbf{i}_z\frac{\epsilon_o}{d}\frac{dv}{dt}(z-d)\frac{v}{d} & \text{for } b < r < a \end{cases} \tag{10}$$

(d) The integral is

$$-\oint \mathbf{S}\cdot d\mathbf{a} = \int_0^r 2\pi r dr[S_z(z=0) - S_z(z=d)] \tag{11}$$

For $r < b$:

$$= \int_0^r 2\pi r dr d\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)\frac{v}{d} = \pi r^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v \tag{12a}$$

For $a < r < b$:

$$\begin{aligned} &= \int_0^b 2\pi r dr d\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)\frac{v}{d} + \int_b^r 2\pi r dr d\frac{\epsilon_o}{d}\frac{dv}{dt}\frac{v}{d} \\ &= \pi b^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v + \pi(r^2 - b^2)\frac{\epsilon_o}{d}\frac{dv}{dt}v \end{aligned} \tag{12b}$$

Equations (12) agree with (6).

(e) The power input at $r = a$ is from (12b)

$$\pi b^2\left(\frac{\sigma v}{d} + \frac{\epsilon}{d}\frac{dv}{dt}\right)v + \pi(a^2 - b^2)\frac{\epsilon_o}{d}\frac{dv}{dt}v = vi \tag{13}$$

where

$$i = \pi b^2\left[\frac{\sigma v}{d} + \epsilon\frac{d}{dt}(v/d)\right] + \pi(a^2 - b^2)\epsilon_o\frac{d}{dt}(v/d)$$

which is the sum of the displacement current and convection current between the two plates.

11.3.4 (a) From the potentials (7.5.4) and (7.5.5) we find the E-field

$$\begin{aligned} \mathbf{E} = -\nabla\Phi = \mathbf{i}_r E_o\cos\phi&\left(1 + \left(\frac{R}{r}\right)^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right) \\ - \mathbf{i}_\phi E_o\sin\phi&\left(1 - \left(\frac{R}{r}\right)^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right) \qquad r < R \end{aligned} \tag{1a}$$

and

$$\frac{2\sigma_a}{\sigma_b + \sigma_a} E_o(\mathbf{i_r} \cos\phi - \mathbf{i}_\phi \sin\phi) \qquad r < R \tag{1b}$$

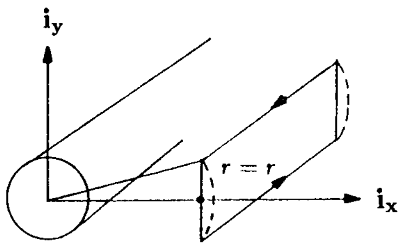


Figure S11.3.4

The **H**-field is z-directed by symmetry and can be found from Ampère's law using a contour in a $z - x$ plane, symmetrically located around the x-axis and of unit width in z-direction. If the contour is picked as shown in Fig. S11.3.4, then

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{a} = 2H_z = 2\int_0^\phi J_r r d\phi$$
$$= \begin{cases} 2r\sigma_a E_o \sin\phi \left(1 + (\frac{R}{r})^2 \frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right) & \text{for } r > R \\ 2r\sigma_b E_o \frac{2\sigma_a}{\sigma_b + \sigma_a} \sin\phi & \text{for } r < R \end{cases} \tag{2}$$

The Poynting vector is

$$\mathbf{E} \times \mathbf{H} = E_\phi H_z \mathbf{i_r} - E_r H_z \mathbf{i}_\phi = -\mathbf{i_r} r\sigma_a E_o^2 \sin^2\phi \left[1 - (\frac{R}{r})^4 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)^2\right]$$
$$- \mathbf{i}_\phi r\sigma_a E_o^2 \sin\phi\cos\phi \left[1 + (\frac{R}{r})^2 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2 \qquad r > R$$
$$= -\mathbf{i_r} r\sigma_b E_o^2 \sin^2\phi \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2$$
$$- \mathbf{i}_\phi r\sigma_b E_o^2 \sin\phi\cos\phi \left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \qquad r < R$$

(b) The alternate power flow vector $\mathbf{S} = \Phi\mathbf{J}$ follows from (7.5.4)-(7.5.5) and (1)

$$\Phi\mathbf{J} = -\mathbf{i_r}\sigma_a E_o^2 r \cos^2\phi \left[1 - (\frac{R}{4})^4 \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)^2\right]$$
$$+ \mathbf{i}_\phi \sigma_a E_o^2 r \sin\phi\cos\phi \left[1 - (\frac{R}{r})^2 \frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right]^2 \qquad r > R$$
$$= -\mathbf{i_r}\sigma_b E_o^2 r \cos^2\phi \left(\frac{2\sigma_a}{\sigma_b + \sigma_a}\right)^2$$
$$+ \mathbf{i}_\phi \sigma_b E_o^2 r \sin\phi\cos\phi (\frac{2\sigma_a}{\sigma_b + \sigma_a})^2 \qquad r < R \tag{4}$$

(c) The power dissipation density P_d is

$$P_d = \sigma\mathbf{E}^2 = \sigma_a E_o^2 \cos^2\phi\left[1 + (\frac{R}{r})^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right]^2 + \sigma_a E_o^2 \sin^2\phi\left[1 - (\frac{R}{r})^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right]^2 \qquad r > R \tag{5a}$$

$$= \sigma_b E_o^2\left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \qquad r < R \tag{5b}$$

(d) We must now evaluate $\nabla\cdot(\mathbf{E}\times\mathbf{H})$ and $\nabla\cdot\Phi\mathbf{J}$ and show that they yield $-P_d$.

$$\begin{aligned}\nabla\cdot\mathbf{S} = \frac{1}{r}\frac{\partial(rS_r)}{\partial r} + \frac{1}{r}\frac{\partial S_\phi}{\partial\phi} &= -2\sigma_a E_o^2\sin^2\phi\left[1 + (\frac{R}{r})^4\left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)^2\right]\\ &- \sigma_a E_o^2(\cos^2\phi - \sin^2\phi)\left[1 + (\frac{R}{r})^2\left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2\\ &= -\sigma_a E_o^2\sin^2\phi\left[1 - (\frac{R}{r})^2\left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2\\ &- \sigma_a E_o^2\cos^2\phi\left[1 + (\frac{R}{r})^2\left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2\end{aligned} \tag{6a}$$

for $r > R$,

$$\begin{aligned}\nabla\cdot S &= -2\sigma_b E_o^2\sin^2\phi\left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2\\ &- (\cos^2\phi - \sin^2\phi)\sigma_b E_o^2\left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2\\ &= -\sigma_b E_o^2\left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2\end{aligned} \tag{6b}$$

for $r < R$. Comparison of (5) and (6) shows that the Poynting theorem is obeyed. Now take the other form of power flow. The analysis is simplified if we note that $\nabla\cdot\mathbf{J} = 0$. Thus

$$\begin{aligned}\nabla\cdot\Phi\mathbf{J} = \mathbf{J}\cdot\nabla\Phi &= J_r\frac{\partial}{\partial r}\Phi + J_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\Phi = -\sigma\mathbf{E}^2\\ &= -\sigma_a E_o^2\cos^2\phi\left[1 + (\frac{R}{r})^2\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a})\right]^2\\ &- \sigma_a E_o^2\sin^2\phi\left[1 - (\frac{R}{r})\left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)\right]^2 \qquad r > R\end{aligned} \tag{7a}$$

and

$$\nabla\cdot\Phi\mathbf{J} = -\sigma_b E_o^2\left(\frac{2\sigma_a}{\sigma_a + \sigma_b}\right)^2 \qquad r < R \tag{7b}$$

Q.E.D.

11.4 ENERGY STORAGE

√

11.4.1 From (8.5.14)-(8.5.15) we find the H-fields. Integrating the energy density we find

$$
\begin{aligned}
w &= \int dv \frac{1}{2}\mu_o \mathbf{H}^2 = \frac{1}{2}\mu_o \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \Big(\frac{Ni}{3R}\Big)^2 \\
&+ \frac{1}{2}\mu_o \int_R^\infty r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \Big(\frac{Ni}{6R}\Big)^2 \Big(\frac{R}{r}\Big)^6 (4\cos^2\theta + \sin^2\theta) \\
&= \frac{1}{2}\mu_o \frac{4\pi R^3}{3}\Big(\frac{Ni}{3R}\Big)^2 + \frac{1}{2}\mu_o 2\pi \times 4\Big(\frac{Ni}{6R}\Big)^2 \times \frac{1}{3}R^3 \\
&= \frac{1}{2}\frac{2\pi N^2 \mu_o R}{9} i^2
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\int_0^\pi \sin\theta d\theta (4\cos^2\theta + \sin^2\theta) &= -\int_0^\pi d(\cos\theta)(3\cos^2\theta + 1) \\
&= \int_{-1}^1 dx(3x^2+1) = (x^3 + x)\Big|_{-1}^1 = 4
\end{aligned}
$$

Because

$$
w = \frac{1}{2}Li^2
$$

we find that

$$
L = \frac{2\pi N^2 \mu_o R}{9}
$$

Q.E.D.

11.4.2 The scalar potential of P9.6.3 is

$$
\Psi = \frac{N}{2}\frac{i\cos\phi}{1+\frac{\mu}{\mu_o}} \begin{cases} R/r & r > R \\ -\frac{\mu}{\mu_o}\frac{r}{R} & r < R \end{cases}
$$

The field is

$$
\mathbf{H} = \frac{N}{2R}\frac{i\cos\phi}{1+\frac{\mu}{\mu_o}} \begin{cases} (\mathbf{i}_r \cos\phi + \mathbf{i}_\phi \sin\phi)(R/r)^2; & r > R \\ \frac{\mu}{\mu_o}(\mathbf{i}_r\cos\phi - \mathbf{i}_\phi \sin\phi); & r < R \end{cases}
$$

The energy is

$$
\begin{aligned}
w_m &= l\int_0^R \frac{1}{2}\mu_o \mathbf{H}^2 r dr d\phi + l\int_R^\infty \frac{1}{2}\mu \mathbf{H}^2 r dr d\phi \\
&= \frac{1}{2}\mu_o \pi R^2 l\left(\frac{Ni}{2R}\frac{\mu/\mu_o}{1+\frac{\mu}{\mu_o}}\right)^2 \\
&\quad + \frac{1}{2}\mu l\left(\frac{Ni}{2R}\frac{1}{1+\frac{\mu}{\mu_o}}\right)^2 2\pi\int_R^\infty \left(\frac{R}{r}\right)^4 r dr \\
&= \frac{1}{2}\mu_o \pi l \frac{N^2(\mu/\mu_o)^2 i^2}{4(1+\frac{\mu}{\mu_o})^2} + \frac{1}{2}\mu\pi l \frac{N^2 i^2}{4(1+\frac{\mu}{\mu_o})^2} \\
&= \frac{1}{2}\frac{\mu\pi l N^2}{(1+\frac{\mu}{\mu_o})} i^2 = \frac{1}{2} L i^2
\end{aligned}
$$

11.4.3 The vector potential is from (8.6.32)

$$
\mathbf{A} = -\frac{\mu_o N i}{3}\left[\left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right)\right]\sin\phi \mathbf{i}_z \qquad r < a \tag{1}
$$

$$
\begin{aligned}
\mu_o \mathbf{H} &= \nabla \times \mathbf{A} \\
&= -\mathbf{i}_z \times \nabla A_z = \frac{Ni}{3a}\mathbf{i}_z \times \left\{[2(r/a) - 1]\sin\phi \mathbf{i}_r + \left(\frac{r}{a} - 1\right)\cos\phi \mathbf{i}_\phi\right\} \\
&= -\frac{\mu_o N i}{3a}\left[\left(\frac{r}{a} - 1\right)\cos\phi \mathbf{i}_r - \left(2\frac{r}{a} - 1\right)\sin\phi \mathbf{i}_\phi\right]
\end{aligned}
$$

The energy is

$$
\begin{aligned}
l\int_0^a\int_0^{2\pi}\frac{1}{2}\mu_o \mathbf{H}^2 r dr d\phi &= \frac{\mu_o}{2} l\left(\frac{Ni}{3a}\right)^2 \pi\int_0^a r dr\left[\left(\frac{r}{a} - 1\right)^2 + \left(2\frac{r}{a} - 1\right)^2\right] \\
&= \frac{\mu_o}{2} l\left(\frac{Ni}{3}\right)^2 \frac{\pi}{4} = \frac{1}{2} L i^2
\end{aligned}
$$

Therefore,

$$
L = \frac{\pi}{36}\mu_o l N^2
$$

11.4.4 The energy differential is

$$
dw_m = i_1 d\lambda_1 + i_2 d\lambda_2 \tag{1}
$$

The coenergy is

$$
\begin{aligned}
dw'_m &= d(i_1\lambda_1) + d(i_2\lambda_2) - dw_m = \lambda_1 di_1 + \lambda_2 di_2 \\
&= (L_{11} i_1 + L_{12} i_2) di_1 + (L_{21} i_1 + L_{22} i_2) di_2
\end{aligned} \tag{2}
$$

with

$$L_{21} = L_{12} \tag{3}$$

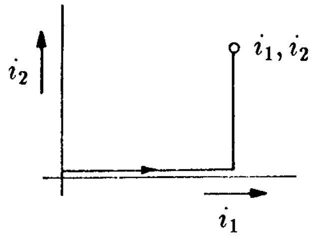


Figure S11.4.4

If we integrate this expression along a conveniently chosen path in the $i_1 - i_2$ plane as shown in Fig S11.4.4, we get

$$\begin{aligned}\int_{\substack{i_1=0\\i_2=0}}^{i_1} L_{11} i_1 di_1 + \int_{\substack{i_2=0\\i_1=\text{const}}}^{i_2} (L_{21} i_1 + L_{22} i_2) di_2 \\ = \frac{1}{2} L_{11} i_1^2 + L_{21} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 \\ = \frac{1}{2}(L_{11} i_1^2 + L_{12} i_1 i_2 + L_{21} i_2 i_1 + L_{22} i_2^2) \\ = \frac{1}{2} L_o (N_1^2 i_1^2 + 2N_1 N_2 i_1 i_2 + N_2^2 i_2^2)\end{aligned} \tag{4}$$

when the last expression is written symmetrically, using (3).

11.4.5 If the gap is small $(a - b) \ll a$, the field is radial and can be evaluated using Ampère's law with the contour shown in Fig. S11.4.5. It is simplest to evaluate the field of stator and rotor separately and then to add. The field vanishes at $\phi = \pi/2$ and thus

$$\oint_C \mathbf{H} \cdot d\mathbf{s} = -(a - b) H_r(\phi) \tag{1}$$

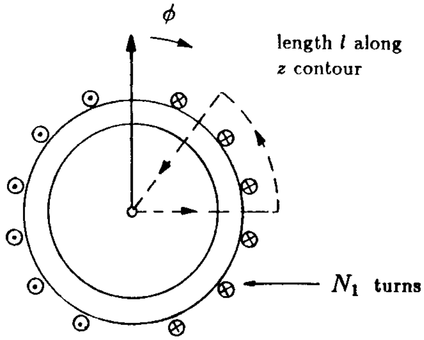


Figure S11.4.5

For the stator field, the integral of the current density is

$$\int_S \mathbf{J}\cdot d\mathbf{a} = -\int_{\phi}^{\pi/2} \frac{N_1 i_1}{2a}\sin\phi a d\phi = -\frac{N_1 i_1}{2}\cos\phi \tag{2}$$

where N_1 is the total number of terms of the stator winding. Therefore, the stator field is given by

$$\mathbf{H} \simeq \mathbf{i}_r H_r = \mathbf{i}_r \frac{N_1 i_1}{2(a-b)}\cos\phi \tag{3}$$

The rotor coil gives the field

$$H_r = \frac{N_2 i_2}{2(a-b)}\cos(\phi-\theta) \tag{4}$$

where N_2 is the total number of turns of the rotor winding. In a linear system, coenergy is equal to energy, only the independent variables have to be chosen properly, i.e. the energy expressed in terms of the currents, is coenergy. When expressed in terms of fluxes, it is energy. The coenergy density is

$$W_m' = \frac{1}{2}\mu_o H_r^2 \tag{5}$$

The coenergy is

$$\begin{aligned} w_m' &= \frac{1}{2}\mu_o(a-b)l\int_0^{2\pi} H_r^2 a d\phi \\ &= \frac{1}{2}\frac{\mu_o l a\pi}{4(a-b)}[(N_1 i_1)^2 + (N_2 i_2)^2 + 2N_1 N_2 i_1 i_2\cos\theta] \end{aligned} \tag{6}$$

We find

$$L_{ii} = \frac{\pi\mu_o a l}{4(a-b)}N_i^2 \tag{7}$$

and

$$L_{12} = L_{21} = \frac{\pi\mu_o a l}{4(a-b)}N_1 N_2\cos\theta$$

11.4.6

$$\mathbf{D} = \left(\frac{\alpha_1}{\sqrt{1+\alpha_2 E^2}} + \epsilon_o\right)\mathbf{E}$$

The coenergy density in the nonlinear medium is [note $\mathbf{E}\cdot d\mathbf{E} = d(\frac{1}{2}\mathbf{E}^2)$]

$$\begin{aligned} W_e' &= \int_0^{\mathbf{E}} \mathbf{D}\cdot d\mathbf{E} = \int \frac{1}{2}\left(\frac{\alpha_1}{\sqrt{1+\alpha_2 E^2}} + \epsilon_o\right)dE^2 \\ &= \frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2 E^2} + \frac{1}{2}\epsilon_o E^2 \end{aligned}$$

In the linear material

$$w_e' = \frac{1}{2}\epsilon_o \mathbf{E}^2$$

Integrating the densities over the respective volumes one finds $(E^2 = v^2/a^2)$

$$w_e' = \left[\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{v^2}{a^2}} + \frac{1}{2}\epsilon_o\frac{v^2}{a^2}\right]\xi ca + \frac{1}{2}\epsilon_o\frac{v^2}{a^2}(b-\xi)ca$$

Q.E.D.

11.4.7 (a) $\mathbf{H} = \mathbf{i}_z i/w$ in both regions. Therefore,

$$\mathbf{B} = \mathbf{i}_z \mu_o i/w$$

in region (a)

$$\mathbf{B} = \mathbf{i}_z\left(\mu_o + \frac{\alpha_1}{\sqrt{1+\alpha_2 i^2/w^2}}\right)i/w$$

in region (b). The coenergy densities are

$$W_m' = \begin{cases} \frac{1}{2}\mu_o\frac{i^2}{w^2} & \text{in region (a)} \\ \frac{1}{2}\left(\mu_o\frac{i^2}{w^2} + 2\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{i^2}{w^2}}\frac{i^2}{w^2}\right) & \text{in region (b)} \end{cases}$$

The coenergy is

$$w_m' = wA_a\frac{1}{2}\mu_o\frac{i^2}{w^2} + wA_b\frac{1}{2}\left(\mu_o + 2\frac{\alpha_1}{\alpha_2}\sqrt{1+\alpha_2\frac{i^2}{w^2}}\right)\frac{i^2}{w^2}$$

11.5 ELECTROMAGNETIC DISSIPATION

11.5.1 From (7.9.16) we find an equation for the complex amplitude $\hat{E}_a$:

$$\hat{E}_a = \frac{j\omega\epsilon_b + \sigma_b}{(j\omega\epsilon_a + \sigma_a)b + (j\omega\epsilon_b + \sigma_b)a}\hat{v} \tag{1}$$

and since

$$a\hat{E}_a + b\hat{E}_b = \hat{v} \tag{2}$$

we find

$$\hat{E}_b = \frac{j\omega\epsilon_a + \sigma_a}{(j\omega\epsilon_a + \sigma_a)b + (j\omega\epsilon_b + \sigma_b)a}\hat{v} \tag{3}$$

(Another way of finding $\hat{E}_b$ from (1) is to note that $\hat{E}_a$ and $\hat{E}_b$ are related to each other by an interchange of a and b and of the subspcripts.) The time average power dissipation is

$$\begin{aligned}\langle p_d\rangle &= \frac{1}{2}\sigma_a|\hat{E}_a|^2 aA + \frac{1}{2}\sigma_b|\hat{E}_b|^2 bA \\ &= \frac{A}{2}\frac{a\sigma_a(\omega^2\epsilon_b^2 + \sigma_b^2) + b\sigma_b(\omega^2\epsilon_a^2 + \sigma_a^2)}{(b\sigma_a + a\sigma_b)^2 + \omega^2(b\epsilon_a + a\epsilon_b)^2}|\hat{v}|^2\end{aligned}$$

11.5.2 (a) The electric field follows from (7.9.36)

$$\hat{E}_b = -\nabla\hat{\Phi} = 3E_p(\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta)\frac{\sigma_a + j\omega\epsilon_a}{2\sigma_a + \sigma_b + j\omega(2\epsilon_a + \epsilon_b)}; \quad r < R \quad (1b)$$

Therefore

$$\langle P_d\rangle = \frac{1}{2}\sigma_b|\hat{E}_b|^2 = \frac{9}{2}|E_p|^2\sigma_b\frac{\sigma_a^2 + \omega^2\epsilon_a^2}{(2\sigma_a + \sigma_b)^2 + \omega^2(2\epsilon_a + \epsilon_b)^2}; \quad r < R \quad (2b)$$

The electric field in region (a) is

$$\begin{aligned}\hat{E}_a = E_p\Big\{\mathbf{i}_r\cos\theta\Big[1 - 2\frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}(R/r)^3\Big] \\ - \mathbf{i}_\theta\sin\theta\Big[1 + \frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}(R/r)^3\Big]\Big\}\end{aligned}$$

If we denote by

$$\hat{A} \equiv \frac{\sigma_a - \sigma_b + j\omega(\epsilon_a - \epsilon_b)}{(2\sigma_a + \sigma_b) + j\omega(2\epsilon_a + \epsilon_b)}$$

we obtain

$$\begin{aligned}\langle P_d\rangle = \frac{1}{2}\sigma_a|\hat{E}_a|^2 = |E_p|^2\{\cos^2\theta[1 - 4(R/r)^3\mathrm{Re}\,\hat{A} + 4(R/r)^6|\hat{A}|^2] \\ + \sin^2\theta[1 + 2(R/r)^3\mathrm{Re}\,\hat{A} + (R/r)^6|\hat{A}|^2]\}\end{aligned}$$

(b) The power dissipated is

$$\langle p_d\rangle = \frac{4\pi R^3}{3}\langle P_d\rangle \quad (3)$$

where $\langle P_d\rangle$ is taken from (2b).

11.5.3 (a) The magnetic field is z-directed and equal to the surface current in the sheet. In region (b)

$$\mathbf{H} = H^b\mathbf{i}_z \quad (1)$$

in region (a) it is

$$\mathbf{H} = \mathbf{i}_z K \quad (2)$$

The field at the sheet is, from Faraday's integral law

$$E_y = b\mu_o\frac{dH^b}{dt} \quad \text{at} \quad x = -b \quad (3)$$

The field at the source is

$$E_y = a\mu_o\frac{dK}{dt} + b\mu_o\frac{dH^b}{dt} \quad (4)$$

The power dissipated in the sheet is, using (3)

$$p_d = \int \sigma E_y^2 dv = \sigma \Delta w d b^2 \mu_o^2 \Big(\frac{dH^b}{dt}\Big)^2 \tag{5}$$

The stored energy is

$$\begin{aligned}\int_V W\,dv &= \frac{1}{2}\mu_o (H^a)^2 a d w + \frac{1}{2}\mu_o (H^b)^2 b d w \\ &= \frac{1}{2}\mu_o d w [b(H^b)^2 + aK^2]\end{aligned} \tag{6}$$

(b) The integral of the Poynting vector gives

$$\oint \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = -E_y H_z w d = -\big(a\mu_o \frac{dK}{dt} + b\mu_o \frac{dH^b}{dt}\big) K w d \tag{7}$$

Now

$$H_b = K - E_y \sigma \Delta = K - b\mu_o \frac{dH_b}{dt}\sigma\Delta \tag{8}$$

When we introduce this into (7) we get

$$\begin{aligned}\oint \mathbf{E} \times \mathbf{H} \cdot d\mathbf{a} = &- \Big\{\frac{1}{2} a\mu_o w d \frac{dK^2}{dt} + \frac{1}{2} b\mu_o w d \frac{dH^{b2}}{dt}\Big\} \\ &- \sigma b^2 w d \mu_o^2 \Big(\frac{dH^b}{dt}\Big)^2 \sigma\Delta\end{aligned} \tag{9}$$

But the last term is p_d; and the term in wavy brackets is the time rate of change of the magnetic energy.

11.5.4 Solving (10.4.13) for $\hat{A}$, under sinusoidal, steady state conditions, gives

$$\begin{aligned}\hat{A} &= \frac{1}{(j\omega\tau_m + 1)}\Big[-j\omega\tau_m + \frac{1 - \frac{\mu_o}{\mu}}{\mu_o \Delta \sigma a}\tau_m\Big] a^2 H_o \\ &= \frac{1}{(j\omega\tau_m + 1)}\Big[-j\omega\tau_m + \frac{\mu - \mu_o}{\mu + \mu_o}\Big] a^2 H_o\end{aligned} \tag{1}$$

From (10.4.11), we obtain $\hat{C}$

$$\hat{C} = -\frac{\mu_o}{\mu}\big(H_o + \frac{\hat{A}}{a^2}\big) = -\frac{\frac{2\mu_o}{\mu + \mu_o}}{1 + j\omega\tau_m} H_o \tag{2}$$

The discontinuity of the tangential magnetic field gives the current flowing in the cylinder. From (10.4.10)

$$\begin{aligned}\Delta \hat{H}_\phi &= -\big(H_o - \frac{\hat{A}}{a^2}\big)\sin\phi - \hat{C}\sin\phi \\ &= -\big[1 + j\omega\tau_m + j\omega\tau_m - \frac{\mu - \mu_o}{\mu + \mu_o} - \frac{2\mu_o}{\mu + \mu_o}\big]\frac{H_o \sin\phi}{1 + j\omega\tau_m} \\ &= -2\frac{j\omega\tau_m}{1 + j\omega\tau_m}\sin\phi H_o = \hat{K}_z\end{aligned} \tag{3}$$

Note the dependence of the current upon ω: when $\omega\tau_m \gg 1$, then the current is just large enough $(-2H_o \sin\phi)$ to cancel the field internal to the cylinder. When $\omega\tau_m \to 0$, of course, the current goes to zero. The jump of H_ϕ is equal to K. The power dissipated is, per unit axial length:

$$p_d = \frac{1}{2}\int \sigma|\hat{E}|^2 dv = \frac{1}{2}\sigma\Delta a \int_0^{2\pi} |\hat{E}_z|^2 d\phi \tag{4}$$

But

$$\sigma\hat{E}_t\Delta = \hat{K}_z \tag{5}$$

and thus

$$p_d = \frac{1}{2}\int_0^{2\pi} d\phi \frac{|\hat{K}_z|^2}{\sigma^2\Delta^2}\sigma\Delta a = \frac{\pi a}{\sigma\Delta}\frac{2\omega^2\tau_m^2 a}{1+\omega^2\tau_m^2}|H_o|^2 \tag{6}$$

11.5.5 (a) The applied field is in the direction normal to the paper, and is equal to

$$H_o \cos\omega t = N i_o \cos\omega t/d \tag{1}$$

The internal field is $H_o + K$ where K is the current flowing in the cylinder. From Faraday's law in complex form

$$\oint \hat{E}\cdot d\mathbf{s} = -j\omega\mu(\hat{H}_o + \hat{K})b^2 \tag{2}$$

Because $\hat{K}$ must be a constant, $\hat{E}$ tangential to the surface of the cylindrical shell must be constant. The path length is $4b$. We have

$$\hat{K} = \sigma\Delta\hat{E} = -\frac{j\omega\mu\sigma\Delta b}{4}(\hat{H}_o + \hat{K}) \tag{3}$$

and solving for $\hat{K}$

$$\hat{K} = -\frac{j\omega\tau_m}{1+j\omega\tau_m}H_o \tag{4}$$

where

$$\tau_m = \frac{\mu\sigma\Delta b}{4} \tag{5}$$

The surface current cancels H_o in the high frequency limit $\omega\tau_m \to \infty$. In the low frequency limit, it approaches zero as $\omega\tau_m$ approaches zero. Thus

$$p_d = \frac{1}{2}\int \sigma|\hat{E}|^2 dv = \frac{1}{2}\frac{4b\Delta d\sigma}{\sigma^2\Delta^2}|\hat{K}|^2 = \frac{2b}{\sigma\Delta d}N^2 i_o^2 \frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2} \tag{6}$$

(b) The time average Poynting flux is

$$\begin{aligned}
-\text{Re}\oint \hat{\mathbf{E}}\times\hat{\mathbf{H}}\cdot d\mathbf{a} &= -\text{Re}\ \frac{1}{2}4bd\hat{E}\hat{H}^* \\
&= -\text{Re}\ \{2bdH_o^*(-j\omega\tau_m)(\hat{H}_o+\hat{K})\} \\
&= \text{Re}\ 2bdj\omega\tau_m\hat{H}_o^*\hat{K} \\
&= \frac{2bd}{\sigma\Delta}\frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2}|H_o|^2 = \frac{2b}{\sigma\Delta d}\frac{\omega^2\tau_m^2}{1+\omega^2\tau_m^2}N^2 i_o^2
\end{aligned} \tag{7}$$

which is the same as above.

11.5.6 (a) When the volume current density is zero, then Ampère's law in the MQS limit becomes

$$\nabla \times \mathbf{H} = 0 \tag{1}$$

and Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu_o(\mathbf{H} + \mathbf{M}) \tag{2}$$

If we introduce complex notation to describe the sinusoidal steady state $\mathbf{E} = \operatorname{Re} \hat{\mathbf{E}}(\mathbf{r})e^{j\omega\tau}$ etc., then we get from the above

$$\nabla \times \hat{\mathbf{H}} = 0 \tag{3}$$

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu_o(\hat{\mathbf{H}} + \hat{\mathbf{M}}) \tag{4}$$

If $\hat{M}$ is linearly related to $\hat{H}$ we may write

$$\hat{\mathbf{M}} = \hat{\chi}_m \hat{\mathbf{H}} \tag{5}$$

where $\hat{\chi}_m$ is, in general, a function of ω, we may define

$$\hat{\mu} = \mu_o(1 + \hat{\chi}_m) \tag{6}$$

and write for (4)

$$\nabla \times \hat{\mathbf{E}} = -j\omega\hat{\mathbf{B}} \tag{7}$$

with

$$\hat{\mathbf{B}} \equiv \hat{\mu}\hat{\mathbf{H}} \tag{8}$$

Because $\nabla \cdot \mu_o(\hat{\mathbf{H}} + \hat{\mathbf{M}}) = 0$, we have

$$\nabla \cdot \hat{\mathbf{B}} = 0 \tag{9}$$

(b) The magnetic dipole moment is, according to (20) of the solution to P10.4.3.

$$\hat{m} = -2\pi R^3 \hat{H}_o \frac{j\omega\tau}{1 + j\omega\tau} \tag{10}$$

with $\tau = \mu_o\sigma\Delta R/3$. As $\omega\tau_m \to \infty$, this reduces to the result (9.5.16). The susceptibility is found from (5):

$$\hat{\chi}_m = -2\pi(R/s)^3 \frac{j\omega\tau}{1 + j\omega\tau}$$

where $1/s^3$ is the density of the dipoles.

(c) The magnetic field at $x = -l$ is

$$\hat{\mathbf{H}} = \mathbf{i}_z \hat{K} \tag{14}$$

The electric field follows from Faraday's law: applied to a contour along the perfect conductor and current generator

$$-a\hat{E}_y(-l) = -j\omega\hat{\mu}\hat{H}_z al \tag{15}$$

and thus

$$\hat{E}_y = j\omega\hat{\mu}l\hat{H}_z \tag{16}$$

The power dissipated is

$$\begin{aligned} p_d &= -\frac{1}{2}\text{Re} \oint \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \cdot d\mathbf{a} \\ &= \frac{1}{2}\text{Re}\ \hat{E}_y\hat{H}_z^*\big|_{x=-l} ad \\ &= \frac{1}{2}\text{Re}\ j\omega\hat{\mu}|\hat{K}|^2 adl \end{aligned} \tag{17}$$

Introducing (12) and (13) we find

$$p_d = \pi(R/s)^3\mu_o\frac{\omega^3\tau}{1+\omega^2\tau^2}|\hat{K}|^2 adl \tag{18}$$

11.5.7 From (10.7.15) we find

$$\hat{H}_z = \hat{K}_s \exp -(1+j)\left(\frac{x+b}{\delta}\right) \tag{1}$$

so that $H_z = K_s$ at the surface at $x = -b$. The current density is

$$\hat{\mathbf{J}} \simeq \nabla \times \hat{\mathbf{H}} = -\mathbf{i_y}\frac{\partial H_z}{\partial x} = \mathbf{i_y}\frac{(1+j)}{\delta}K_s \exp -(1+j)\left(\frac{x+b}{\delta}\right) \tag{2}$$

The power dissipation density is

$$P_d = \frac{1}{2}\frac{|\hat{J}_y|^2}{\sigma} \tag{3}$$

and thus the power dissipated per unit area is

$$\int_{x=-b}^{x=0} P_d dx \simeq \frac{|\hat{K}_s|^2}{\sigma}\int_{x=-b}^{\infty} \exp -\frac{2(x+b)}{\delta}dx = \frac{|\hat{K}_s|^2}{2\sigma\delta} \quad \text{watts/m}^2$$

11.5.8 (a) From (10.7.10) we find $\hat{H}_z$ everywhere. The current density is

$$\hat{J} = (\nabla \times \hat{\mathbf{H}})_y = -\frac{\partial H_z}{\partial x} = \frac{(1+j)}{\delta}\hat{K}_s \frac{e^{-(1+j)\frac{x}{\delta}} + e^{(1+j)\frac{x}{\delta}}}{e^{(1+j)\frac{b}{\delta}} - e^{-(1+j)\frac{b}{\delta}}} \tag{1}$$

The density of dissipated power is:

$$\begin{aligned} P_d = \frac{1}{2}\frac{|\hat{J}|^2}{\sigma} &= \frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{e^{-2x/\delta} + 2\cos\frac{2x}{\delta} + e^{2x/\delta}}{e^{2b/\delta} - 2\cos\frac{2b}{\delta} + e^{-2b/\delta}} \\ &= \frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{\cosh\frac{2x}{\delta} + \cos\frac{2x}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}} \end{aligned} \tag{2}$$

The total dissipated power is

$$\begin{aligned} p_d = ad\int_{x=-b}^{0} P_d dx &= ad\frac{1}{\sigma\delta^2}|\hat{K}_s|^2 \frac{\delta}{2}\frac{\sinh\frac{2x}{\delta} + \sin\frac{2x}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}}\Bigg|_{-b}^{0} \\ &= ad\frac{|\hat{K}_s|^2}{2\sigma\delta}\frac{\sinh\frac{2b}{\delta} + \sin\frac{2b}{\delta}}{\cosh\frac{2b}{\delta} - \cos\frac{2b}{\delta}} \end{aligned} \tag{3}$$

(b) Take the limit $\delta \ll b$. Then $\sinh\frac{2b}{\delta} \simeq \cosh\frac{2b}{\delta} = \frac{1}{2}e^{2b/\delta}$ and the sines and cosines are negligible.

$$p_d = \frac{ad}{2\sigma\delta}|\hat{K}_s|^2 \tag{4}$$

which is consistent with P11.5.7. When $2b/\delta \ll 1$, then

$$\cosh\left(\frac{2b}{\delta}\right) - \cos\left(\frac{2b}{\delta}\right) \approx 1 + \frac{1}{2}\left(\frac{2b}{\delta}\right)^2 - \left(1 - \frac{1}{2}\left(\frac{2b}{\delta}\right)^2\right) = \left(\frac{2b}{\delta}\right)^2 \tag{5}$$

$$\sinh\left(\frac{2b}{\delta}\right) + \sin\left(\frac{2b}{\delta}\right) \simeq \frac{4b}{\delta} \tag{6}$$

and thus

$$p_d = ad\frac{1}{2\sigma\delta}|\hat{K}_s|^2\frac{\delta}{b} = \frac{ad|\hat{K}_s|^2}{2\sigma b} \tag{7}$$

The total current is

$$\hat{i} = \hat{K}_s d \tag{8}$$

The resistance is

$$R = \frac{a}{\sigma b d} \tag{9}$$

and

$$\frac{1}{2}|\hat{i}|^2 R = ad\frac{|\hat{K}_s|^2}{2\sigma b} \tag{10}$$

Q.E.D.

11.5.9 The constitutive law

$$\frac{\partial \mathbf{M}}{\partial t} = \gamma \mathbf{H} \tag{1}$$

gives for complex vector amplitudes

$$j\omega \hat{\mathbf{M}} = \gamma \hat{\mathbf{H}} \tag{2}$$

and thus

$$\hat{\chi}_m = \frac{\gamma}{j\omega} \tag{3}$$

and

$$\hat{\mu} = \mu_o(1 + \hat{\chi}_m) = \mu_o\left(1 + \frac{\gamma}{j\omega}\right) \tag{4}$$

The flux is

$$\mathbf{B} = \hat{\mu}\hat{\mathbf{H}} = \mu_o\left(1 + \frac{\gamma}{j\omega}\right)\hat{\mathbf{H}} \tag{5}$$

The induced voltage is

$$v = \frac{d\lambda}{dt} \Rightarrow \hat{v} = j\omega\hat{\lambda} \tag{6}$$

and

$$\hat{\lambda} = N_1 \frac{\pi w^2}{4} \hat{B}_\phi \tag{7}$$

But

$$\hat{H}_\phi = \frac{N_1 \hat{i}}{2\pi R} \tag{8}$$

and thus

$$\hat{\lambda} = \hat{\mu} \frac{N_1^2 w^2}{8R} \hat{i} \tag{9}$$

and thus

$$\hat{v} = j\omega\hat{\lambda} = j\omega\mu_o \frac{N_1^2 w^2}{8R} \hat{i} + \mu_o \frac{\gamma N_1^2 w^2 \hat{i}}{8R} = (j\omega L + R_m)\hat{i} \tag{10}$$

Thus

$$L = \mu_o \frac{N_1^2 w^2}{8R} \qquad R_m = \frac{\mu_o \gamma N_1^2 w^2}{8R} \tag{11}$$

11.5.10 (a) The peak H field is

$$H_{\text{peak}} = \frac{N_1 i_{\text{peak}}}{2\pi R} = \frac{N_1}{2\pi R}\frac{2H_c 2\pi R}{N_1} = 2H_c \tag{1}$$

Thus (see Fig. S11.5.10a).

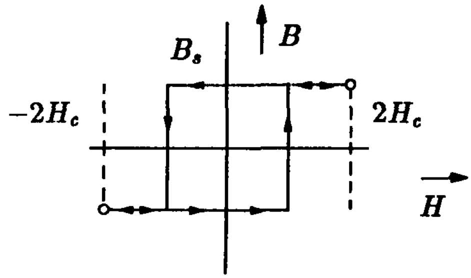


Figure S11.5.10a

(b) The terminal voltage is

$$v = \frac{d}{dt}N_1\frac{\pi w^2}{4}B \propto \frac{dB}{dt} \tag{2}$$

The B field jumps suddenly, when $H = H_c$. This is shown in Fig. S11.5.10b. The voltage is impulse like with content equal to the flux discontinuity: $2N_1\frac{\pi w^2}{4}B_s$.

(c) The time average power input is $\int vidt$ integrated over one period. Contributions come only at impulses of voltage and are equal to

$$\int vidt = 2 \times 2N_1\frac{\pi w^2}{4}B_s \cdot i(t_o) \tag{3}$$

But

$$\frac{N_1 i(t_o)}{2\pi R} = H_c \tag{4}$$

and thus

$$\int vidt = 4N_1\frac{\pi w^2}{4}B_s H_c\frac{2\pi R}{N_1} = \left(2\pi R\frac{\pi w^2}{4}\right)4B_s H_c \tag{5}$$

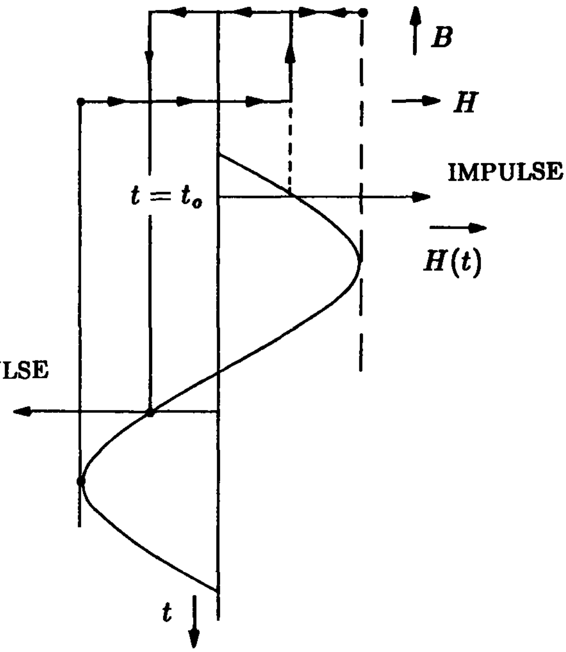


Figure S11.5.10b

(d) The energy fed into the magnetizable material per unit volume within time dt is

$$dt\mathbf{H} \cdot \frac{\partial}{\partial t}\mu_o(\mathbf{H}+\mathbf{M}) = dt\mathbf{H} \cdot \frac{\partial}{\partial t}\mathbf{B} = \mathbf{H} \cdot d\mathbf{B} \tag{6}$$

As one goes through a full cycle,

$$\oint \mathbf{H} \cdot d\mathbf{B} = \text{area of hysteresis loop} \tag{7}$$

This is $4H_cB_s$. Thus the total energy fed into the material in one cycle is

$$\text{volume} \oint \mathbf{H} \cdot d\mathbf{B} = \left(2\pi R\frac{\pi w^2}{4}\right)4B_sH_c \tag{8}$$

11.6 ELECTRICAL FORCES ON MACROSCOPIC MEDIA

11.6.1 The capacitance of the system is

$$C = \frac{\epsilon_o(b-\xi)d}{a}$$

The force is

$$f_e = \frac{1}{2}v^2\frac{dC}{d\xi} = -v^2\frac{\epsilon_o d}{2a}$$

11.6.2 The capacitance per unit length is from (4.6.27)

$$C = \frac{\pi\epsilon_o}{ln(\frac{l}{R} + \sqrt{(l/R)^2 - 1})} \tag{1}$$

where the distance between the two cylinders is $2l$. Thus replacing l by $\xi/2$, we can find the force per unit length on one cylinder by the other from

$$\begin{aligned} f_e &= \frac{1}{2}v^2\frac{dC}{d\xi} = \frac{1}{2}v^2\frac{d}{d\xi}\left[\frac{\pi\epsilon_o}{ln[\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}]}\right] \\ &= -\frac{1}{2}v^2\frac{\pi\epsilon_o}{ln^2[(\xi/2R) + \sqrt{(\xi/2R)^2 - 1}]}\frac{\frac{1}{2R} + \frac{\xi}{(2R)^2}\frac{1}{\sqrt{(\xi/2R)^2 - 1}}}{\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}} \end{aligned} \tag{2}$$

This expression can be written in a form, in which it is more recognizable. Using the fact that $\lambda_l = Cv$ we may write

$$f_o = -\frac{\lambda_l^2}{4\pi\epsilon_o R}\frac{1 + (\xi/2R)/\sqrt{(\xi/2R)^2 - 1}}{\frac{\xi}{2R} + \sqrt{(\xi/2R)^2 - 1}} \tag{3}$$

When $\xi/2R \gg 1$, and the cylinder radii are much smaller than their separation, the above becomes

$$f_e = -\frac{\lambda_l^2}{2\pi\epsilon_o 2\xi} \tag{4}$$

This is the force on a line charge λ_l in the field $\lambda_l/(2\pi\epsilon_o 2\xi)$.

11.6.3 The capacitance is made up of two capacitors connected in parallel.

$$C = \frac{2\pi\epsilon_o(l - \xi)}{\ln(a/b)} + \frac{2\pi\epsilon\xi}{ln(a/b)}$$

(a) The force is

$$f_e = \frac{1}{2}v^2\frac{dC}{d\xi} = v^2\frac{\pi(\epsilon - \epsilon_o)}{ln(a/b)}$$

(b) The electric circuit is shown in Fig. S11.6.3. Since R is very small, the output voltage is

$$v_o = iR$$

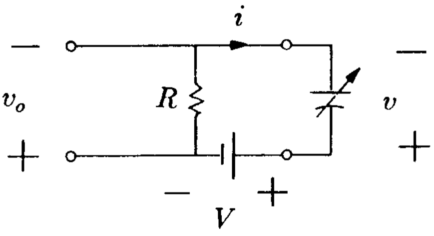


Figure S11.6.3

From Kirchoff's voltage law

$$iR + V = v$$

Now

$$q = Cv$$

and

$$i = \frac{dq}{dt} = \frac{d}{dt}(Cv) = \frac{dC}{dt}v + C\frac{dv}{dt}$$

If R is small, then v is still almost equal to V and dv/dt is much smaller than $(vdC/dt)/C$. Then

$$-i \approx V\frac{dC}{dt}$$

and

$$v_o = Ri = -2\pi RV(\epsilon - \epsilon_o)\frac{d\xi}{dt}/ln(a/b)$$

11.6.4 The capacitance is determined by the region containing the electric field

$$C = \frac{2\pi\epsilon_o(l - \xi)}{ln(a/b)}$$

(a) The force is

$$f_e = \frac{1}{2}V^2\frac{dC}{d\xi} = -\frac{\pi\epsilon_o}{ln(a/b)}V^2$$

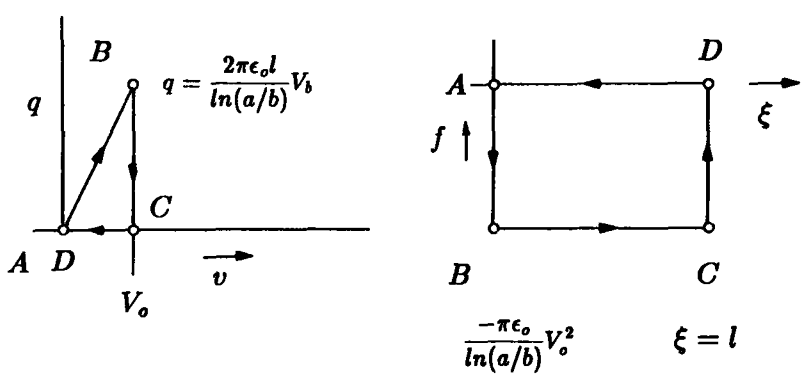


Figure S11.6.4

(b) See Fig. S11.6.4. When $\xi = 0$, then the value of capacitance is maximum. Going from A to B in the $f - \xi$ plane changes the force from 0 to a finite negative value by application of a voltage. Travel from B to C maintains the force while ξ is increasing. Thus ξ increases at constant voltage. The motion from C to D is done at constant ξ by decreasing to voltage from a finite value to zero. Finally as one returns from D to A the inner cylinder is pushed back in. In the $q - v$ plane, the point A is one of zero voltage and maximum capacitance. As the voltage is increased to V_o, the charge increases to

$$q = CV_o = \frac{2\pi\epsilon_o l}{ln(a/b)} V_o$$

The trajectory from B to C keeps the voltage fixed while increasing ξ, decreasing the capacitance. Thus the charge decreases. As one moves from C to D at constant ξ decreasing the voltage to zero, one moves back to the origin. Changing ξ to zero at zero voltage does not change the charge so that D and A coincide in the $q - v$ plane.

(c) The energy input is evaluated as the areas in the $q - v$ plane and the $\xi - f$ plane. The area in the $\xi - f$ plane is

$$\frac{\pi\epsilon_o l}{ln(a/b)} V_o^2$$

and the area in the $v - q$ plane is

$$\frac{1}{2}\frac{2\pi\epsilon_o l}{ln(a/b)} V_o^2$$

which is the same.

11.6.5 Using the coenergy value obtained in P11.4.6, we find the force is

$$f_e = \frac{\partial w_e'}{\partial \xi}\Big|_v = \left[\frac{\alpha_1}{\alpha_2}\left(\sqrt{1 + \frac{\alpha^2 v^2}{\alpha^2}} - 1\right) + \frac{1}{2}\epsilon_o \frac{v^2}{\alpha^2}\right] ca - \frac{1}{2}\frac{\epsilon_o v^2}{a} c$$

11.7 MACROSCOPIC MAGNETIC FORCES

11.7.1 The magnetic coenergy is

$$w_m' = \frac{1}{2}(L_{11} i_1^2 + 2L_{12} i_1 i_2 + L_{22} i_2^2)$$

The force is

$$\begin{aligned} f_m = \frac{\partial w_m'}{\partial x}\Big|_{i_1, i_2} &= \frac{1}{2}\left(\frac{dL_{11}}{dx} i_1^2 + 2\frac{dL_{12}}{dx} i_1 i_2 + \frac{dL_{22}}{dx} i_2^2\right) \\ &= \frac{1}{2}\left(N_1^2 \frac{dL_o}{dx} i_1^2 + 2N_1 N_2 \frac{dL_o}{dx} i_1 i_2 + N_2^2 \frac{dL_o}{dx} i_2^2\right) \end{aligned}$$

Since

$$L_o = \frac{aw\mu_o}{x(1 + \frac{a}{b})}$$

we have

$$f_m = -\frac{1}{2}(N_1^2 i_1^2 + 2N_1 N_2 i_1 i_2 + N_2^2 i_2^2) \frac{aw\mu_o}{x^2(1 + \frac{a}{b})}$$

11.7.2 The inductance of the coil is, according to the solution to (9.7.6)

$$f_m = \frac{1}{2}i^2\frac{dL}{dx} = -\frac{1}{2}i^2\frac{\mu_o N^2}{\left[\frac{x}{\pi a^2}+\frac{g}{2\pi a d}\right]^2}\frac{1}{\pi a^2}$$

√

11.7.3 We first compute the inductance of the circuit. The two gaps are in series so that Ampère's law for the electric field gives

$$y(H_1 + H_2) = ni \tag{1}$$

where H_1 is the field on the left, H_2 is the field on the right. Flux conservation gives

$$H_1(a-x)d = H_2 x d \tag{2}$$

Thus

$$H_1 = \frac{ni}{y}\frac{x}{a}$$

The flux is

$$\Phi_\lambda = \frac{\mu_o ni}{y}\left(\frac{a-x}{a}\right)xd$$

The inductance is

$$L = n\Phi_\lambda = \frac{\mu_o n^2}{y}\frac{xd(a-x)}{a}$$

The force is

$$\mathbf{f}_m = \frac{1}{2}i^2\left(\frac{\partial L}{\partial x}\mathbf{i}_x + \frac{\partial L}{\partial y}\mathbf{i}_y\right) = \frac{1}{2}i^2\frac{\mu_o n^2 d}{a}\left\{\frac{(a-2x)}{y}\mathbf{i}_x - \frac{x(a-x)}{y^2}\mathbf{i}_y\right\}$$

11.7.4 Ampère's law applied to the fields H_o and H at the inner radius in the media μ_o and μ, respectively, gives

$$H_o\int_b^a \frac{b}{r}dr = H\int_b^a \frac{b}{r}dr = Ni \tag{1}$$

and thus

$$H_o = H = \frac{Ni}{b\, ln\frac{a}{b}} \tag{2}$$

The flux is composed of the two individual fluxes

$$\Phi_\lambda = 2\pi\frac{Ni}{ln\frac{a}{b}}[\mu_o(l-\xi)+\mu\xi] \tag{3}$$

The inductance is

$$L = N\Phi_\lambda/i = \frac{2\pi}{ln(a/b)}N^2\{\mu\xi + \mu_o(l-\xi)\} \tag{4}$$

The force is

$$f(i,\xi) = \frac{1}{2}i^2\frac{dL}{d\xi} = \frac{\pi(\mu-\mu_o)}{ln(a/b)}N^2 i^2 \tag{5}$$

11.7.5 The H-field in the two gaps follows from Ampère's integral law

$$2H\Delta = 2Ni \tag{1}$$

The flux is

$$\Phi_\lambda = \mu_o H d(2\alpha - \theta)R = \mu_o N i d(2\alpha - \theta)R/\Delta \tag{2}$$

and the inductance

$$L = \frac{2N\Phi_\lambda}{i} = 2N^2\mu_o \frac{dR(2\alpha - \theta)}{\Delta} \tag{3}$$

The torque is

$$\tau = \frac{1}{2}i^2\frac{dL}{d\theta} = -\mu_o dRN^2 i^2/\Delta \tag{4}$$

11.7.6 The coenergy is

$$\begin{aligned} w'_m &= \int [\lambda_a di_a + \lambda_b di_b + \lambda_r di_r] \\ &= \frac{1}{2}L_s i_a^2 + \frac{1}{2}L_s i_b^2 + \frac{1}{2}L_r i_r^2 \\ &\quad + M\cos\theta i_a i_r + M\sin\theta i_r i_b \end{aligned} \tag{1}$$

where we have taken advantage of the fact that the integral is independent of path. We went from $i_a = i_b = i_r = 0$ first to i_a, then raised i_b to its final value and then i_r to its final value.

(b) The torque is

$$\tau = \frac{\partial w'_m}{\partial \theta} = i_r(-M\sin\theta i_a + M\cos\theta i_b)$$

(c) The two coil currents i_a and i_b produce effective z-directed surface currents with the spatial distributions $\sin\phi$ and $\sin(\phi - \frac{\pi}{2}) = -\cos\phi$ respectively. If they are phased as indicated, the effective surface current is proportional to

$$\cos(\omega t)\sin\phi - \sin\omega t\cos\phi = \sin(\phi - \omega t)$$

Thus the rate of change of the maximum of the current density is $d\phi/dt = \omega$.

(d) The torque is

$$\begin{aligned} \tau &= I_r[-M\sin(\Omega t - \gamma)I\cos\omega t + M\cos(\Omega t - \gamma)I\sin\omega t] \\ &= I_r I(-M\sin(\Omega t - \gamma - \omega t) \end{aligned}$$

But if $\Omega = \omega$, then

$$\tau = I_r I M \sin\gamma$$

11.8 FORCES ON MACROSCOPIC ELECTRIC AND MAGNETIC DIPOLES

11.8.1 (a) The potential obeys Laplace's equation and must vanish for $y \rightarrow \infty$. Thus the solution is of the form $e^{-\beta y} \cos \beta x$. The voltage distribution of $y = 0$ picks the amplitude as V_o. The E field is

$$\mathbf{E} = \beta V_o (\sin \beta x \mathbf{i_x} + \cos \beta x \mathbf{i_y}) e^{-\beta y}$$

(b) The force on a dipole is

$$\mathbf{f} = \mathbf{p} \cdot \nabla \mathbf{E} = 4\pi\epsilon_o R^3 (\mathbf{E} \cdot \nabla)\mathbf{E}$$

It behooves us to compute $(\mathbf{E} \cdot \nabla)\mathbf{E}$. We first construct the operator

$$\mathbf{E} \cdot \nabla = \beta V_o e^{-\beta y} (\sin \beta_x \frac{\partial}{\partial x} + \cos \beta x \frac{\partial}{\partial y})$$

Thus

$$\begin{aligned}
\mathbf{E} \cdot \nabla \mathbf{E} = {} & \beta V_o e^{-\beta y} \Big\{ \sin \beta x \frac{\partial}{\partial x} \big[\beta V_o (\sin \beta_x \mathbf{i_x} + \cos \beta x \mathbf{i_y}) e^{-\beta y}\big] \\
& + \cos \beta x \frac{\partial}{\partial y} \big[\beta V_o (\sin \beta x \mathbf{i_x} + \cos \beta x \mathbf{i_y}) e^{-\beta y}\big] \\
= {} & \beta^2 V_o^2 \beta \Big[(\sin \beta x \cos \beta x \mathbf{i_x} - \sin^2 \beta x \mathbf{i_y}) e^{-\beta y} \\
& - (\cos \beta x \sin \beta x \mathbf{i_x} + \cos^2 \beta x \mathbf{i_y}) e^{-\beta y}\Big] \\
= {} & -\beta^2 V_o^2 \beta \mathbf{i_y} e^{-\beta y}
\end{aligned}$$

and thus

$$\mathbf{f} = -4\pi\epsilon_o R^3 (\beta V_o)^2 \beta \mathbf{i_y} e^{-\beta y}$$

11.8.2 Again we compute, as in P11.8.1,

$$(\mathbf{E} \cdot \nabla)\mathbf{E}$$

in spherical coordinates

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_o r^2} \mathbf{i_r} \tag{1}$$

and the gradient operator is

$$\nabla = \mathbf{i_r} \frac{\partial}{\partial r} + \mathbf{i_\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{i_\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \qquad)2)$$

Thus,

$$\mathbf{E} \cdot \nabla = \frac{Q}{4\pi\epsilon_o r^2} \frac{\partial}{\partial r} \tag{3}$$

and

$$\mathbf{E}\cdot\nabla\mathbf{E} = -\frac{Q}{4\pi\epsilon_o r^2}\frac{2Q}{4\pi\epsilon_o r^3} = -\frac{2Q^2}{(4\pi\epsilon_o)^2 r^5} \tag{4}$$

and the force is

$$\mathbf{f} = \mathbf{p}\cdot\nabla\mathbf{E} = -4\pi\epsilon_o R^3 \frac{2Q^2}{(4\pi\epsilon_o)^2 r^5} = -\frac{2Q^2 R^3}{4\pi\epsilon_o r^5} \tag{5}$$

Note that the computation was simple, because $(\partial/\partial r)\mathbf{i}_r = 0$. In general, derivatives of the unit vectors in spherical coordinates are not zero.

11.8.3 The magnetic potential Ψ is of the form

$$\Psi = \begin{cases} A\cos\beta x e^{-\beta y} & y > 0 \\ A\cos\beta x e^{\beta y} & y < 0 \end{cases}$$

At $y = 0$, the potential has to be continuous and the normal component of $\mu_o\mathbf{H}$ has to be discontinuous to account for the magnetic surface charge density

$$\rho_m = \nabla\cdot\mu_o\mathbf{M} \Rightarrow \mu_o M_o \cos\beta x$$

Thus

$$\Psi = \frac{M_o}{2\beta}\cos\beta x e^{-\beta y}$$

This is of the same form as Φ of P11.8.1 with the correspondence

$$V_o \leftrightarrow M_o/2\beta$$

The infinitely permeable particle must have $H = 0$ inside. Thus, in a uniform field $H_o\mathbf{i}_z$, the potential around the particle is (We use, temporarily, the conventional orientation of the spherical coordinate, $\theta = 0$ axis as along z. Later we shall identify it with the orientation of the dipole moment.)

$$\Psi = -H_o R\cos\theta\left[\frac{r}{R} - (R/r)^2\right]$$

The particle produces a dipole field

$$\frac{H_o R^3}{r^3}(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta) = \frac{m}{4\pi r^3}(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta)$$

Thus the magnetic dipole is

$$\mu_o m = 4\pi\mu_o H_o R^3$$

This is analogous to the electric dipole with the correspondence

$$\mathbf{p} \leftrightarrow \mu_o\mathbf{m}$$

$$H_o \leftrightarrow E_o$$

$$\mu_o \leftrightarrow \epsilon_o$$

Since the force is

$$\mathbf{f} = \mu_o\mathbf{m}\cdot\nabla\mathbf{H}$$

we find perfect correspondence.

11.8.4 The field of a magnetic dipole $\mu_o \mathbf{m} \parallel \mathbf{i}_z$ is

$$\mathbf{H} = \frac{\mu_o m}{4\pi\mu_o r^3}(2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta)$$

The image dipole is at distance $-Z$ below the plane and has the same orientation. According to P11.8.3, we must compute

$$\mathbf{f} = \mu_o \mathbf{m} \cdot \nabla \mathbf{H} = \mu_o \mathbf{m} \cdot \nabla \frac{\mu_o m}{4\pi\mu_o r^3}(2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta)$$

where we identify

$$r = 2Z$$

after the differentiation. Now

$$\mu_o \mathbf{m} \cdot \nabla = \mu_o m \frac{\partial}{\partial r}$$

$\mathbf{i}_r$ and $\mathbf{i}_\theta$ are independent of r and thus

$$\mathbf{f} = -\mu_o m \frac{4\mu_o m}{4\pi\mu_o r^4}\mathbf{i}_r$$

since $\theta = 0$. But

$$\mu_o m = 4\pi\mu_o R^3 H_o$$

and thus

$$\mathbf{f} = -\frac{\pi\mu_o R^6}{Z^4}H_o^2 \mathbf{i}_r$$

11.9 MACROSCOPIC FORCE DENSITIES

11.9.1 Starting with (11.9.14) we note that $J = 0$ and thus

$$\mathbf{f} = \int \mathbf{F} dv = -\int \frac{1}{2}H^2 \nabla \mu dv \tag{1}$$

The gradient of μ of the plunger is directed to the right, is singular (unit impulse-like) and of content $\mu - \mu_o$. The only contribution is from the flat end of the plunger (of radius a). We take advantage of the fact that μH is constant as it passes from the outside into the inside of the plunger. Denote the position just outside by x_-, that just inside by x_+.

$$\begin{aligned} -\int \frac{1}{2}H^2 \nabla \mu dv &= -\mathbf{i}_x \pi a^2 \int_{x_-}^{x_+} H^2 \frac{d\mu}{dx}dx \\ &\simeq -\mathbf{i}_x \frac{\pi a^2}{2}\left[\mu H^2\Big|_{x_-}^{x_+} - \int \mu \frac{d}{dx}H^2 dx\right] \end{aligned} \tag{2}$$

where we have integrated by parts. The integrand in the second term can be written

$$\mu \frac{d}{dx} H^2 = 2\mu H \frac{dH}{dx} \tag{3}$$

and the integral is

$$\int_{x_-}^{x_+} \mu H \frac{dH}{dx} = \mu H H \Big|_{x_-}^{x_+} = -\mu_o H^2 \Big|_{x_-} \tag{4}$$

where we have taken into account that μH is x-independent and that $H(x_+) = 0$. Combining (2), (3), and (4), we find

$$\mathbf{f} = -\mathbf{i_x} \frac{\pi a^2}{2} \mu_o H^2 \tag{5}$$

Using the H-field of Prob. 9.7.6, we find

$$\mathbf{f} = -\mathbf{i_x} \frac{\pi a^2}{2} \frac{\mu_o N^2 i^2}{\left(x + \frac{g\pi a^2}{2\pi a d}\right)^2} \tag{6}$$

This is the same as found in Prob. 11.7.2.

11.9.2 (a) From (11.9.14) we have

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{1}$$

Now B varies from $\mu_o H_o$ to $\mu_o H_i$ in a linear way, whereas $\mathbf{J}$ is constant

$$\begin{aligned} \mathbf{i_r} T_r &= \int_a^{a+\Delta} \mathbf{J} \times \mathbf{B} dr = \int_a^{a+\Delta} dr J \mu_o H (\mathbf{i}_\phi \times \mathbf{i_z}) \\ &= \mathbf{i_r} \mu_o K \Big(\frac{H_o + H_i}{2}\Big) \end{aligned} \tag{2}$$

where

$$\int_a^{a+\Delta} dr J \equiv K \tag{3}$$

Now, both J and H_i are functions of time. We have from (10.3.11)-(10.3.12)

$$T_r = -\mathbf{i_r} \frac{1}{2} \mu_o H_o e^{-t/\tau_m} [H_o + H_o(1 - e^{t/\tau_m})] = -\mathbf{i_r} \frac{1}{2} \mu_o H_o^2 (2 - e^{-t/\tau_m}) e^{-t/\tau_m}$$

11.9.3 (a) Here the first step is analogous to the first three equations of P11.9.2. Because $\mathbf{J}$ is constant and H varies linearly

$$\mathbf{i_r}T_r = \mu_o K \frac{(H_o + H_i)}{2}(\mathbf{i_z} \times \mathbf{i_\phi}) \tag{1}$$

(b) If we introduce the time dependence of A from (10.4.16), with $\mu = \mu_o$,

$$A = -H_m a^2 e^{-t/\tau_m} \tag{2}$$

and of K_z from (19)

$$K_z = H_\phi^o - H_\phi^i = 2\frac{A}{a^2}\sin\phi = -2H_m \sin\phi e^{-t/\tau_m} \tag{3}$$

Further note that $H_\phi^i = 0$ at $t = 0$. Therefore from (3) and (2)

$$H_\phi^o = -2H_m \sin\phi \quad \text{at} \quad t = 0 \tag{4}$$

At $t = \infty$

$$H_\phi^o = -H_m \sin\phi \tag{5}$$

because the field has fully penetrated. Thus

$$H_\phi^o = -H_m \sin\phi[1 + e^{-t/\tau_m}] \tag{6}$$

From (6) and (3) we find

$$H_\phi^i = -H_m \sin\phi[1 - e^{-t/\tau_m}] \tag{7}$$

Thus we find from (1), (3), (6), and (7)

$$\begin{aligned}
\mathbf{i_r}T_r &= -\mathbf{i_r}\frac{\mu_o}{2}[(H_\phi^o)^2 - (H_\phi^i)^2] \\
&= -\mathbf{i_r}\frac{\mu_o}{2}H_m^2 \sin^2\phi[(1 + e^{-t/\tau_m})^2 - (1 - e^{-t/\tau_m})^2] \\
&= -\mathbf{i_r}2\mu_o H_m^2 \sin^2\phi e^{-t/\tau_m}
\end{aligned}$$

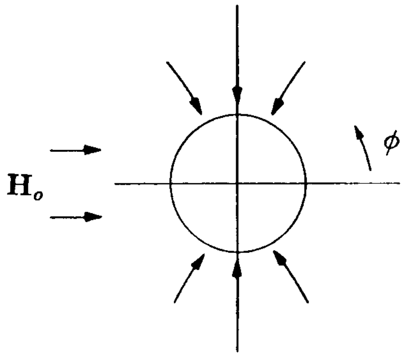


Figure S11.9.2

The force is inward, peaks at $t = 0$ and then decays. This shows that the cylinder will get crushed when a magnetic field is applied suddenly (Fig. S11.9.2).

SOLUTIONS TO CHAPTER 12

12.1 ELECTRODYNAMIC FIELDS AND POTENTIALS

12.1.1 The particular part of the E-field obeys

$$\nabla \times \mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{B} \tag{1}$$

$$\nabla \cdot \epsilon_o \mathbf{E}_p = 0 \tag{2}$$

If we set

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

then

$$\nabla \times \left(\mathbf{E}_p + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{4}$$

or

$$\mathbf{E}_p = -\frac{\partial}{\partial t}\mathbf{A} - \nabla \Phi_p \tag{5}$$

Because of (2),

$$\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} + \nabla^2 \Phi_p = 0 \tag{6}$$

But, because we use the Coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0 \tag{7}$$

and thus

$$\nabla^2 \Phi_p = 0 \tag{8}$$

There is no source for the scalar potential of the particular solution. Further

$$\nabla^2 \mathbf{A} = -\mu_o J_u \tag{9}$$

Conversely,

$$\nabla \cdot \epsilon_o \mathbf{E}_h = \rho_u \tag{10}$$

and

$$\nabla \times \mathbf{E}_h = 0 \tag{11}$$

Therefore,

$$\mathbf{E}_h = -\nabla \Phi_h \tag{12}$$

and from (10)

$$\nabla^2 \Phi_h = -\frac{\rho_u}{\epsilon_o} \tag{13}$$

Thus (9) and (13) look like the inhomogeneous wave equation with $\partial^2/\partial t^2$ terms omitted.

12.1.2 $\frac{\partial^2}{\partial t^2}\mathbf{A}$ is of order $1/\tau^2\mathbf{A}$, $\nabla^2\mathbf{A}$ is of order $\mathbf{A}/L^2$. Thus, $\mu\epsilon\frac{\partial^2}{\partial t^2}\mathbf{A}$ is of order $\frac{\mu\epsilon}{\tau^2}L^2$ compared with $\nabla^2\mathbf{A}$. It is negligible if $\mu\epsilon L^2/\tau^2 = L^2/c^2\tau^2 \ll 1$. The same approach shows that $\mu\epsilon(\partial^2/\partial t^2)\Phi$ can be neglected compared with $\nabla^2\Phi$ if $L^2/c^2\tau^2 \ll 1$.

12.2 ELECTRODYNAMIC FIELDS OF SOURCE SINGULARITIES

12.2.1 The time dependence of $q(t)$ is the same as that of Fig. 12.2.5, except that it now extends over one full period.

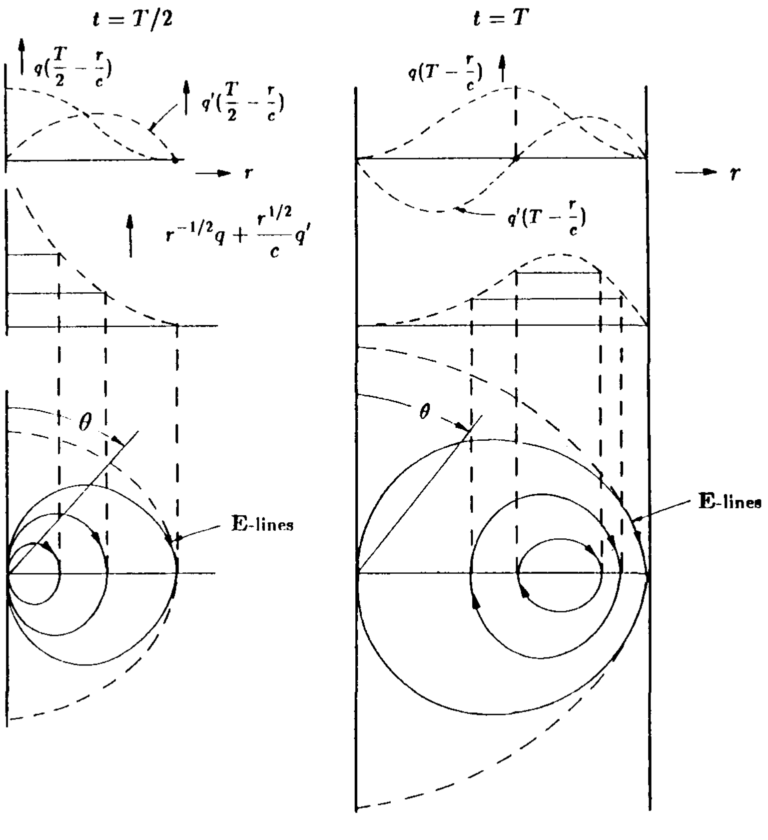


Figure S12.2.1a

Plot of Electric Dipole Field. Any set of field lines that close upon themselves may be considered to be lines of equal height of a potential. The potential does not necessarily reproduce the field intensity at every point. i.e.

$$\mathbf{E} = -\underbrace{(\mathbf{i}_\phi \times \nabla\Phi)}\cdot\overbrace{f(r,\theta)} \tag{1}$$

The "underbrace" gives the pattern. The "overbrace" is the multiplier. It does not change the direction of the field. Take

$$\mathbf{E} = \frac{d}{4\pi\epsilon}\left\{2\cos\theta\left[\frac{q}{r^3}+\frac{q'}{cr^2}\right]\mathbf{i_r} + \left[\frac{q}{r^3}+\frac{q'}{cr^2}+\frac{q''}{c^2r}\right]\sin\theta\mathbf{i}_\theta\right\} \tag{2}$$

where

$$q = q(t-\frac{r}{2})$$

If one defines

$$\Phi = \left(\frac{d}{4\pi\epsilon}\right)2\sin\theta\left(qr^{-1/2}+\frac{q'}{c}r^{1/2}\right) \tag{3}$$

Then

$$\begin{aligned}\nabla\Phi = (\frac{d}{4\pi\epsilon})\Big[&2\sin\theta\big(-\frac{1}{2}r^{-3/2}-q'\frac{1}{c}r^{-1/2}+\frac{1}{2}q'\frac{1}{c}r^{1/2}-\frac{q''}{c^2}r^{-1/2}\big)\mathbf{i_r}\\ &+\mathbf{i}_\theta 2\cos\theta\big(qr^{-3/2}+\frac{q'}{c}r^{-1/2}\big)\Big]\end{aligned} \tag{4}$$

One constructs a vector perpendicular to $\nabla\Phi$, $\mathbf{i}_\phi\times\nabla\Phi$, by interchanging the θ and r components and reversing the sign of one of them

$$-\mathbf{i}_\phi\times\nabla\Phi = (\frac{d}{4\pi\epsilon})r^{-3/2}\{2\cos\theta(q+\frac{r}{c}q')\mathbf{i_r}+\sin\theta(q+\frac{r}{c}q'+\frac{r^2}{c^2}q'')\mathbf{i}_\theta\} \tag{5}$$

Thus if we choose $f(r,\theta) = r^{-3/2}$, we reproduce the E-field of the dipole by expression (1).

We can sketch the function Φ for $\theta = \pi/2$.

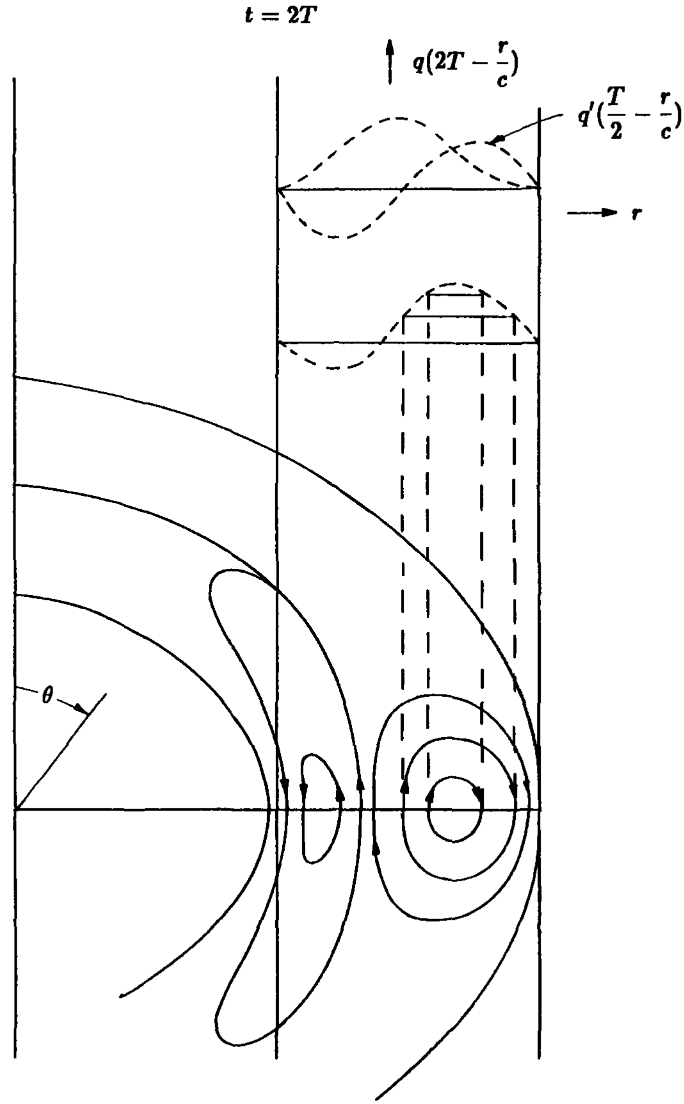


Figure S12.2.1b

12.2.2 Interchange $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$ and $\mu_o \rightarrow \epsilon_o$. From (23)

$$d\hat{i} = j\omega \hat{q} d \rightarrow j\omega \hat{q}_m d = j\omega \mu_o \hat{m} \tag{1}$$

where q_m is the magnetic charge. We obtain

$$\hat{E}_\phi = -\frac{jkj\omega\mu_o\hat{m}}{4\pi} \sin\theta \frac{e^{-jkr}}{r} \tag{2}$$

and from (24)

$$\hat{H}_\theta = -\sqrt{\frac{\epsilon_o}{\mu_o}}\hat{E}_\phi = -\frac{k^2\hat{m}}{4\pi}\sin\theta\frac{e^{-jkr}}{r} \qquad \text{QED} \tag{3}$$

12.2.3 Because $\mu_o m(t) = q_m d \rightarrow qd$ in the electric dipole case, the time dependence of $q(t)d$ and $\mu_o m(t)$ correspond to each other. With $\mathbf{E} \rightarrow \mathbf{H}$ and $\mathbf{H} \rightarrow -\mathbf{E}$ we must obtain mutually corresponding field patterns.

12.2.4 We can use the field sketch of Problem 12.2.1 with proper interchange of variables.

12.3 SUPERPOSITION INTEGRAL FOR ELECTRODYNAMIC FIELDS

12.4 ANTENNAE RADIATION FIELDS IN THE SINUSOIDAL STEADY STATE

12.4.1 From (4)

$$\begin{aligned}\psi_o(\theta) &= \frac{\sin\theta}{l}\int_0^l e^{-jkz'}e^{jkz'\cos\theta}dz' \\ &= \frac{\sin\theta}{l}\frac{1}{jk(\cos\theta-1)}\{e^{-jk(1-\cos\theta)l}-1\} \\ &= \frac{\sin\theta}{l}\frac{2}{k(1-\cos\theta)}\sin\left[\frac{kl}{2}(1-\cos\theta)\right]e^{-jk(1-\cos\theta)l/2}\end{aligned} \tag{1}$$

The radiation pattern is

$$\begin{aligned}\Psi(\theta) = |\psi_o(0)|^2 &= \frac{4\sin^2\theta\sin^2\frac{kl}{2}(1-\cos\theta)}{k^2l^2(1-\cos\theta)^2} \\ &= \frac{\sin^2\theta}{k^2l^2\sin^4(\theta/2)}\sin^2\left(kl\sin^2\frac{\theta}{2}\right)\end{aligned} \tag{2}$$

With $kl = 2\pi$

$$\Psi(\theta) \equiv \frac{\sin^2\theta}{4\pi^2\sin^4(\theta/2)}\left(\sin^2 2\pi\sin^2\frac{\theta}{2}\right) \tag{3}$$

The radiation pattern peaks near $\theta = 60°$.

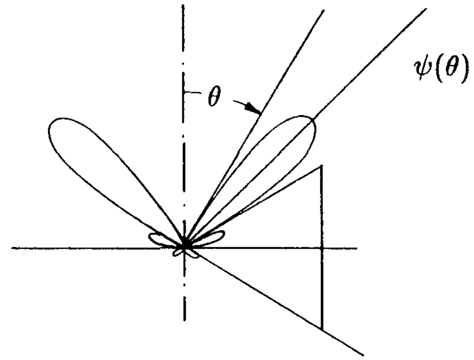


Figure S12.4.1

12.4.2 By analogy with (3) one replaces $H_\phi \to E_\phi$, $\mu \leftrightarrow \epsilon$ and $i(z')dz' = j\omega(qz)dz' \to j\omega(q_m d)dz' = j\omega\mu\mu(z')dz'$ where we interpret qd and $q_m d$ as assigned to unit length. Thus, from (2) of Prob. 12.2.2, with $\mu_o \to \mu$, $\epsilon_o \to \epsilon$,

$$\begin{aligned} E_\phi &= \frac{k^2}{4\pi}\sin\theta\frac{e^{-jkr}}{r}\sqrt{\frac{\mu}{\epsilon}}\int \mathcal{M}(z')e^{jk\mathbf{r}'\cdot\mathbf{i}_r}dz' \\ &= \frac{k^2 l}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\frac{e^{-jkr}}{4}\mathcal{M}_o e^{j\alpha_o}\epsilon_o(\theta) \end{aligned}$$

where

$$\psi_o(\theta) \equiv \frac{\sin\theta}{l}\int\frac{\mathcal{M}(r')}{\mathcal{M}_o}e^{j(k\mathbf{r}'\cdot\mathbf{i}_r-\alpha_o)}dz'$$

12.4.3

$$\begin{aligned} \psi_o(\theta) &= -\frac{\sin\theta}{l}\int_0^l \frac{\sin\beta(z'-l)}{\sin\beta l}e^{jkz'\cos\theta}dz' \\ &= -\frac{\sin\theta}{\beta l\sin\beta l}\int_0^l \frac{1}{2j}\left\{\left(e^{j\beta(z'-l)} - e^{-j\beta(z'-l)}\right)e^{jkz'\cos\theta}d(\beta z')\right. \\ &= -\frac{\sin\theta}{\beta l\sin\beta l}\frac{1}{2j}\left\{\frac{e^{j(\beta+k\cos\theta)l}-1}{j\left(1+\frac{k}{\beta}\cos\theta\right)}e^{-j\beta l} - \frac{e^{-j(\beta-k\cos\theta)l}-1}{-j\left(1-\frac{k}{\beta}\cos\theta\right)}e^{j\beta l}\right\} \\ &= -\frac{\sin\theta}{\beta l\sin\beta l}\frac{2}{1-\frac{k^2}{\beta^2}\cos^2\theta}\left\{\cos\beta l + j\sin\beta l\frac{k}{\beta}\cos\theta - e^{jk\cos\theta l}\right\} \end{aligned}$$

12.4.4 (a) From (12), and with $\mathbf{a}_n = n\frac{\lambda}{4}\mathbf{i}_x$,

$$\begin{aligned} \psi_a &= \sum_{n=0}^{3} e^{jk\mathbf{a}_n\cdot\mathbf{i}_r}e^{j(\alpha_n-\alpha_o)} \qquad (1) \\ &= 1 + e^{j(\frac{\pi}{2}\cos\phi\sin\theta+\alpha_1-\alpha_o)} + e^{j(\pi\cos\phi\sin\theta+\alpha_2-\alpha_o)} \end{aligned}$$

(b) Since $\psi_o = \sin\theta$, and $\alpha_i = 0$

$$|\psi_o||\psi_a| = \left|1 + 2\cos\left(\frac{\pi}{2}\cos\phi\sin\theta\right)\right|\sin\theta \tag{2}$$

(c)

$$\begin{aligned}\psi_a &= 1 + e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)} + e^{j\pi(\cos\phi\sin\theta+1)}\\ &= e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\left\{e^{-j\frac{\pi}{2}(\cos\phi\sin\theta+1)} + 1 + e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\right\}\\ &= e^{j\frac{\pi}{2}(\cos\phi\sin\theta+1)}\left[2\cos\frac{\pi}{2}(\cos\phi\sin\theta+1)+1\right]\end{aligned} \tag{3}$$

$$|\psi_o||\psi_a| = \left|\left[1 + 2\cos\frac{\pi}{2}(\cos\phi\sin\theta+1)\right]\sin\theta\right| \tag{4}$$

12.4.5 (a)

$$\psi_a(\theta) = \sum_{n=0}^{1} e^{jk\mathbf{a}_n\cdot\mathbf{1}_r}e^{j(\alpha_n-\alpha_o)} = 1 + e^{j[\pi\cos\theta+\alpha_1-\alpha_o]} \tag{1}$$

(b)

$$|\psi_a|^2|\psi_0|^2 = 4\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta \tag{2}$$

(c)

$$G = \frac{4\pi\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta}{\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\sin\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta} \tag{3}$$

Define

$$\cos\theta = u \tag{4}$$

$$\int_0^{\pi} d\theta\sin^3\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right) = \int_{-1}^{1} du(1-u^2)\cos^2\left(\frac{\pi}{2}u\right) \tag{5}$$

Now consider integral

$$\int dx x^2\cos^2 x = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)x^2 - \frac{x^3}{3} + \frac{2x}{8}\cos 2x - \frac{1}{8}\sin 2x \tag{6}$$

The integral is

$$\begin{aligned}\int_{-1}^{1} du(1-u^2)\cos^2\frac{\pi}{2}u &= 1 - \left(\frac{2}{\pi}\right)^3\left\{\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)x^2\right.\\ &\quad\left. - \frac{x^3}{3} + \frac{2x}{8}\cos 2x - \frac{1}{8}\sin 2x\right\}_{-\pi/2}^{\pi/2}\\ &= \frac{2}{3} + \frac{1}{2}\left(\frac{2}{\pi}\right)^2\end{aligned} \tag{7}$$

The gain is

$$G = \frac{4\pi \cos^2\left(\frac{\pi}{2}\cos\theta\right)\sin^2\theta}{2\pi\{\frac{2}{3} + \frac{2}{\pi^2}\}} \tag{8}$$

(d) We find for $\Psi(\theta)$ of array

$$\Psi(\theta) = \{|\psi_o(\theta)||\psi_1(\theta)||\psi_2(\theta)|\}^2 \tag{9}$$

with

$$\psi_2(\theta) = 1 - e^{jka\sin\theta\cos\phi} \tag{10}$$

In order to get maximum superposition in the direction $\phi = 0$, one needs $ka = \pi$ or $a = \lambda/2$. Thus

$$|\psi_2(\theta)| = |2\sin\left(\frac{\pi}{2}\sin\theta\cos\phi\right)|$$

12.5 COMPLEX POYNTING'S THEOREM AND RADIATION RESISTANCE

12.5.1 The radiation field Poynting vector of the antenna is from 12.4.2, 3.4.5

$$\frac{1}{2}(\hat{E}_\theta \hat{H}_\phi^*) = \frac{1}{2}\frac{(kl)^2}{(4\pi)^2}\sqrt{\frac{\mu_o}{\epsilon_o}}|I_o|^2(\psi_o(\theta))^2 \tag{1}$$

where $\psi_o(\theta)$ is from 12.4.28

$$\begin{aligned} \psi_o(\theta) &= \frac{1}{\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)} \frac{\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta} \\ &= \frac{2}{3\pi}\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta} \end{aligned} \tag{2}$$

The radiated power is

$$\begin{aligned} \frac{1}{2}|I_o|^2 R_{\text{rad}} &= \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{2}\hat{E}_\theta \hat{H}_\phi^* \\ &= \frac{1}{2}\frac{(3\pi)^2}{(4\pi)^2}\sqrt{\mu_o/\epsilon_o}|I_o|^2\left(\frac{2}{3\pi}\right)^2 2\pi \int_0^\pi \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta}\sin\theta d\theta \\ &= \frac{1}{2}|I_o|^2 \frac{\sqrt{\mu_o/\epsilon_o}}{4\pi^2}(2\pi)\int_0^\pi d\theta \sin\theta \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta} \end{aligned} \tag{3}$$

Therefore

$$\begin{aligned} R_{\text{rad}} &= \frac{\sqrt{\mu_o/\epsilon_o}}{2\pi}\int_0^\pi d\theta \sin\theta \frac{\cos^2\left(\frac{3\pi}{2}\cos\theta\right)}{\sin^2\theta} \\ &= \frac{1}{2\pi}\sqrt{\mu_o/\epsilon_o}\int_{-1}^1 dx \frac{\cos^2\left(\frac{3\pi}{2}x\right)}{1-x^2} \\ &= 104\Omega \end{aligned} \tag{4}$$

12.5.2 The scalar potential of the spherical coil is (see Eq. 8.5.17)

$$\Psi = \frac{NI_oR^2}{6r^2}\cos\theta = \frac{m}{4\pi r^2}\cos\theta \tag{1}$$

This identifies

$$\hat{m} = \frac{2\pi}{3}N\hat{I}_oR^2 \tag{2}$$

We have for the θ component of the H-field

$$\hat{H}_\theta = \frac{\hat{m}}{4\pi r^3}\sin\theta[1 + jkr + (jkr)^2] \tag{3}$$

and thus the radiation field is

$$\hat{H}_\theta \approx -\frac{k^2\hat{m}}{4\pi r}\sin\theta \tag{4}$$

The power radiated is

$$\begin{aligned} -\frac{1}{2}\int\int \hat{E}_\phi\hat{H}_\theta^* r^2\sin\theta d\theta d\phi &= \frac{1}{2}2\pi\frac{4}{3}\sqrt{\mu_o/\epsilon_o}\left(\frac{k^2m}{4\pi}\right)^2 \\ &= \frac{1}{2}|\hat{I}_o|^2R_{\text{rad}} \end{aligned} \tag{5}$$

Therefore,

$$R_{\text{rad}} = \frac{2\pi}{27}\sqrt{\mu_o/\epsilon_o}N^2(kR)^4 \tag{6}$$

The inductance of the coil is from (8.5.20)

$$L = \frac{2\pi}{9}\mu_oRN^2 \tag{7}$$

and therefore

$$R_{\text{rad}} = \frac{1}{3}\omega L(kR)^3 \tag{8}$$

12.6 PERIODIC SHEET-SOURCE FIELDS: UNIFORM AND NONUNIFORM PLANE WAVES

12.6.1 (a) From continuity:

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma}_s = 0$$

Taking into account the x-dependence:

$$-jk_x\hat{K}_x + j\omega\hat{\sigma}_s = 0 \tag{2}$$

and therefore

$$\hat{K}_x = \frac{\omega}{k_x}\sigma_o e^{-jk_x x} \tag{3}$$

and

$$K_x = \text{Re}\left(\frac{\omega\sigma_o e^{jk_x x}}{k_x}\right)e^{j\omega t}$$

(b) The boundary condition on the tangential **H** is:

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \qquad \mathbf{n} \parallel \mathbf{i}_y$$

Since

$$\mathbf{H} \parallel \mathbf{i}_z \tag{4}$$

and thus

$$\hat{H}_z^a - \hat{H}_z^b = \hat{K}_x \tag{5}$$

H_z is antisymmetric, of opposite sign on the two sides of current sheet.

$$2\hat{H}_z^a = \hat{K}_x \tag{6}$$

and thus

$$\hat{H}^{\binom{a}{b}} = \mathbf{i}_z \text{Re}\left[\pm\frac{\omega\sigma_o}{2k_x}e^{\mp j\beta y}e^{j(\omega t - k_x x)}\right] \tag{7}$$

From (12.6.6) and (12.6.7)

$$\mathbf{E} = \text{Re}\left[\mathbf{i}_x\left(-\frac{\beta\sigma_o}{2\epsilon_o k_x}\right) + \mathbf{i}_y\left(\pm\frac{\sigma_o}{2\epsilon_o}\right)\right]e^{\mp j\beta y}e^{j(\omega t - k_x x)} \tag{8}$$

(c) As in Problem 12.2.1, a plot of a divergence-free field can be done by defining a potential Φ and obtaining the field

$$\mathbf{E} = -\mathbf{i}_z \times \nabla\Phi f(x, y) \tag{9}$$

Now, it is clear that the potential necessary to produce (8) is

$$\Phi = \pm\frac{1}{jk_x}\left(\frac{\sigma_o}{2\epsilon_o}\right)e^{\mp j\beta y}e^{j(\omega t - k_x x)}$$

Then

$$-\mathbf{i}_z \times \nabla\Phi = \mathbf{i}_x \frac{\partial \Phi}{\partial y} - \mathbf{i}_y \frac{\partial \Phi}{\partial x}$$

and is found to be equal to (8) with $f(x, y)$ equal to unity. By visualizing the potential, one may plot E lines.

k_y imaginary:

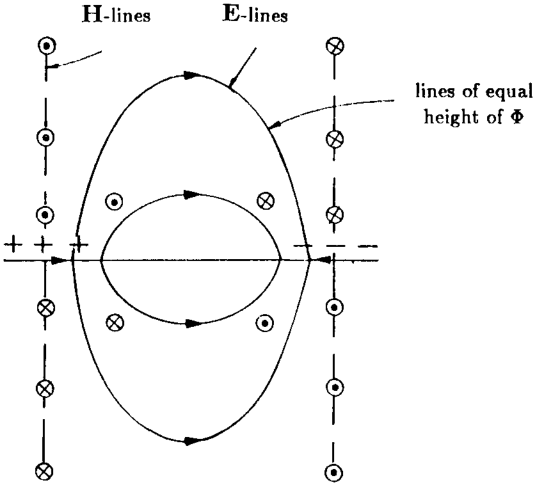


Figure S12.6.1a

At $\omega t = 0$, the potential is

$$\mathrm{Re} - \frac{1}{jk_x}\frac{\sigma_o}{2\epsilon_o} e^{\mp j\beta y} e^{-jk_x x} = \frac{\sigma_o}{2\epsilon_o k_x}\sin(k_x x) e^{\mp|\beta| y}$$

k_y real:

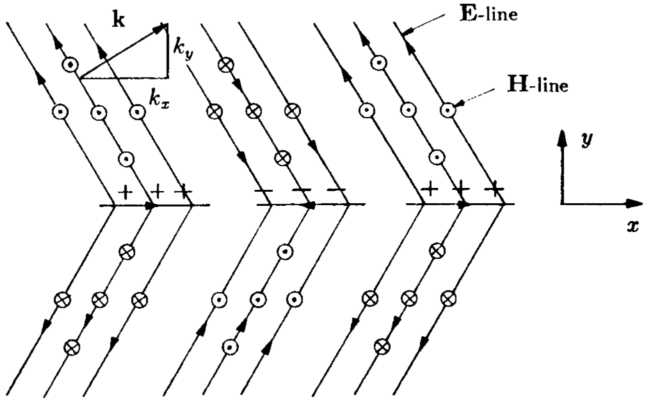


Figure S12.6.1b

At $\omega t = 0$, the potential is

$$\frac{\sigma_o}{2\epsilon_o k_x}\sin(k_x x \pm k_y y)$$

12.6.2 (a) The E-field will be z-directed, the H-field is in the $x-y$ plane

$$\hat{E}_z = A\sin(k_x x)e^{\mp jk_y y} \tag{1}$$

From (12.6.29)

$$\hat{H}_x = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp jk_y)A\sin k_x x \tag{2}$$

The discontinuity of tangential H gives:

$$\mathbf{n}\times(\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K} \tag{3}$$

in $x-z$ plane. And thus, combining (2) and (3)

$$2\frac{k_y}{\omega\mu}A\sin k_x x = -K_o\sin k_x x \tag{4}$$

and therefore

$$A = -\frac{\omega\mu K_o}{2k_y} \tag{5}$$

From (2) and (5)

$$\hat{H}_x = \mp\frac{K_o}{2}\sin(k_x x)e^{\mp jk_y y} \tag{6}$$

and from (12.6.30)

$$\hat{H}_y = j\frac{k_x}{k_y}\frac{K_o}{2}\cos(k_x x)e^{\mp jk_y y} \tag{7}$$

(b) Again we can use a potential Φ to which the H lines are lines of equal height. If we postulate

$$\Phi = \left(\frac{1}{jk_y}\right)\frac{K_o}{2}\sin k_x x e^{\mp jk_y y} \tag{8}$$

Then

$$-\mathbf{i}_z\times\nabla\Phi = \mathbf{i}_x\frac{\partial\Phi}{\partial y}\frac{K_o}{2}\frac{k_x}{jk_y}\cos k_x x \tag{9}$$

$$-\mathbf{i}_y\frac{\partial\Phi}{\partial x} = \mp\mathbf{i}_x\frac{K_o}{2}\sin k_x x - \mathbf{i}_y\frac{K_o}{2}\frac{k_x}{jk_y}\cos k_x x$$

The potential hill at $\omega t = 0$ is

$$\mathrm{Re}[\Phi] = \mp\frac{K_o}{2}\sin k_x x\sin k_y y \tag{10}$$

$$\omega t = 0$$

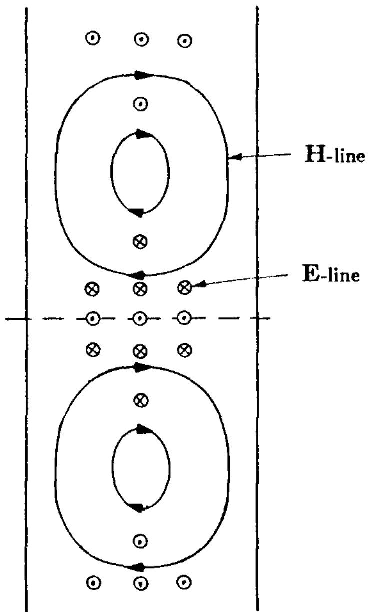


Figure S12.6.2

(c) We may write (1)

$$\hat{E}_z = -\frac{\omega\mu K_o}{2k_y}\frac{1}{2j}\left\{e^{jk_x x \mp jk_y y} - e^{-jk_x x \mp jk_y y}\right\} \tag{11}$$

and for (6) and (7)

$$\begin{aligned}\hat{\mathbf{H}} =& j\frac{K_o}{4}\Big\{ \pm \mathbf{i_x}\left(e^{jk_x x \mp jk_y y} - e^{-jk_x x \mp jk_y y}\right) \\ &+ \frac{k_x}{k_y}\mathbf{i_z}\left(e^{jk_x x \mp jk_y y} + e^{-jk_x x \mp jk_y y}\right)\Big\}\end{aligned} \tag{12}$$

12.6.3 (a) At first it is best to find the field E_z due to a single current sheet at $y = 0$. We have

$$\hat{E}_z = Ae^{-jk_x x}e^{\mp j\beta y} \tag{1}$$

From (12.6.29)

$$\hat{H}_x = -\frac{1}{j\omega\mu}\frac{\partial \hat{E}_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp j\beta)Ae^{-jk_x x}e^{\mp j\beta y} \tag{2}$$

From the boundary condition

$$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^{(b)}) = \mathbf{K} \tag{3}$$

we get

$$2\frac{\beta}{\omega\mu}Ae^{-jk_x x} = -Ke^{-jk_x x}$$

and thus

$$A = -\frac{\omega\mu K}{2\beta} \tag{4}$$

Now we can add the fields due to each source

$$\begin{aligned}\hat{E}_z = -\frac{\omega\mu}{2\beta}[\hat{K}^a e^{-jk_x x} &\begin{Bmatrix} e^{-j\beta(y-\frac{d}{2})} \\ e^{j\beta(y-\frac{d}{2})} \\ e^{j\beta(y-\frac{d}{2})} \end{Bmatrix} \\ + \hat{K}^b e^{-jk_x x} &\begin{Bmatrix} e^{-j\beta(y+\frac{d}{2})} \\ e^{-j\beta(y+\frac{d}{2})} \\ e^{j\beta(y+\frac{d}{2})} \end{Bmatrix}\end{aligned} \tag{5}$$

(b) When

$$\hat{K}_a e^{-j\beta\frac{d}{2}} + \hat{K}_b e^{j\beta\frac{d}{2}} = 0 \tag{6}$$

Then

$$\hat{K}_b = -\hat{K}_a e^{-j\beta d} \tag{7}$$

there is cancellation at $y < -d/2$

(c)

$$\begin{aligned}\hat{E}_z &= -\frac{\omega\mu}{2\beta}\left[\hat{K}_a e^{-jk_x x} e^{-j\beta(y-\frac{d}{2})} - \hat{K}_a e^{-jk_x x} e^{-j\beta(y+\frac{3d}{2})}\right] \\ &= -j\frac{\omega\mu\hat{K}_a}{\beta} e^{-jk_x x} e^{-j\beta(y-\frac{d}{2})} \sin\beta d e^{-j\beta d}\end{aligned} \tag{8}$$

(d) In order to produce maximum radiation we want the endfire array situation of $\beta d = \pi/2$. (Indeed, $\sin\beta d = 1$ in this case.) Because

$$\beta = \sqrt{\omega^2\mu\epsilon - k_x^2} \tag{9}$$

we have

$$\omega = \frac{1}{\sqrt{\mu\epsilon}}\left[k_x^2 - \left(\frac{\pi}{2d}\right)^2\right]^{1/2} \tag{10}$$

The direction is

$$\mathbf{k} = k_x \mathbf{i_x} + \beta \mathbf{i_y} = k_x \mathbf{i_x} + \frac{\pi}{2d}\mathbf{i_y}$$

12.6.4 (a) If we want cancellations, we again want (compare P12.6.3)

$$\hat{\sigma}_b = -\hat{\sigma}_a e^{-jk_y d} \tag{1}$$

(b) A single sheet at $y = d/2$ gives

$$H_z = \pm A e^{-jk_x x} e^{\mp jk_y (y - \frac{d}{2})} \tag{2}$$

Now,

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma} = 0 \tag{3}$$

gives

$$\hat{K}_x = \frac{\omega}{k_x}\hat{\sigma}_a \tag{4}$$

and

$$2\hat{H}_z^a\Big|_{y=0_+} = \frac{\omega}{k_x}\hat{\sigma}_a \tag{5}$$

Therefore

$$A = \frac{\omega}{2k_x}\hat{\sigma}_a \tag{6}$$

and the field of both sheets is

$$H_z = j\frac{\omega}{k_x}\hat{\sigma}_a e^{-jk_x x} e^{-jk_y \left(y + \frac{d}{2}\right)} \sin k_x d \tag{7}$$

(c) $k_y d = \pi/2$. Therefore, as in P12.6.3,

$$\omega = \frac{1}{\sqrt{\mu\epsilon}}\left[k_x^2 - \left(\frac{\pi}{2d}\right)^2\right]^{1/2} \tag{8}$$

12.7 ELECTRODYNAMIC FIELDS IN THE PRESENCE OF PERFECT CONDUCTORS

12.7.1 The field of the antenna is that of a current distribution $|\cos kz|$. We may treat it in terms of an array factor of three antennae spaced $\lambda/2$ apart along the z-axis. From 12.4.12

$$\begin{aligned} |\psi_a(\theta)| &= \Big|\sum_{i=0}^{3} e^{ik\frac{\lambda}{2}\cos\theta}\Big| = \left|1 + e^{j\pi\cos\theta} + e^{2j\pi\cos\theta}\right| \\ &= \left|e^{-j\pi\cos\theta} + 1 + e^{j\pi\cos\theta}\right| \\ &= 1 + 2\cos(\pi\cos\theta) \end{aligned} \tag{1}$$

The function $\psi_o(\theta)$ follows from 12.4.8 with $kl = \pi$

$$\psi_o(\theta) = \frac{2}{\pi}\cos\left(\frac{\pi}{2}\cos\theta\right)/\sin\theta \tag{2}$$

Combining (1) and (2) we complete the proof.

12.7.2 The current distribution, with image, is proportional to $|\sin kz|$. The point at which the current is fed into the antenna calls for zero current. Since the radiated power is finite, $R_{\rm rad}$ is infinite. In practice, because of the finite losses, it is not infinite but much larger than $\sqrt{\mu_o/\epsilon_o}$.

12.7.3 (a) We have a surface current $\hat{K}_x$

$$\frac{\partial \hat{K}_x}{\partial x} + j\omega\hat{\sigma}_s = 0 \tag{1}$$

Therefore

$$\hat{K}_x = -\frac{j\omega}{\pi/a}\sigma_o \sin\left(\frac{\pi x}{a}\right) \tag{2}$$

The **H**-field is z-directed and antisymmetric with respect to y.

$$\hat{H}_z = \pm A \sin\left(\frac{\pi x}{a}\right) e^{\mp jk_y y} \tag{3}$$

From the boundary condition

$$\mathbf{n} \times (\hat{\mathbf{H}}^a - \hat{\mathbf{H}}^b) = \hat{\mathbf{K}} \tag{4}$$

with $\mathbf{n} \parallel \mathbf{i}_y$

$$2A\sin\left(\frac{\pi x}{a}\right) = -\frac{j\omega}{\pi/a}\sigma_o \sin\left(\frac{\pi x}{a}\right) \tag{5}$$

$$A = -\frac{j\omega}{2\pi/a}\sigma_o \tag{6}$$

The E-field is from (12.6.6)

$$\begin{aligned}\hat{E}_x &= \frac{1}{j\omega\epsilon_o}\frac{\partial \hat{H}_z}{\partial y} = \pm\frac{1}{j\omega\epsilon_o}\left(-\frac{j\omega}{2\pi/a}\sigma_o\right)(\mp jk_y)\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \\ &= \frac{jk_y\sigma_o}{\epsilon_o(2\pi/a)}\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\end{aligned} \tag{7}$$

and from 12.6.7

$$\begin{aligned}\hat{E}_y &= -\frac{1}{j\omega\epsilon_o}\frac{\partial \hat{H}_z}{\partial x} = \left(-\frac{1}{j\omega\epsilon_o}\right)\left(\mp\frac{j\omega\sigma_o}{2(\pi/a)}\right)\frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right)e^{\mp jk_y y} \\ &= \pm\frac{\sigma_o}{2\epsilon_o}\cos\left(\frac{\pi}{a}x\right)e^{\mp jk_y y}\end{aligned} \tag{8}$$

(b) On the plate at $x = -a/2$

$$\hat{\sigma}_s = \epsilon_o\hat{E}_x|_{x=-a/2} = -\frac{jk_y\sigma_o}{2\pi/a}e^{\mp jk_y y} \tag{9}$$

At $x = a/2$ it is of opposite sign. The surface current is

$$\hat{K}_y = -\hat{H}_z|_{x=-a/2} = \pm \frac{j\omega\sigma_o}{2\pi/a} e^{\mp jk_y y} \tag{10}$$

and is the negative of that at $x = a/2$.

(c)

$$k_x^2 + k_y^2 = \omega^2 \mu_o \epsilon_o \tag{11}$$

and thus

$$k_y = \sqrt{\omega^2 \mu_o \epsilon_o - \left(\frac{\pi}{a}\right)^2} \tag{12}$$

Again we may identify a potential whose lines of equal height give **E**. Indeed,

$$\Phi = \mp \frac{\sigma_o}{\epsilon_o (2\pi/a)} \sin\left(\frac{\pi x}{a}\right) e^{\mp jk_y y} \tag{13}$$

gives

$$\begin{aligned} -\mathbf{i}_z \times \nabla\Phi &= \mathbf{i}_x \frac{\partial \Phi}{\partial y} - \mathbf{i}_y \frac{\partial \Phi}{\partial x} \\ &= \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} e^{\mp jk_y y} \left[\mp jk_y \mathbf{i}_x \sin\left(\frac{\pi x}{a}\right) - \frac{\pi}{a} \mathbf{i}_y \cos\left(\frac{\pi x}{a}\right) \right] \end{aligned} \tag{14}$$

(d) For k_y imaginary and $\omega t = 0$

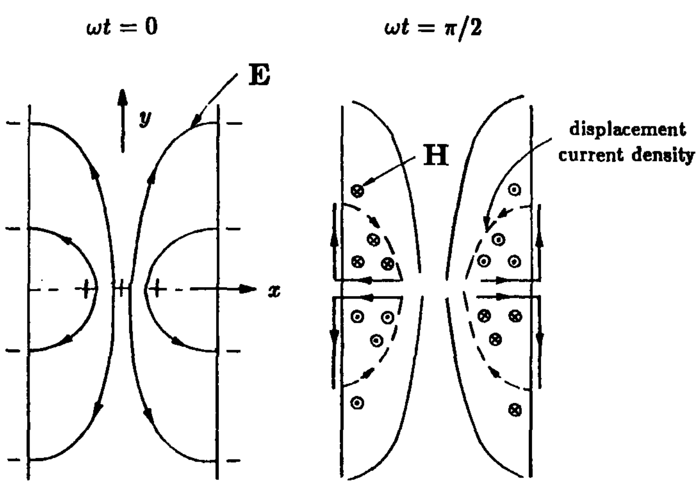


Figure S12.7.3a

$$\mathrm{Re}[\Phi] = \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} \sin\left(\frac{\pi x}{a}\right) e^{\mp |k_y| y}$$

For k_y real, $\omega t = 0$

$$\mathrm{Re}[\Phi] = \mp \frac{\sigma_o}{\epsilon_o(2\pi/a)} \sin\left(\frac{\pi x}{a}\right) \cos k_y y$$

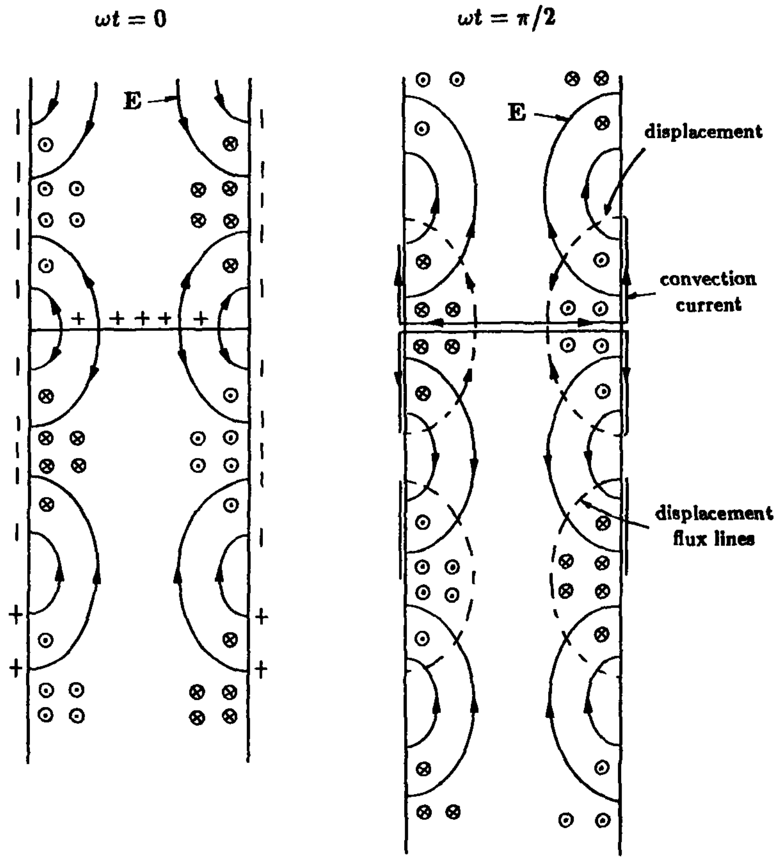


Figure S12.7.3b

12.7.4 (a) We now have a TE field with

$$\hat{E}_z = A\cos\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \tag{1}$$

From (12.6.29)

$$\begin{aligned}\hat{H}_x &= -\frac{1}{j\omega\mu}\frac{\partial \hat{E}_z}{\partial y} = -\frac{1}{j\omega\mu}(\mp jk_y)A\cos\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \\ &= \pm\frac{k_y}{\omega\mu}A\cos\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\end{aligned} \tag{2}$$

and the boundary condition

$$\mathbf{n} \times (\hat{\mathbf{H}}^a - \hat{\mathbf{H}}^b) = \hat{\mathbf{K}} \tag{3}$$

we obtain relation for A:

$$-2\frac{k_y}{\omega\mu} A \cos\frac{\pi x}{a} = K_o \cos\frac{\pi x}{a} \tag{4}$$

or

$$A = -\frac{\omega\mu K_o}{2k_y} \tag{5}$$

and thus

$$\hat{H}_x = \mp\frac{K_o}{2}\cos\left(\frac{\pi x}{2}\right)e^{\mp jk_y y} \tag{6}$$

From (12.6.60)

$$\begin{aligned}\hat{H}_y = \frac{1}{j\omega\mu}\frac{\partial \hat{E}_z}{\partial x} &= -\frac{\pi/a}{j\omega\mu}A\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \\ &= -j\frac{\pi/a}{2k_y}K_o\sin\left(\frac{\pi x}{a}\right)e^{\mp jk_y y}\end{aligned} \tag{7}$$

(b) Since the E-field is z-directed, it vanishes at the walls and there is no surface charge density. On wall at $x = -a/2$

$$\hat{K}_z = \hat{H}_y \tag{8}$$

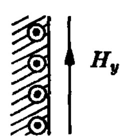


Figure S12.7.4a

and thus

$$\hat{K}_z = j\frac{\pi/a}{2k_y}K_o e^{\mp jk_y y} \tag{9}$$

On the other wall, the current is opposite.

(c)

$$k_y = \sqrt{\omega^2\mu_o\epsilon_o - (\pi/a)^2} \tag{10}$$

since $k_x = \pi/a$. Again we have a potential Φ, the lines of equal height of which give $\mathbf{H}$.

$$\Phi = \frac{1}{jk_y}\frac{K_o}{2}\cos\left(\frac{\pi x}{a}\right)e^{\mp jk_y y} \tag{11}$$

(d) For k_y imaginary:

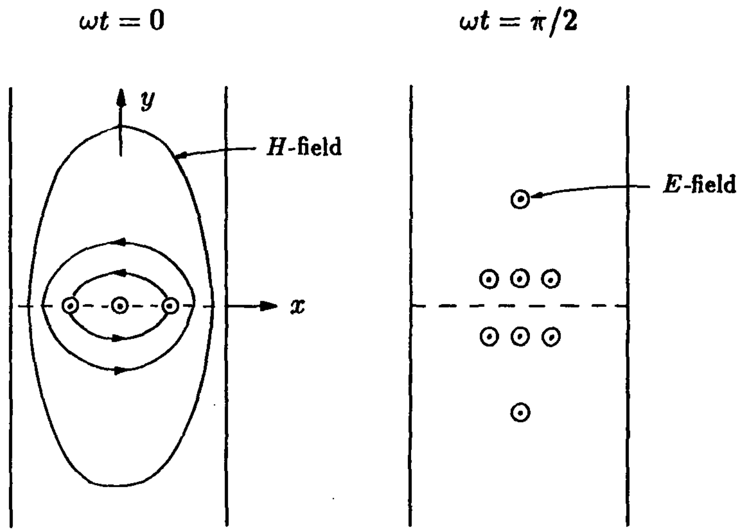


Figure S12.7.4b

for k_y real:

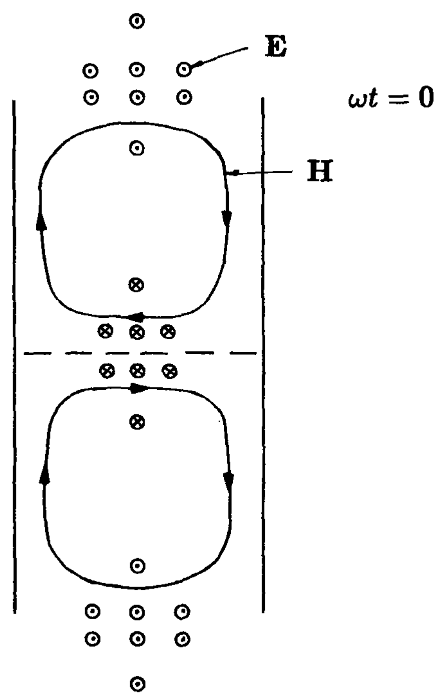


Figure S12.7.4c

SOLUTIONS TO CHAPTER 13

13.1 INTRODUCTION TO TEM WAVES

13.1.1 (a) From (13.1.3):

$$\frac{\partial E_x}{\partial y} = \beta \text{Re}[A\cos(\beta y)\exp(j\omega t)] = \mu \frac{\partial H_z}{\partial t}$$
$$= \beta |A| \cos\beta y \cos(\omega t + \phi) \tag{1}$$

where ϕ is the phase angle of A. Integrating the above yields

$$H_z = \frac{\beta}{\omega\mu}|A|\cos\beta y \sin(\omega t + \phi) = -\text{Re}\, j\frac{\beta}{\omega\mu}A\cos\beta y e^{j\omega t} \tag{2}$$

Introducing (2) and the expression for E_x into (13.1.2) gives

$$-\frac{\beta^2}{\omega\mu}|A|\sin\beta y \sin(\omega t + \phi) = -\omega\epsilon|A|\sin\beta y \sin(\omega t + \phi) \tag{3}$$

from which the dispersion relation follows $\beta^2 = \omega^2\mu\epsilon$.

(b) From (13.1.13)

$$H_z(-b,t) = -\text{Re}\,\hat{K}_o e^{j\omega t}$$

This gives, using (2),

$$-\text{Re}\, j\frac{\beta}{\omega\mu}A\cos\beta y e^{j\omega t} = -\text{Re}\,\hat{K}_o e^{j\omega t} \tag{4}$$

and thus

$$A = -j\frac{\omega\mu\hat{K}_o}{\beta\cos\beta b} = -j\hat{K}_o\sqrt{\frac{\mu}{\epsilon}}\frac{1}{\cos\beta b} \tag{5}$$

Using (2) we find

$$H_z = -\text{Re}\,\hat{K}_o\frac{\cos\beta y}{\cos\beta b}e^{j\omega t} \tag{6}$$

and putting the value of A from (5) into the expression for E_x gives

$$E_x = -\text{Re}\, jK_o\sqrt{\frac{\mu}{\epsilon}}\frac{\sin\beta y}{\cos\beta b}e^{j\omega t} \tag{7}$$

13.1.2 (a) The standing wave

$$H_z = \text{Re}\, A \sin \beta y e^{j\omega t}$$

satisfies the boundary conditions of zero H_z at $y = 0$. From (13.1.2)

$$\frac{\partial H_z}{\partial y} = \beta \text{Re}\, A \cos \beta y e^{j\omega t} = \epsilon \frac{\partial E_x}{\partial t} \tag{1}$$

Integrating to find E_x gives

$$E_x = -\frac{\beta}{\omega\epsilon} \text{Re}\, jA \cos \beta y e^{j\omega t} \tag{2}$$

From (13.1.3) we find

$$\frac{\partial E_x}{\partial y} = \frac{\beta^2}{\omega\epsilon} \text{Re}\, jA \sin \beta y e^{j\omega t} = \mu \frac{\partial H_z}{\partial t} = \omega\mu \text{Re}\, jA \sin \beta y e^{j\omega t} \tag{3}$$

and thus

$$\beta^2 = \omega^2 \mu\epsilon \tag{4}$$

(b) Turning to the boundary conditions,

$$E_x(-b, t) = \text{Re}\, \hat{V}_d e^{j\omega t}/a \tag{5}$$

and thus from (2)

$$-\frac{\beta}{\omega\epsilon} \text{Re}\, jA \cos \beta b e^{j\omega t} = \text{Re}\, \hat{V}_d e^{j\omega t}/a \tag{6}$$

and hence

$$A = j\frac{\omega\epsilon}{\beta} \frac{\hat{V}_d}{a} \frac{1}{\cos \beta b} = j\sqrt{\frac{\epsilon}{\mu}} \frac{\hat{V}_d}{a} \frac{1}{\cos \beta b} \tag{7}$$

We find

$$H_z = \text{Re}\, j\sqrt{\frac{\epsilon}{\mu}} \frac{\hat{V}_d}{a} \frac{\sin \beta y}{\cos \beta b} e^{j\omega t}$$

$$E_x = \text{Re}\, \frac{V_d}{a} \frac{\cos \beta y}{\cos \beta b} e^{j\omega t}$$

13.1.3 Using the identity

$$\sin x = (e^{jx} - e^{-jx})/2j \tag{1}$$

one finds from (13.1.17)

$$\begin{aligned} E_x &= -\text{Re}\, j\hat{K}_o \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\cos \beta b} \frac{1}{2j} (e^{j\beta y} - e^{-j\beta y}) e^{j\omega t} \\ &= -\text{Re}\, \frac{1}{2} K_o \sqrt{\frac{\mu}{\epsilon}} [e^{j(\omega t - \beta y)} - e^{-j(\omega t + \beta y)}]/\cos \beta b \end{aligned} \tag{2}$$

The exponentials in the brackets represent waves that retain constant amplitude when $dy = \pm \frac{\omega}{\beta} dt$ exhibiting the (phase) velocities $\pm\omega/\beta = \pm 1/\sqrt{\mu\epsilon}$.

13.1.4 (a) The EQS potential in a coax is a solution of Laplace's equation. The field with rotational symmetry is

$$\Phi = A ln\frac{r}{a} \tag{1}$$

satisfying $\Phi = 0$ on outer conductor of radius a. The field is z-independent with a constant potential difference. The potential difference is

$$A ln(b/a) = V \tag{2}$$

The field is

$$\mathbf{E} = -\nabla\Phi = -\mathbf{i_r}\frac{\partial}{\partial r}A ln(r/a) = -\mathbf{i_r}\frac{A}{r} = \mathbf{i_r}\frac{V}{r ln(a/b)} \tag{3}$$

(b) The field has cylindrical symmetry with field-lines parallel to $\mathbf{i}_\phi$. The potential Ψ is

$$\Psi = A\phi \tag{4}$$

The H field is

$$\mathbf{H} = -\mathbf{i}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\Psi = -\mathbf{i}_\phi\frac{A}{r} \tag{5}$$

Ampère's integral law gives

$$\oint \mathbf{H}\cdot d\mathbf{s} = \int \mathbf{J}\cdot d\mathbf{a} = I \tag{6}$$

Since H is z independent, $I =$ constant and at $z = -l$

$$-\frac{A}{r}2\pi r = -2\pi A = I \tag{7}$$

Therefore

$$\mathbf{H} = \mathbf{i_y}\frac{I}{2\pi r} \tag{8}$$

(c) The preceding analysis suggests that

$$\mathbf{E} = \mathbf{i_r}\frac{V(z,t)}{ln(a/b)r} \tag{9a}$$

and

$$\mathbf{H} = \mathbf{i}_\phi\frac{I(z,t)}{2\pi r} \tag{9b}$$

can be solutions of Maxwell's equations. To show this it is advantageous to separate the ∇ operator into

$$\nabla = \nabla_T + \mathbf{i_z}\frac{\partial}{\partial z} \tag{10}$$

where

$$\nabla_T = \mathbf{i_r}\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{i}_\phi\frac{\partial}{\partial \phi} \tag{11}$$

is the transverse part of the operator. Then

$$\nabla \times \mathbf{E} = \nabla_T \times \mathbf{E} + \mathbf{i_z} \times \frac{\partial}{\partial z}\mathbf{E} \tag{12}$$

Now ∇_T differentiates only r and ϕ. The EQS field, which is z independent, has $\nabla_T \times \mathbf{E} = 0$. Hence we conclude that the same holds for the "Ansatz" (9). But $\mathbf{i_z} \times \mathbf{i_r} = \mathbf{i}_\phi$ and $\mathbf{i_z} \times \mathbf{i}_\phi = -\mathbf{i_r}$. We obtain from Faraday's law

$$\frac{1}{ln(a/b)}\frac{1}{r}\frac{\partial}{\partial a}V = -\mu\frac{1}{2\pi r}\frac{\partial I}{\partial t} \tag{13}$$

The common r-dependence can be eliminated, and we find

$$\frac{\partial}{\partial z}V = -L\frac{\partial I}{\partial t} \tag{14}$$

where

$$L = \frac{\mu ln(b/a)}{2\pi} \tag{15}$$

A similar reasoning applied to $\nabla \times \mathbf{H}$ and Ampère's law yields

$$-\mathbf{i_r}\frac{1}{2\pi r}\frac{\partial I}{\partial z} = \mathbf{i_r}\frac{\epsilon}{ln(b/a)r}\frac{\partial V}{\partial t} \tag{16}$$

or

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t} \tag{17}$$

with

$$C = \frac{2\pi\epsilon}{ln(b/a)} \tag{18}$$

13.1.5 (a) With the time dependence $\exp j\omega t$, we get for the transmission line equations of (14) and (17) of Prob. 13.1.4

$$\frac{d\hat{V}}{dz} = -j\omega L\hat{I} \tag{1}$$

$$\frac{d\hat{I}}{dz} = -j\omega C\hat{V} \tag{2}$$

where

$$V = \mathrm{Re}\, \hat{V} e^{j\omega t}$$

and

$$I = \text{Re}\ \hat{I}e^{j\omega t}$$

Eliminating $\hat{V}$ from (1) and (2) one obtains

$$\frac{d^2\hat{V}}{dz^2} = -j\omega L\frac{d\hat{I}}{dz} = -\omega^2 LC\hat{V} \tag{3}$$

with the solutions

$$\hat{V} \propto \begin{cases} \cos\beta z \\ \sin\beta z \end{cases} \tag{4}$$

with

$$\beta = \omega\sqrt{LC} \tag{5}$$

We pick the solution

$$\hat{V} = A\sin\beta z \tag{6}$$

because the short forces $\hat{V}$ to be zero at $z = 0$. From (1) we find

$$\hat{I} = \frac{j}{\omega L}\frac{d\hat{V}}{dz} = \frac{j\beta}{\omega L}A\cos\beta z \tag{7}$$

and since $I = \text{Re}\ I_o e^{j\omega t}$ at $z = -l$,

$$A\cos\beta l = -j\frac{\omega L}{\beta}I_o \tag{8}$$

or

$$A = -j\sqrt{L/C}\frac{I_o}{\cos\beta l} \tag{9}$$

where we used (5). We find for the current and voltage as functions of z and t:

$$I(z,t) = \text{Re}\ \frac{I_o}{\cos\beta l}\cos\beta z e^{j\omega t} \tag{10}$$

$$\hat{V}(z,t) = -\text{Re}\ j\sqrt{L/C}I_o\frac{\sin\beta z}{\cos\beta l}e^{j\omega t} \tag{11}$$

(b) At low frequencies $\cos\beta z \simeq 1$ for all $-l < z < 0$ and $\sin\beta z \simeq \beta z = \omega\sqrt{LC}z$. Using (9) of the preceding problem,

$$\mathbf{H}(z,t) = \mathbf{i}_\phi \text{Re}\ \frac{I_o}{2\pi r}e^{j\omega t} \tag{12}$$

For the E-field we find from the preceding problem and (11) above

$$\mathbf{E} = -\mathbf{i}_r \text{Re} j\omega\frac{LzI_o e^{j\omega t}}{ln(a/b)r} = -\mathbf{i}_r \text{Re}\ j\omega\frac{\mu}{2\pi}zI_o e^{j\omega t} \tag{13}$$

This gives the voltage at $z = -l$

$$\int_b^a \mathbf{E}\cdot\mathbf{i}_r dr = \mathrm{Re}[j\omega L l I_o e^{j\omega t}] \tag{14}$$

The inductance is Ll because L, as defined here, is the inductance per unit length. Thus we have shown that, in the limit of low frequencies, the structure behaves as a single-turn inductor.

(c) The H-field in the space between the conductors is the gradient of a potential $\Psi \propto \phi$ that is a solution of Laplace's equation. Thus,

$$\mathbf{H} = \mathrm{Re}\,\frac{I_o}{2\pi r}\mathbf{i}_\phi e^{j\omega t} \tag{15}$$

We obtain $\mathbf{E}$ from Faraday's law

$$\nabla\times\mathbf{E} = -\frac{\mu\partial\mathbf{H}}{\partial t} = -\mu\mathrm{Re}\,j\omega\frac{I_o}{2\pi r}\mathbf{i}_\phi e^{j\omega t} \tag{16}$$

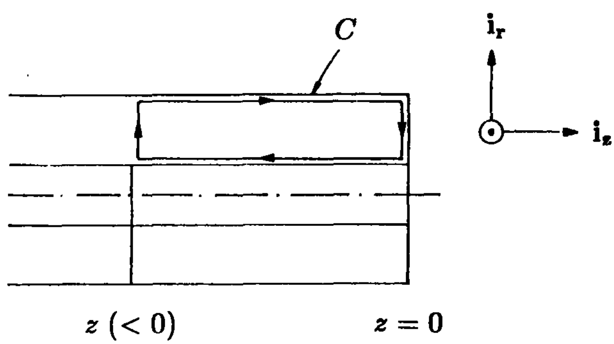


Figure S13.1.5

With the line integral along the contour C shown Fig. S13.1.5, we may find from the integral form of Faraday's law

$$\int_b^a E_r dr = \mu\mathrm{Re}\,\left\{j\omega I_o e^{j\omega t}|z|\int_b^a \frac{dr}{2\pi r}\right\} \tag{17}$$

Integrals over the radial coordinate appear on both sides. Thus, comparing the integrands we find

$$E_r = -\mathrm{Re}\,\frac{j\omega\mu I_o}{2\pi} z e^{j\omega t} \tag{18}$$

which is the same as (13).

13.1.6 (a) From the solutions (4) in Prob. 13.1.5 we pick the $\cos\beta z$ dependence, because the magnetic field, proportional to $\hat{I}$, is zero at $z = 0$ according to (7) of the same problem. Indeed, if $\hat{V} = A\cos\beta z$, then

$$\hat{I} = \frac{j}{\omega L}\frac{d\hat{V}}{dz} = -\frac{j\beta}{\omega L}A\sin\beta z \tag{1}$$

Since

$$\mathrm{Re}[A\cos\beta z\exp j\omega t]_{z=-l} = \mathrm{Re}[V_o\exp j\omega t] \tag{2}$$

we find

$$A = \frac{V_o}{\cos\beta l} \tag{3}$$

and

$$\hat{I} = -j\sqrt{C/L}\frac{V_o}{\cos\beta l}\sin\beta z \tag{4}$$

Therefore,

$$V(z,t) = \mathrm{Re}\left[\frac{V_o}{\cos\beta l}\cos\beta z\exp j\omega t\right] \tag{5}$$

$$I(z,t) = -\mathrm{Re}\, j\sqrt{C/L}\frac{V_o}{\cos\beta l}\sin\beta z e^{j\omega t} \tag{6}$$

$$\mathbf{E} = V(z,t)\nabla_T\frac{ln(r/a)}{ln(a/b)} = \mathbf{i}_r\frac{V(z,t)}{ln(a/b)}\frac{1}{r} \tag{7}$$

where ∇_T is the transverse gradient operator,

$$\nabla_T = \mathbf{i}_r\frac{\partial}{\partial r} + \mathbf{i}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}$$

and we use the result of Prob. 13.1.4. In a similar vein

$$\mathbf{H} = \mathbf{i}_\phi\frac{I(z,t)}{2\pi r} \tag{8}$$

(b) At low frequencies, $\cos\beta z \simeq 1, \sin\beta z \simeq \beta z$ and $V(z,t) \simeq \mathrm{Re}\, V_o\exp j\omega t$. Then, assuming V_o to be real,

$$\mathbf{E} = \frac{\mathbf{i}_r}{ln(a/b)}\frac{1}{r}V_o\cos\omega t \tag{9}$$

$$\mathbf{H} = \frac{\mathbf{i}_\phi}{2\pi r}\sqrt{C/L}\beta z V_o\sin\omega t = \mathbf{i}_\phi\frac{\omega\epsilon}{r ln(a/b)}zV_o\sin\omega t \tag{10}$$

(c) At low frequencies, using EQS directly

$$\mathbf{E} = \frac{\mathbf{i}_r}{ln(a/b)}\frac{1}{r}V_o\cos\omega t \tag{11}$$

namely the gradient of a Laplacian potential $\propto ln(r/a)$. The H-field follows from

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{12}$$

with

$$\mathbf{H} = \mathbf{i}_\phi \frac{A}{r} z \tag{13}$$

introduced into (12)

$$\nabla \times \mathbf{H} = -\mathbf{i}_r \frac{\partial}{\partial z} H_\phi = -\mathbf{i}_r \frac{A}{r} = -\omega \epsilon \frac{\mathbf{i}_r}{ln(a/b)} \frac{1}{r} V_o \sin \omega t$$

and therefore

$$A = \frac{\omega}{ln(a/b)} V_o \sin \omega t \tag{14}$$

which gives the same result as (10).

13.2 TWO-DIMENSIONAL MODES BETWEEN PARALLEL-PLATES

13.2.1 We can write

$$\cos \frac{n\pi}{a} x = \frac{1}{2}\left(\exp j \frac{n\pi}{a} x + \exp -j \frac{n\pi}{a} x \right)$$

and

$$\sin \frac{n\pi}{a} x = \frac{1}{2j}\left(\exp j \frac{n\pi}{a} x - \exp -j \frac{n\pi}{a} x \right)$$

Introducing these expressions into (13.2.19)-(13.2.20) we find four terms of the form

$$\exp \mp j\beta_n y \exp \mp j \frac{n\pi}{a} x = \exp \mp j\left(\beta_n y \pm \frac{n\pi}{a} x\right) = \exp -j\mathbf{k} \cdot \mathbf{r}$$

where

$$\mathbf{k} = \pm \frac{n\pi}{a} \mathbf{i}_x \pm \beta_n \mathbf{i}_y$$

and

$$\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y$$

This proves the assertion that the solution consists of four waves of the stated nature. These waves are phased so as to yield x-dependences of the form $\cos \frac{n\pi}{a} x$ and $\sin \frac{n\pi}{a} x$ to satisfy the boundary conditions.

13.2.2 We can start with the solutions (13.2.19) and (13.2.20) shifting x so that

$$x' = x - \frac{a}{2}$$

Considering TM modes first we note that

$$\begin{aligned} H_z \propto \cos\frac{n\pi}{a}x &= \cos\left(\frac{n\pi x'}{a} + \frac{n\pi}{2}\right) \\ &= \cos\frac{n\pi x'}{a}\cos\frac{n\pi}{2} - \sin\frac{n\pi x'}{a}\sin\frac{n\pi}{2} \\ &= \begin{cases} (-1)^{n/2}\cos\left(\frac{n\pi x'}{a}\right) & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\sin\left(\frac{n\pi x'}{a}\right) & n \text{ odd} \end{cases} \end{aligned}$$

We see that the modes with even n are even with respect to the symmetry plane of the guide, the modes with n-odd are odd.

Next studying the TE-modes,

$$\begin{aligned} E_z \propto \sin\frac{n\pi}{a}x &= \sin\left(\frac{n\pi x'}{a}\right)\cos\frac{n\pi}{2} + \cos\left(\frac{n\pi x'}{a}\right)\sin\frac{n\pi}{2} \\ &= \begin{cases} (-1)^{n/2}\sin\frac{n\pi x'}{a} & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\cos\frac{n\pi x'}{a} & n \text{ odd} \end{cases} \end{aligned}$$

We find that E_z is even for n odd, odd for n even.

(a) When $x' = \pm a/2$ and the modes are odd, $H_z = (-1)^{(n-1)/2}\sin\frac{n\pi}{2}, E_z = (-1)^{n/2}\sin\frac{n\pi}{2}$; in the first case n is odd and H_z is an extremum at $x' = \pm a/2$, and in the second case n is even and E_z is zero at both boundaries.

(b) When $x' = \pm a/2$ and the modes are even then $H_z = (-1)^{n/2}\cos(\frac{n\pi}{2})$ and $E_z = (-1)^{(n-1)/2}\cos\frac{n\pi}{2}$ we see that both boundary conditions are in both cases, because n is odd in the first case and H_z is an extrenum, n is even in the second case, and E_z is zero.

13.3 TE AND TM STANDING WAVES BETWEEN PARALLEL PLATES

13.3.1 We multiply (13.3.1) by $\hat{h}^*_{zm}$ and integrate over the interval from 0 to a.

$$\begin{aligned} \int_0^a dx\left(\hat{h}^*_{zm}\frac{d^2\hat{h}_{zn}}{dx^2} + p_n^2\hat{h}^*_{zm}\hat{h}_{zn}\right) &= \int_0^a dx\frac{d}{dx}\left(\hat{h}^*_{zm}\frac{d\hat{h}_{zn}}{dx}\right) \\ &- \int_0^a dx\left(\frac{d}{dx}\hat{h}^*_{zm}\right)\left(\frac{d}{dx}\hat{h}_{zn}\right) \\ &+ p_n^2\int_0^a \hat{h}^*_{zm}\hat{h}_{zn}dx = 0 \end{aligned} \tag{1}$$

where we have integrated by parts. Because $dh_{zn}/dx = 0$ at $x = 0$ and $x = a$, the integral of the integrand containing the total derivative vanishes.

Next take the complex conjugate of (13.3.1) applied to $\hat{h}_{zm}$ multiply by $\hat{h}_{zn}$ and integrate. The result is

$$\int_0^a dx \frac{d}{dx}\hat{h}_{zm}^* \frac{d}{dx}\hat{h}_{zn} = p_m^2 \int_0^a \hat{h}_{zm}^* \hat{h}_{zn} dx \tag{2}$$

Subtraction of (1) and (2) gives

$$(p_m^2 - p_n^2)\int_0^a \hat{h}_{zm}^* \hat{h}_{zn} dx = 0$$

Thus

$$\int_0^a \hat{h}_{zm}^* \hat{h}_{zn} dx = 0$$

when $p_m^2 \neq p_n^2$ and orthogonality is proven. The steps involving $\hat{e}_{zn}$ are identical. The only difference is that

$$\int_0^a dx \frac{d}{dx}\Big(\hat{e}_{zm}^* \frac{d\hat{e}_{zn}}{dx}\Big)$$

vanishes because $\hat{e}_{zm}^*$ vanishes at $x = 0$ and $x = a$.

13.3.2 (a) The charge in the bottom plate is

$$q = \int_0^w \int_{(a-\Delta)/2}^{(a+\Delta)/2} \epsilon E_y dx dz \tag{1}$$

Using (13.3.15)

$$q = \mathrm{Re}\left[\sum_{\substack{n=1\\ \mathrm{odd}}} \frac{4n\pi\epsilon}{a}\frac{\hat{v}}{\beta_n a}\frac{1}{\sin\beta_n b} e^{j\omega t}\right]\frac{2wa}{n\pi}(-1)^{\frac{n-1}{2}} \sin\frac{n\pi\Delta}{2a} \tag{2}$$

where we have used the fact that

$$\begin{aligned}\int_0^w \int_{(a-\Delta)/2}^{(a+\Delta)/2} \sin\Big(\frac{n\pi}{a}x\Big) dx dz &= -\frac{wa}{n\pi}\Big[\cos\Big(\frac{n\pi}{a}\frac{a+\Delta}{2}\Big) - \cos\Big(\frac{n\pi}{a}\frac{a-\Delta}{2}\Big)\Big]\\ &= 2\frac{wa}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi\Delta}{2a}\\ &= \frac{2wa}{n\pi}(-1)^{\frac{n-1}{2}}\sin\frac{n\pi\Delta}{2a}\end{aligned} \tag{3}$$

$$\begin{aligned}v_o &= -\mathrm{Re}\, j\omega\hat{q}e^{j\omega t}R\\ &= -\mathrm{Re}\left[j\omega 8\epsilon w R\hat{v}\sum_n \frac{(-1)^{\frac{n-1}{2}}}{\beta_n a}\frac{\sin\frac{n\pi\Delta}{2a}}{\sin\beta_n b}e^{j\omega t}\right]\end{aligned} \tag{4}$$

When $\beta_n b = \pi$ we have a resonance. Now

$$\beta_n = \sqrt{\omega^2\mu\epsilon - \left(\frac{n\pi}{a}\right)^2}$$

The resonance frequency of the n-th mode occurs at

$$\beta_n \frac{b}{a} a = \sqrt{4\omega^2\mu\epsilon a^2 - (n\pi)^2} = \pi \tag{5}$$

or

$$\omega\sqrt{\mu\epsilon}a = \sqrt{(n^2+1)}\frac{\pi}{2} \tag{6}$$

(b) For $n = 1$ this is at π. The next mode resonates at $\sqrt{5/4}\pi$. Thus, in this range, two resonances occur for which the response goes to infinity. Of course, in this limit, losses have to be taken into account which will maintain the response finite. The low frequency limit is when

$$\omega\sqrt{\mu\epsilon} \ll n\frac{\pi}{a}$$

Then

$$\beta_n \sin\beta_n b \to -\frac{n\pi}{a}\sinh\frac{n\pi}{a}b$$

and

$$v_o = \mathrm{Re}\left[j\omega 8\pi\epsilon w R\hat{v}\sum_n \frac{(-1)^{\frac{n-1}{2}}\sin\frac{n\pi\Delta}{2a}}{\frac{n\pi}{a}\sinh\frac{n\pi}{a}b}e^{j\omega t}\right] \tag{7}$$

(c) From (13.3.13), when only one mode predominates,

$$H_z \simeq \mathrm{Re}\left[\frac{4j\omega\epsilon\hat{v}}{\beta_n a}\frac{\cos\beta_n y}{\sin\beta_n b}\cos\frac{n\pi}{a}\right]e^{j\omega t}$$

where $n = 1$ at $\omega\sqrt{\mu\epsilon}a = \pi$ and $n = 2$ at $\omega\sqrt{\mu\epsilon}a = \sqrt{5/4}\pi$. To get a finite answer, we need $\hat{v}/\sin\beta_n b$ to remain finite as the resonance frequency is approached.

13.3.3 (a) H_z at $x = 0$ and $x = a$ gives the surface currents in the bottom and top electrodes. Because the voltage sources push currents into the structure in opposite directions, the surface currents, and H_z, have to vanish at the symmetry plane.

(b) The x-component of the E field can be found directly from (13.3.14), replacing the $\sin\beta_n y/\sin\beta_n b$ by $\cos\beta_n y/\cos\beta_n b$ to take into account the changed symmetry of the field

$$E_x = \mathrm{Re}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty} -\frac{4\hat{v}}{a}\frac{\cos\beta_n y}{\cos\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

Because $\partial H_z/\partial y = j\omega\epsilon E_x$ we find

$$H_z = \mathrm{Re}\left[\sum_{\substack{n=1\\ n\text{odd}}}^{\infty} \frac{4j\omega\epsilon\hat{v}}{\beta_n a}\frac{\sin\beta_n y}{\cos\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

13.3.4 (a) The flux linkage λ is

$$\lambda = \mu H_z A \tag{1}$$

and the voltage is

$$v_o = \mu A \frac{dH_z}{dt} \tag{2}$$

(b) From (13.3.13) we find that $|\hat{H}_z|$ is a maximum for $x = 0$ and $x = a$.

(c) From the detailed expression (13.3.13), using (2)

$$v_o = -\mathrm{Re}\Big[\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{\infty} \frac{4\omega^2\mu\epsilon\hat{v}}{\beta_n a} A \frac{1}{\sin\beta_n b}\cos\frac{n\pi}{a}X\Big]e^{j\omega t}$$

(d) The loop should lie in the $y - z$ plane. Then it links H_x that is tangential to the bottom plate.

13.3.5 The E_x field is derivable from a potential that is a square wave as shown in Fig. S13.3.5. We have

$$\Phi(x) = \sum_m A_m \sin\frac{m\pi}{a}x \tag{1}$$

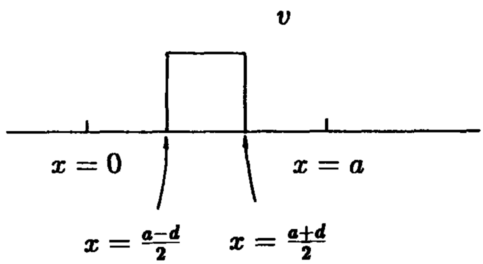


Figure S13.3.5

and using orthogonality, multiplication of both sides by $\sin\frac{n\pi}{a}x$ and integration gies

$$\frac{a}{2}A_n = v\int_{\frac{a-d}{2}}^{\frac{a+d}{2}} \sin\frac{n\pi}{a}x dx = -\frac{av}{n\pi}\Big[\cos\big(\frac{n\pi}{a}\frac{a+d}{2}\big) - \cos\big(\frac{n\pi}{a}\frac{a-d}{2}\big)\Big]$$
$$= 2\frac{av}{n\pi}\sin\big(\frac{n\pi}{2}\big)\sin\big(\frac{n\pi d}{2a}\big)$$

We find

$$A_n = \frac{4v}{n\pi}\sin\big(\frac{n\pi}{2}\big)\sin\big(\frac{n\pi d}{2a}\big)$$

We may adapt (13.3.13)-(13.3.15) for this case by replacing $4v/n\pi$ by

$$4v/n\pi \sin\big(\frac{n\pi}{2}\big)\sin\frac{n\pi d}{2a}$$

$$H_z = \mathrm{Re}\left[\sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} \frac{4j\omega\epsilon\hat{v}}{\beta_n a}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2a}d\right)\frac{\cos\beta_n y}{\sin\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

$$E_x = \mathrm{Re}\left[\sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} -\frac{4v}{a}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi d}{2a}\right)\frac{\sin\beta_n y}{\sin\beta_n b}\cos\frac{n\pi}{a}x\right]e^{j\omega t}$$

$$E_y = \mathrm{Re}\left[\sum_{\substack{n=1\\ \mathrm{odd}}}^{\infty} \frac{4n\pi}{a}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi d}{2a}\right)\frac{\cos\beta_n y}{\sin\beta_n b}\sin\frac{n\pi}{a}x\right]e^{j\omega t}$$

13.3.6 In (13.3.5) we recognized that E_x at $y = b$ must be the derivative of a potential that is a square wave. This, of course, is equivalent to the statement that E_x possesses two impulse functions. In a similar manner, H_y can be considered the derivative of a flux function $\int_0^x \mu H_y dx$. Note the analogy between (13.3.19) and (13.3.14). We may, therefore, adapt the expansion of P13.3.5 to this problem, because the flux function of Example 13.3.2 is the same as the potential of example 13.3.1. From (13.3.17)-(13.3.19):

$$E_z = \mathrm{Re}\left[\sum_{\substack{m=1\\ \mathrm{odd}}}^{\infty} -\frac{4j\hat{\Lambda}\omega}{m\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi d}{2a}\right)\frac{\sin\beta_m y}{\sin\beta_m b}\sin\frac{m\pi}{a}x\right]e^{j\omega t}$$

$$H_x = \mathrm{Re}\left[\sum_{\substack{m=1\\ \mathrm{odd}}}^{\infty} \frac{4\beta_m\hat{\Lambda}}{\mu m\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2a}d\right)\frac{\cos\beta_m y}{\sin\beta_m b}\sin\frac{m\pi}{a}x\right]e^{j\omega t}$$

$$H_y = \mathrm{Re}\left[\sum_{\substack{m=1\\ \mathrm{odd}}}^{\infty} -\frac{4\hat{\Lambda}}{\mu a}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2a}d\right)\frac{\sin\beta_m y}{\sin\beta_m b}\cos\frac{m\pi}{a}x\right]e^{j\omega t}$$

13.4 RECTANGULAR WAVEGUIDE MODES

13.4.1 The loop in the $y - z$ plane produces H-field lines along the x-direction. If placed in the center of the waveguide, at $x = a/2$, these field lines have the same symmetry as those of the TE_{10} mode and thus excite this mode. The detection loop links these fields lines as well. Of course, the position of the exciting loop must be displaced along y by one quarter wavelength compared to the capacitive probe for maximum excitation.

13.4.2 The cutoff frequencies are given by

$$\frac{\omega_c}{c} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{w}\right)^2}$$

The dominant mode has $n = 0$ and thus it has the (lowest) cutoff frequency

$$\frac{\omega_c}{c}\Big|_{m=1,n=0} = \left(\frac{\pi}{a}\right)$$

The higher order modes have cutoff frequencies

$$\frac{\omega_c}{c}\Big|_{mn} = \left(\frac{\pi}{a}\right)\sqrt{m^2 + n^2(a/w)^2} = \frac{\pi}{a}\sqrt{m^2 + n^2(4/3)^2}$$

The cuttoff frequencies are in the ratio to that of the dominant mode:

TE_{01}	1.33
TE_{11} and TM_{11}	1.66
TE_{20}	2.0
TE_{21} and TM_{21}	2.4

13.4.3 (a) TM-modes have all three E-field components. They approach the quasistatic fields of Ex. 5.10.1 which imposes the same boundary conditions as this example. Hence the modes are TM. From (9) we find that $\hat{e}_x \propto -jk_y \frac{\partial e_y}{\partial x} = \frac{\partial^2 e_y}{\partial y \partial x}$ and $\hat{e}_z \propto -jk_y \frac{\partial e_y}{\partial z} = \frac{\partial^2 e_y}{\partial y \partial z}$. Since $\hat{e}_x$ and $\hat{e}_z$ must vanish at $y = 0$, $\hat{e}_y$ must behave as a cosine function of y, so that $\hat{e}_x$ and $\hat{e}_z$ are sine functions of y. Therefore,

$$\begin{aligned} E_y &= \mathrm{Re} \sum_m \sum_n (A^+_{mn} e^{-j\beta_{mn} y} + A^-_{mn} e^{j\beta_{mn} y}) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \\ &= \mathrm{Re} \sum_m \sum_n 2A^+_{mn} \cos \beta_{mn} y \sin \frac{m\pi}{a} x \sin \frac{n\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \end{aligned} \qquad (1)$$

where

$$\beta_{mn} = \sqrt{\omega^2 \mu\epsilon - (m\pi/a)^2 - (n\pi/w)^2} \qquad (2)$$

From (13.4.9):

$$\begin{aligned} E_x &= \mathrm{Re} \sum_m \sum_n \frac{-j\beta_{mn}\left(\frac{m\pi}{a}\right)}{\omega^2\mu\epsilon - \beta^2_{mn}} (A^+_{mn} e^{-j\beta_{mn} y} - A^-_{mn} e^{j\beta_{mn} y}) \\ &\quad \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \\ &= \mathrm{Re} \sum_m \sum_n \frac{\beta_{mn} \frac{m\pi}{a}}{\omega^2\mu\epsilon - \beta^2_{mn}} 2A^+_{mn} \sin \beta_{mn} y \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \end{aligned} \qquad (3)$$

and similarly,

$$E_z = \mathrm{Re} \sum_m \sum_n \frac{\beta_{mn} \frac{n\pi}{w}}{\omega^2 \mu\epsilon - \beta_{mn}^2} 2A_{mn}^+ \sin \beta_{mn} y \sin \frac{m\pi}{a} x \cos \frac{n\pi}{w} z e^{j\omega t} \tag{4}$$

(b) At $y = b$, E_x as a function of x must possess two equal and opposite unit impulse functions of content $v(t)/\Delta$ to give the proper voltage drop at the edges. The integral of E_x, $-\int_0^x E_x dx$ must be a square wave function of amplitude v. The same holds with regard to the integral of E_z with respect to z. In summary, E_x and E_z at $y = b$ must be derivable from a potential that is a two-dimensional square wave with the Fourier expansion (5.10.15) (compare 5.10.11):

$$\Phi(x, y) = \mathrm{Re} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \tag{5}$$

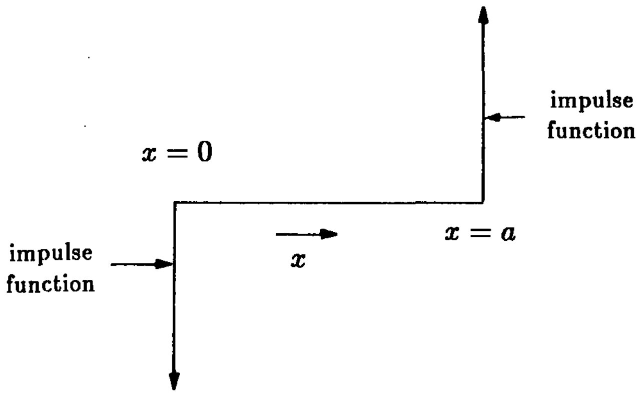


Figure S13.4.3

Thus, at $y = b$

$$E_x(y = b) = -\mathrm{Re} \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n=1 \\ n \text{ odd}}} \frac{16\hat{v}}{mn\pi^2} \left(\frac{m\pi}{a}\right) \cos \frac{m\pi}{a} x \sin \frac{n\pi}{w} z e^{j\omega t} \tag{6}$$

Comparison with (3) gives

$$2A_{mn} \beta_{mn} \frac{m\pi}{a} / (\omega^2 \mu\epsilon - \beta_{mn}^2) \sin \beta_{mn} b = \frac{16\hat{v}}{mn\pi^2} \frac{m\pi}{a} \tag{7}$$

for m and n odd. This gives the quoted result. An analogous relation may be obtained for E_z which yields the same result.

(c) The amplitudes go to infinity when $\sin \beta_{mn} b = 0$ or

$$\sqrt{\omega^2 \mu\epsilon - (m\pi/a)^2 - (n\pi/w)^2} b = p\pi$$

or

$$\omega\sqrt{\mu\epsilon}a = \pi\sqrt{m^2 + \left(\frac{a}{w}n\right)^2 + \left(\frac{a}{b}p\right)^2}$$

(d) We have already used the fact that the distribution of E_x and E_z in the $y = b$ plane is the same as in the quasistatic case. The only difference lies in the y-dependence which, for low frequencies gives the propagation constant

$$\beta_{mn} \simeq j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{w}\right)^2}$$

and is pure imaginary. The EQS solution according to (5.10.11) and (5.10.15) is

$$\Phi = \text{Re} \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2} \frac{\sinh k_{mn}y}{\sinh k_{mn}b} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{w}z e^{j\omega t}$$

and gives for E_x:

$$E_x = -\text{Re} \sum_{\substack{m=1 \\ m\,\text{odd}}}^{\infty} \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{16\hat{v}}{mn\pi^2}\left(\frac{m\pi}{a}\right)\frac{\sinh k_{mn}y}{\sinh k_{mn}b} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{w}z e^{j\omega t}$$

This is the same expression as the EQS result.

13.4.4

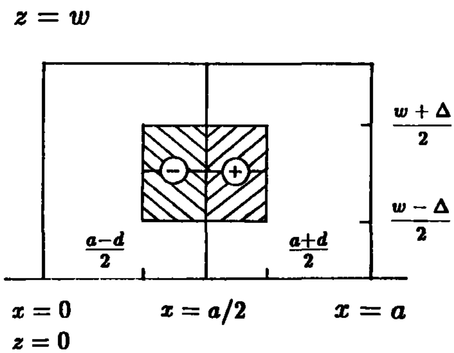


Figure S13.4.4a

The excitation produces a H_y. It looks like TE-modes are going to satisfy all the boundary conditions. H_y must be zero at $y = 0$ and thus from (25) of text

$$\begin{aligned} H_y &= \text{Re} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (C_{mn}^{+} e^{-j\beta_{mn}y} + C_{mn}^{-} e^{j\beta_{mn}y}) \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{w}z\right) e^{j\omega t} \\ &= -\text{Re} \sum_{m=0} \sum_{n=0} 2jC_{mn}^{+} \sin\beta_{mn}y \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{w}z\right) e^{j\omega t} \end{aligned} \tag{1}$$

At $y = b$ we must represent the two dimensional square-wave in the x-direction and in Fig. S13.4.4b in the z-direction as shown in Fig S13.4.4c.

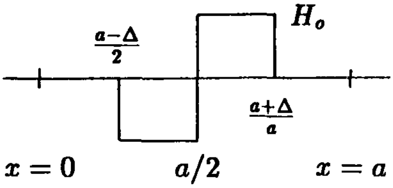


Figure S13.4.4b

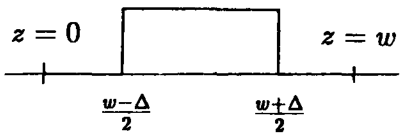


Figure S13.4.4c

We have, setting

$$-2jC_{mn}^{+}\sin\beta_{mn}b \equiv A_{mn} \tag{2}$$

$$\int_0^w\int_0^a \sum\sum A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{w}z\right)\cos\left(\frac{p\pi}{a}x\right)\cos\left(\frac{q\pi}{w}z\right)dxdz = -\frac{1}{4}A_{pq}(aw)$$

$$= -H_o\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dz\int_{\frac{a-d}{2}}^{\frac{a}{2}} dx\cos\left(\frac{p\pi}{a}x\right)\cos\frac{q\pi}{w}z$$

$$+ H_o\int_{\frac{w-\Delta}{2}}^{\frac{w+\Delta}{2}} dz\int_{\frac{a}{2}}^{\frac{a+d}{2}} dx\cos\left(\frac{p\pi}{a}x\right)\cos\frac{q\pi}{w}z$$

$$= -\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\left(\frac{p\pi}{a}\frac{a}{2}\right) - \sin\left(\frac{p\pi}{a}\frac{a-d}{2}\right)\right]$$

$$\left[\sin\left(\frac{q\pi}{w}\frac{w+\Delta}{2}\right) - \sin\left(\frac{q\pi}{w}\frac{w-\Delta}{2}\right)\right]$$

$$+\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\frac{p\pi}{a}\frac{a+d}{2} - \sin\frac{p\pi}{a}\frac{a}{2}\right]$$

$$\left[\sin\left(\frac{q\pi}{w}\frac{w+\Delta}{2}\right) - \sin\left(\frac{q\pi}{w}\frac{w-\Delta}{2}\right)\right.$$

$$= -\frac{H_o}{\left(\frac{p\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[\sin\left(\frac{p\pi}{a}\frac{a}{2}\right)\right.$$

$$\left. - \sin\left(\frac{p\pi}{a}\frac{a-d}{2}\right) - \sin\left(\frac{p\pi}{a}\frac{a+d}{2}\right) + \sin\left(\frac{p\pi}{a}\frac{a}{2}\right)\right]$$

$$\left[\sin\frac{q\pi}{w}\frac{w+\Delta}{2} - \sin\frac{q\pi}{w}\frac{w-\Delta}{2}\right]$$

$$= -\frac{H_o}{\left(p\frac{\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\left[2\sin\left(\frac{p\pi}{2}\right) - 2\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi}{2a}\Delta\right)\right]2\cos\frac{q\pi}{2}\sin\frac{q\pi\Delta}{2w}$$

$$= -\frac{4H_o}{\left(p\frac{\pi}{a}\right)\left(\frac{q\pi}{w}\right)}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{q\pi}{2}\right)\sin\frac{q\pi\Delta}{2w}\left[1-\cos\frac{p\pi}{2a}\Delta\right] \tag{3}$$

We find that p must be odd and q must be even for a finite amplitude to result.

$$A_{pq} = \frac{H_o}{pq\pi^2}(-1)^{p-1}(-1)^{\frac{q}{2}-1}\left[1-\cos\frac{p\pi}{2a}\Delta\right] \tag{4}$$

The case $q = 0$ must be handled separately.

$$\frac{1}{2}A_{po}(aw) = -\frac{H_o}{(p\frac{\pi}{a})}\left[\sin\left(\frac{p\pi}{a}\frac{a}{2}\right) - \sin\left(\frac{p\pi}{a}\frac{a-d}{2}\right)\right]w$$
$$+\frac{H_o}{(\frac{p\pi}{a})}\left[\sin\frac{p\pi}{a}\left(\frac{a+d}{2}\right) - \sin\left(\frac{p\pi}{a}\frac{a}{2}\right)\right]w \tag{5}$$
$$= -\frac{2wH_o}{(p\frac{\pi}{a})}\left[1-\cos\frac{p\pi}{2a}\right]\sin\frac{p\pi}{2}$$

and thus

$$A_{po} = \frac{H_o}{p\pi}\left[1-\cos\frac{p\pi}{2a}\Delta\right](-1)^{p-1}$$

From (13.4.7) and (13.4.8), one finds

$$H_x = \mathrm{Re}\sum_m\sum_n \frac{2jC_{mn}^{+}\beta_{mn}\left(\frac{m\pi}{a}\right)}{\omega^2\mu\epsilon - \beta_{mn}^2}\cos\beta_{mn}y\sin\frac{m\pi}{a}x\cos\frac{n\pi}{w}ze^{j\omega t} \tag{7}$$

$$H_z = \mathrm{Re}\sum_m\sum_n \frac{2jC_{mn}^{+}\beta_{mn}\left(\frac{n\pi}{w}\right)}{\omega^2\mu\epsilon - \beta_{mn}^2}\cos\beta_{mn}y\cos\frac{m\pi}{a}x\sin\frac{n\pi}{w}ze^{j\omega t} \tag{8}$$

with C_{mn}^{+} expressed in terms of the A_{mn}'s by (2)

13.5 DIELECTRIC WAVEGUIDES: OPTICAL FIBERS

13.5.1 (a) To get an odd function of x for $\hat{e}_z$ one uses the Ansatz

$$\hat{e}_z = \begin{cases} Ae^{-\alpha_x(x-d)} & d < x < \infty \\ A\frac{\sin k_x x}{\sin k_x d} & -d < x < d \\ -Ae^{\alpha_x(x+d)} & -\infty < x < -d \end{cases} \tag{1}$$

which has been adjusted so that $\hat{e}_z$ is continuous at $x = \pm d$. Since

$$\frac{\partial \hat{e}_z}{\partial x} = j\omega\mu\hat{h}_y \tag{2}$$

and thus

$$\hat{h}_y = \frac{1}{j\omega\mu}\begin{cases} -\alpha_x Ae^{-\alpha_x(x-d)} \\ k_x A\frac{\cos k_x x}{\sin k_x d} \\ -\alpha_x Ae^{\alpha_x(x+d)} \end{cases} \tag{3}$$

$\hat{h}_y$ and $\hat{e}_z$ are continuous at $x = d$. The continuity of $\hat{e}_z$ has already been established. From the continuity of h_y:

$$\alpha_x = -k_x\cot k_x d \tag{4}$$

The cutoffs are at $k_x d = (2n-1)\frac{\pi}{2}$ (see Fig. S13.5.1a).

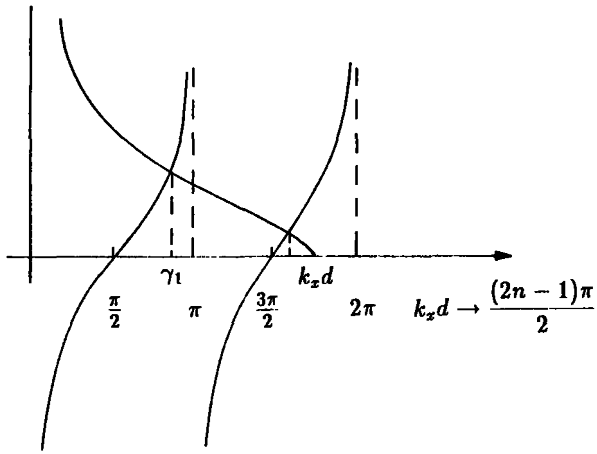


Figure S13.5.1a

(c) When according to 13.5.3

$$k_x d = \sqrt{\omega^2 \mu \epsilon_i - k_y^2} = (2n-1)\frac{\pi}{2}$$

and ω goes to infinity, then k_y must approach $\omega\sqrt{\mu\epsilon_i}$ asymptotically.

(d) See Fig. S13.5.1b (Fig. 6.4 from **Waves and Fields in Optoelectronics**, H. A. Haus, Prentice-Hall, 1984).

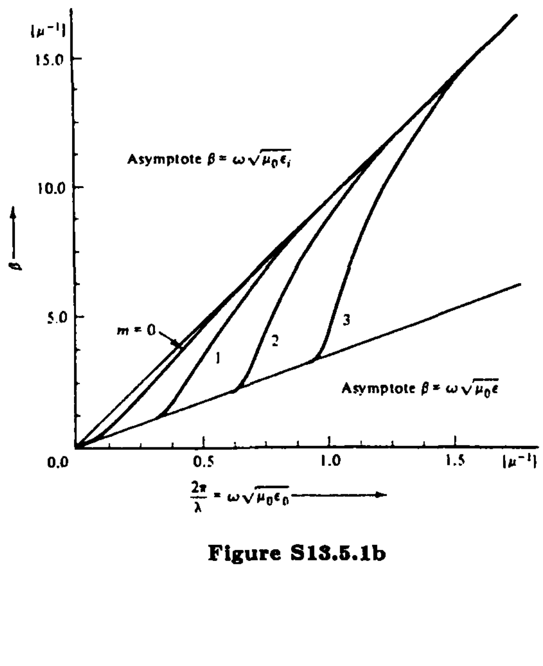


Figure S13.5.1b

13.5.2 The antisymmetric mode comes in when

$$\frac{\alpha_x}{k_x} = 0 \qquad \text{and} \qquad k_x d = \frac{\pi}{2}$$

and from (13.5.8)

$$\sqrt{\frac{\omega^2 \mu\epsilon_i d^2}{(k_x d)^2}\left(1 - \frac{\epsilon}{\epsilon_i}\right) - 1} = 0$$

or

$$\omega\sqrt{\mu\epsilon_i}d = \frac{\pi/2}{\left(1 - \frac{\epsilon}{\epsilon_i}\right)}$$

or

$$\omega = \frac{1}{\sqrt{\mu\epsilon_i}d}\frac{\pi/2}{\sqrt{\left(1 - \frac{\epsilon}{\epsilon_i}\right)}} = \sqrt{\frac{\epsilon_o}{\epsilon}}\frac{c}{d}\sqrt{\frac{\epsilon}{\epsilon_i}}\frac{\pi/2}{\sqrt{1 - \epsilon/\epsilon_i}}$$

$$= \frac{3\times 10^8}{10^{-2}}\frac{1}{\sqrt{2.5}}\frac{\pi/2}{\sqrt{1 - \frac{1}{2.5}}} = 3.85\times 10^{10}$$

$$f = \frac{\omega}{2\pi} = 6.1\times 10^9 \text{ Hz}$$

13.5.3 (a) For TE modes

$$e_z = \begin{cases} Ae^{-\alpha_x(x-d)} & & x > d \\ A\frac{\cos k_x x}{\cos k_x d} \quad \text{or} \quad A\frac{\sin k_x x}{\sin k_x d} & & -d < x < d \\ Ae^{\alpha_x(x+d)} \quad \text{or} \quad -Ae^{\alpha_x(x+d)} & & x < -d \end{cases}$$

where we have allowed for symmetric and antisymmetric modes. Continuity of $\hat{e}_z$ has been assured on both boundaries. The magnetic field follows from

$$\hat{h}_y = \frac{1}{j\omega\mu}\frac{de_z}{dx} \tag{2}$$

and thus

$$\hat{h}_y = \frac{1}{j\omega}\begin{cases} -\frac{\alpha_x}{\mu}Ae^{-\alpha_x(x-d)} & & x > d \\ -\frac{k_x}{\mu_i}A\frac{\sin k_x x}{\cos k_x d} \quad \text{or} \quad \frac{k_x}{\mu_i}\frac{\cos k_x x}{\sin k_x d} & & -d < x < d \\ \frac{\alpha_x}{\mu}Ae^{\alpha_x(x+d)} \quad \text{or} \quad -\frac{\alpha_x}{\mu}Ae^{\alpha_x(x+d)} & & x < -d \end{cases} \tag{3}$$

Continuity of $\hat{h}_y$ gives

$$\frac{\alpha_x}{\mu} = \frac{k_x}{\mu_i}\tan k_x d \tag{4a}$$

for even modes, and

$$\frac{\alpha_x}{\mu} = -\frac{k_x}{\mu_i} \cot k_x d \tag{4b}$$

for odd modes. Here

$$\alpha_x = \sqrt{k_y^2 - \omega^2 \mu\epsilon} \tag{5}$$

$$k_x = \sqrt{\omega^2 \mu_i \epsilon_i - k_y^2} \tag{6}$$

and thus, eliminating k_y,

$$k_x^2 = \omega^2 (\mu_i\epsilon_i - \mu\epsilon) - \alpha_x^2$$

and

$$\frac{\alpha_x}{k_x} = \sqrt{\frac{\omega^2 \mu_i \epsilon_i d^2}{k_x^2 d^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1} \tag{7}$$

(b) Cutoff occurs when $\alpha_x/k_x = 0$ and $k_x d$ is fixed. We find that when μ_i is increased above μ, ω must be lowered.

(c) The constitutive law (a) for symmetric modes has the graphic solution of Fig. 13.5.2. The only change is the expression for α_x/k_x but its $k_x d$ dependence is qualitatively the same; α_x/k_x increases when μ_i/μ increases at constant ω. This means that the intersection point moves to greater $k_x d$ values. k_y^2 increases directly with increasing μ_i/μ according to (6) and decreases with increasing k_x. The intersection point of $k_x d$ does not change as fast, in particular, at high frequencies it does not move at all. Hence, the direct dependence on μ_i predominates, k_y goes up and λ decreases.

13.5.4 (a) The fields are now

$$\hat{h}_z = \begin{cases} Ae^{-\alpha_x(x-d)} & & x > d \\ A\frac{\cos k_x x}{\cos k_x d} \quad \text{or} \quad A\frac{\sin k_x x}{\sin k_x d} & & -d < x < d \\ Ae^{\alpha_x(x+d)} \quad \text{or} \quad -Ae^{\alpha_x(x+d)} & & x < -d \end{cases} \tag{1}$$

where we have allowed for both symmetric and antisymmetric solutions. Cointinuity of $\hat{h}_z$ has been asured on both boundaries. Further,

$$\alpha_x = \sqrt{k_y^2 - \omega^2\mu\epsilon} \tag{2}$$

$$k_x = \sqrt{\omega^2 \mu_i \epsilon_i - k_y^2} \tag{3}$$

Since

$$\hat{e}_y = -\frac{1}{j\omega\epsilon}\frac{d\hat{h}_z}{dx} \tag{4}$$

$$\hat{e}_y = -\frac{1}{j\omega}\begin{cases} -\frac{\alpha_x}{\epsilon}Ae^{-\alpha_x(x-d)} \\ -\frac{k_x}{\epsilon_i}A\frac{\sin k_x x}{\cos k_x d} \quad \text{or} \quad \frac{k_x}{\epsilon_i}\frac{\cos k_x x}{\sin k_x d} \\ \frac{\alpha_x}{\epsilon}Ae^{\alpha_x(x+d)} \quad \text{or} \quad -\frac{\alpha_x}{\epsilon}Ae^{\alpha_x(x+d)} \end{cases} \tag{5}$$

Continuity of $\hat{e}_y$ at $x = \pm d$ gives

$$\frac{\alpha_x}{\epsilon} = \frac{k_x}{\epsilon_i}\tan k_x d \tag{6a}$$

for even modes, and

$$\frac{\alpha_x}{\epsilon} = -\frac{k_x}{\epsilon_i}\cot k_x d \tag{6b}$$

for odd modes. Further,

$$k_y^2 = \alpha_x^2 + \omega^2\mu\epsilon = -k_x^2 + \omega^2\mu_i\epsilon_i \tag{7}$$

Thus

$$\alpha_x^2 = \omega^2(\mu_i\epsilon_i - \mu\epsilon) - k_x^2 \tag{8}$$

and

$$\frac{\alpha_x}{k_x} = \sqrt{\frac{\omega^2\mu_i\epsilon_i d^2}{k_x^2 d^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1} \tag{9}$$

(b) The cutoff frequencies are determined by $k_x d = m\frac{\pi}{2}$ and $\alpha_x = 0$. From (9)

$$\frac{\omega_c^2\mu_i\epsilon_i d^2}{\left(m\frac{\pi}{2}\right)^2}\left[1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}\right] - 1 = 0$$

or

$$\omega_c = \frac{m\frac{\pi}{2}}{\sqrt{\mu_i\epsilon_i}d}\frac{1}{\sqrt{1 - \frac{\mu\epsilon}{\mu_i\epsilon_i}}}$$

SOLUTIONS TO CHAPTER 14

14.1 DISTRIBUTED PARAMETER EQUIVALENTS AND MODELS

14.1.1 The fields are approximated as uniform in each of the dielectric regions. The integral of $\mathbf{E}$ between the electrodes must equal the applied voltage and D is continuous at the interface. Thus,

$$E_a a + E_b b = -V; \qquad \epsilon_a E_a = \epsilon_b E_b \tag{1}$$

and it follows that

$$E_a = -\frac{V}{[a + b(\epsilon_a/\epsilon_b)]} \tag{2}$$

so that the charge per unit length on the upper electrodes is

$$\lambda_l = -w\epsilon_a E_a = CV; \qquad C \equiv \frac{w\epsilon_a}{[a + b(\epsilon_a/\epsilon_b)]} \tag{3}$$

where C is the desired capacitance per unit length. Because the permeability of the region is uniform, $H = I/w$ between the electrodes. Thus,

$$\lambda = (a+b)\mu_o H = LI; \qquad L \equiv \mu_o(a+b)/w \tag{4}$$

where L is the inductance per unit length. Note that $LC \neq \mu\epsilon$ (which permittivity) unless $\epsilon_a = \epsilon_b$.

14.1.2 The currents at the node must sum to zero, with that through the inductor related to the voltage by $V = Ldi_{\text{conductor}}/dt$

$$\frac{L}{\Delta z}\frac{\partial}{\partial t}[I(z) - I(z+\Delta z)] = V \tag{1}$$

and C times the rate of change of the voltage drop across the capacitor must be equal to the current through the capacitor.

$$\frac{C}{\Delta z}\frac{\partial}{\partial t}[V(z) - V(z+\Delta z)] = I \tag{2}$$

In the limit where $\Delta z \to 0$, these become the given backward-wave transmission line equations.

14.2 TRANSVERSE ELECTROMAGNETIC WAVES

14.2.1 (a) From Ampère's integral law, (1.4.10),

$$H_\phi = I/2\pi r \tag{1}$$

and the vector potential follows by integration

$$H_\phi = -\frac{1}{\mu_o}\frac{\partial A_z}{\partial r} \Rightarrow A_z(r) - A_z(a) = -\frac{\mu_o I}{2\pi} ln(r/a) \tag{2}$$

and evaluating the integration coefficient by using the boundary condition on A_z on the outer conductor, where $r = a$. The electric field follows from Gauss' integral law, (1.3.13),

$$E_r = \lambda_l/2\pi\epsilon r \tag{3}$$

and the potential follows by integrating.

$$E_r = -\frac{\partial \Phi}{\partial r} \Rightarrow \Phi(r) - \Phi(a) = \frac{\lambda_l}{2\pi\epsilon} ln\Big(\frac{a}{r}\Big) \tag{4}$$

Using the boundary condition at $r = a$ then gives the potential.

(b) The inductance per unit length follows from evaluation of (2) at the inner boundary.

$$L \equiv \frac{\Lambda}{I} = \frac{A_z(b) - A_z(a)}{I} = \frac{\mu_o}{2\pi} ln(a/b) \tag{5}$$

Similarly, the capacitance per unit length follows from evaluating (5) at the inner boundary.

$$C \equiv \frac{\lambda_l}{V} = \frac{2\pi\epsilon}{ln(a/b)} \tag{6}$$

14.2.2 The capacitance per unit length is as given in the solution to Prob. 4.7.5. The inductance per unit length follows by using (8.6.14), $L = 1/Cc^2$.

14.3 TRANSIENTS ON INFINITE TRANSMISSION LINES

14.3.1 (a) From the values of L and C given in Prob. 14.2.1, (14.3.12) gives

$$Z_o = \sqrt{\mu/\epsilon} ln(a/b)/2\pi$$

(b) From $\mu = \mu_o = 4\pi \times 10^{-7}$ and $\epsilon = 2.5\epsilon_o = (2.5)(8.8.5 \times 10^{-12}), Z_o = (37.9) ln(a/b)$. Because the only effect of geometry is through the ratio a/b and that is logarithmic, the range of characteristic impedances encoutered in practice for coaxial cables is relatively small, typically between 50 and 100 ohms. For example, for the four ratios of $a/b, Z_o = 26, 87, 175$ and 262 Ohms, respectively. To make $Z_o = 1000$ Ohms would require that $a/b = 2.9 \times 10^{11}$!

14.3.2 The characteristic impedance is given by (14.3.13). Presuming that we will find that $l/R \gg 1$, the expression is approximated by

$$Z_o = \sqrt{\mu/\epsilon}\, ln(2l/R)/\pi$$

and solved for l/R.

$$l/R = \frac{1}{2}\exp[\pi Z_o/\sqrt{\mu/\epsilon}] = \exp[\pi(300)/377]$$

Evaluation then gives $l/R = 6.1$.

14.3.3 The solution is analogous to that of Example 14.3.2 and shown in the figure.

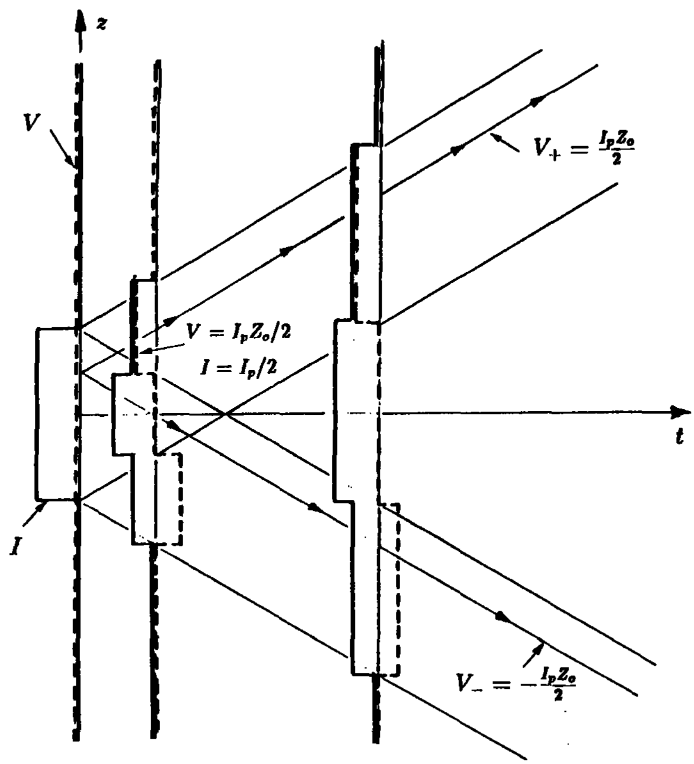


Figure S14.3.3

14.3.4 From (14.3.18) and (14.3.19), it follows that

$$V_\pm = V_o \exp(-z^2/2a^2)/2$$

Then, from (9) and (10)

$$V = \frac{1}{2}V_o\{\exp[-(z-ct)^2/2a^2] + \exp[-(z+ct)^2/2a^2]\}$$

$$I = \frac{1}{2}\frac{V_o}{Z_o}\{\exp[-(z-ct)^2/2a^2] - \exp[-(z+ct)^2/2a^2]\}$$

14.3.5 In general, the voltage and current can be represented by (14.3.9) and (14.3.10). From these it follows that

$$VI = (V_+ + V_-)\frac{1}{Z_o}(V_+ - V_-) = \frac{1}{Z_o}(V_+^2 - V_-^2)$$

14.3.6 By taking the $\partial()/\partial t$ and $\partial()/\partial z$ of the second equation in Prob. 14.1.2 and substituting it into the first, we obtain the partial differential equation that plays the role played by the wave equation for the conventional transmission line

$$LC\frac{\partial^4 V}{\partial t^2 \partial x^2} = V \tag{1}$$

Taking the required derivatives on the left amounts to combining (14.3.6). Thus, substitution of (14.3.3) into (1), gives

$$c^2 V_{\pm}'''' = V$$

By contrast with the wave-equation, this expression is not identically satisfied. Waves do not propagate on this line without dispersion.

14.4 TRANSIENTS ON BOUNDED TRANSMISSION LINES

14.4.1 When $t = 0$, the initial conditions on the line are

$$V = V_o; \qquad I = 0 \quad \text{for} \quad 0 < z < l$$

From (14.4.4) and (14.4.5), it follows that for those characteristics originating on the $t = 0$ axis of the figure

$$V_+ = V_o/2; \qquad V_- = V_o/2$$

For those lines originating at $z = l$, it follows from (14.4.8) with $R_L = \infty(\Gamma_L = 1)$ that

$$V_- = V_+$$

Similarly, for those lines originating at $z = 0$, it follows from (14.4.10) with $\Gamma_g = 0$ and $V_g = 0$ that

$$V_+ = 0$$

Combining these invarients in accordance with (14.1.1) and (14.1.2) at each location gives the (z, t) dependence of V and I shown in the figure.

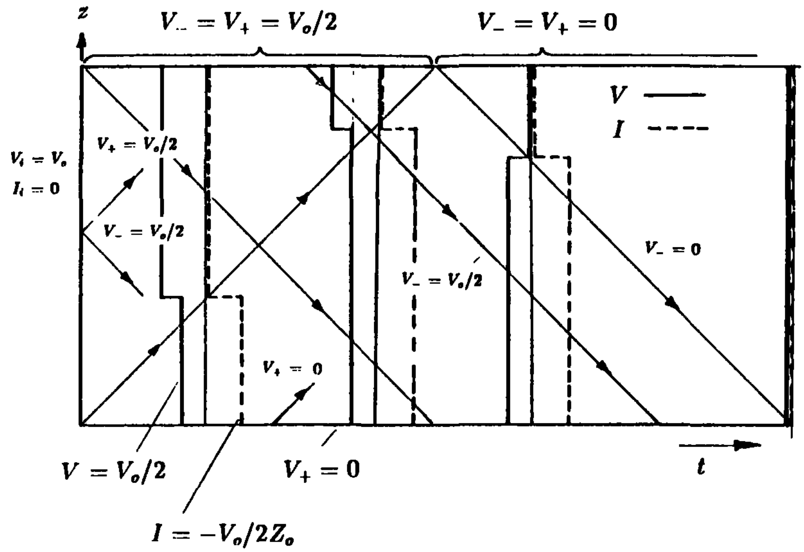


Figure S14.4.1

14.4.2 When $t = 0$, the initial conditions on the line are

$$V = 0; \qquad I = V_o/Z_o \quad \text{for} \quad 0 < z < l$$

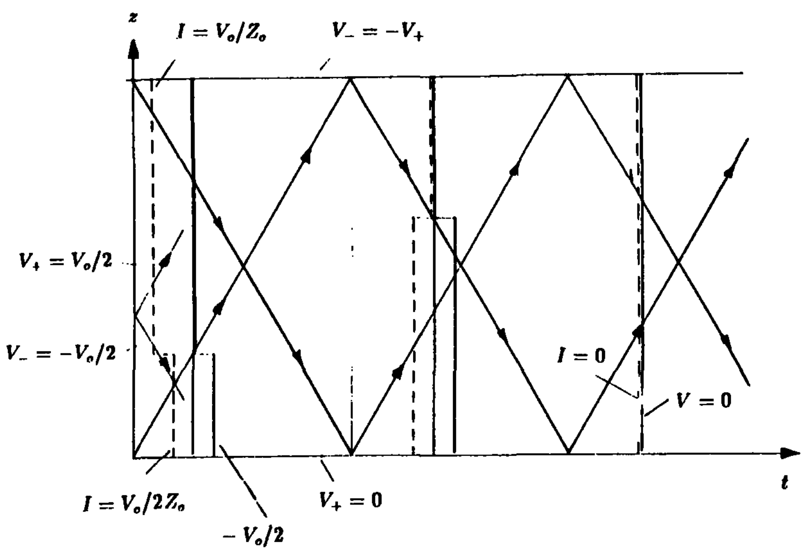


Figure S14.4.2

From (14.4.4) and (14.4.5), it follows that for those characteristics originating on the $t = 0$ axis of the figure

$$V_+ = V_o/2; \qquad V_- = -V_o/2$$

For those lines originating at $z = l$, it follows from (14.4.8) with $R_L = 0$ ($\Gamma_L = -1$) that

$$V_- = -V_+$$

Similarly, for those lines originating at $z = 0$, it follows from (14.4.10) that

$$V_+ = 0$$

Combining these invarients in accordance with (14.4.1) and (14.4.2) at each location gives the (z, t) dependence of V and I shown in the figure.

14.4.3 If the voltage and current on the line are initially zero, then it follows from (14.4.5) that $V_- = 0$ on those characteristic lines $x + ct =$ constant that originate on the $t = 0$ axis. Because $R_L = Z_o$, it follows from (14.4.8) that $V_- = 0$ for all of the other lines $x + ct =$ constant, which originate at $z = l$. Thus, at $z = 0$, (14.4.1) and (14.4.2) become

$$V = V_+; \qquad I = V_+/Z_o$$

and the ratio of these is the terminal relation $V/I = Z_o$, the relation for a resistance equal in value to the characteristic impedance. Implicit to this equivalence is the condition that the initial voltage and current on the line be zero.

14.4.4

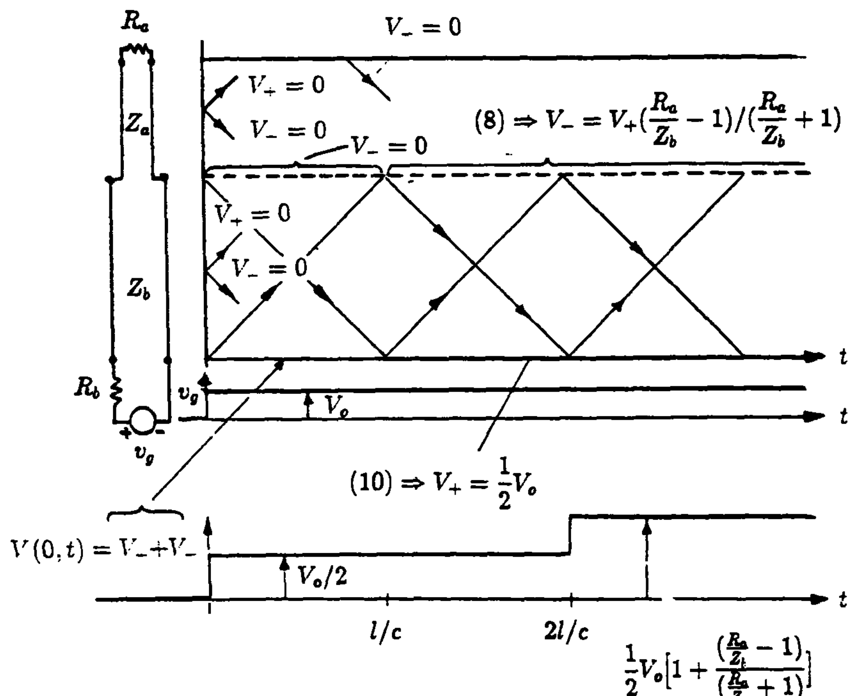


Figure S14.4.4

The solution is constructed in the $z-t$ plane as shown by the figure. Because the upper transmission line is both terminated in its characteristic impedance and free of initial conditions, it is equivalent to a resistance R_a connected to the terminals of the lower line (see Prob. 14.4.3). The values of V_+ and V_- follow from (14.4.4) and (14.4.5) for the characteristic lines originating when $t = 0$ and from (14.4.8) and (14.4.10) for those respectively originating at $z = l$ and $z = 0$.

14.4.5 When $t < 0$, a steady current flows around the loop and the initial voltage and current distribution are uniform over the length of the two line-segments.

$$V_i = \frac{R_a V_o}{R_a + R_b}; \qquad I_i = \frac{V_o}{R_a + R_b}$$

In the upper segment, shown in the figure, it follows from (14.4.5) that $V_- = 0$. Thus, for these particular initial conditions, the upper segment is equivalent to a termination on the lower segment equal to $Z_a = R_a$. In the lower segment, V_+ and V_- originating on the z axis follow from the initial conditions and (14.4.4) and (14.4.5) as being the values given on the $z-t$ diagram. The conditions relating the incident to the reflected waves, given respectively by (14.4.8) and (14.4.10), are also summarized in the diagram. Use of (14.4.1) to find $V(0,t)$ then gives the function of time shown at the bottom of the figure.

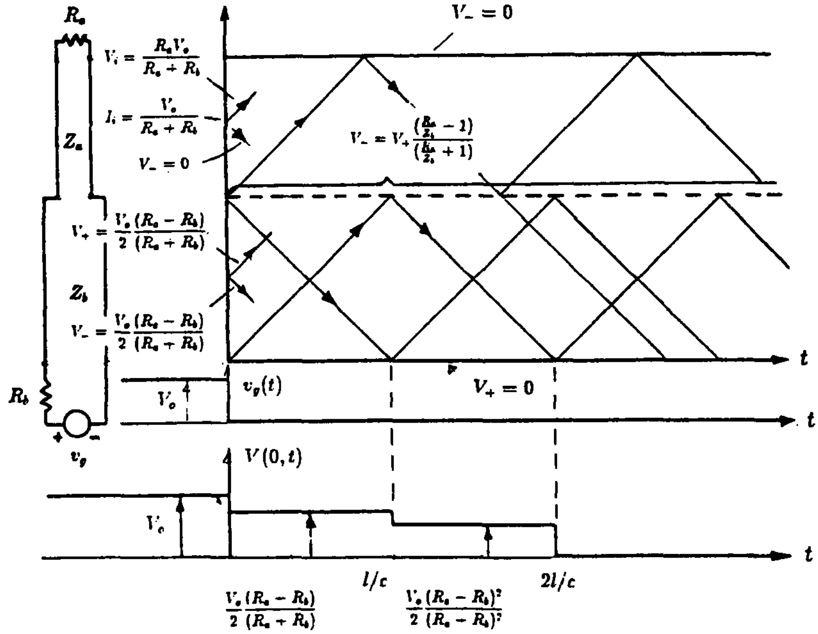


Figure S14.4.5

14.4.6 From (14.4.4) and (14.4.5), it follows from the initial conditions that V_+ and V_- are zero on lines originating on the $t = 0$ axis. The value of V_+ on lines coming from the $z = 0$ axis is determined by requiring that the currents at the input terminal sum to zero.

$$i_g = \frac{1}{R_g}(V_+ + V_-) + \frac{1}{Z_o}(V_+ - V_-)$$

or

$$V_+ = \frac{i_g}{(1/R_g + 1/Z_o)} + V_- \frac{(1/Z_o - 1/R_g)}{(1/Z_o + 1/R_g)}$$

It follows that for $0 < t < T, V_+ = I_o R_g/2$ while for $T < t, V_+ = 0$. At $z = l$, (14.4.8) shows that $V_+ = -V_-$. Thus, the solution is as summarised in the figure.

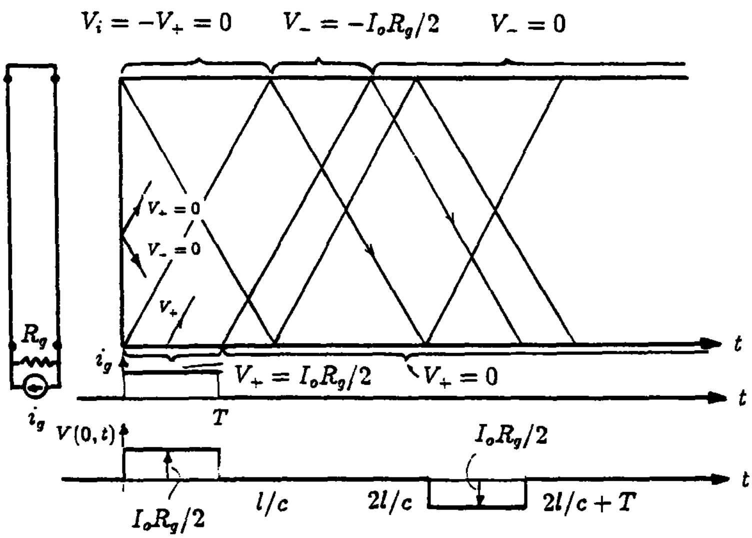


Figure S14.4.6

14.4.7 (a) By replacing $V_+/Z_o \rightarrow I_+, V_-/Z_o \rightarrow -I_-$, the general solutions given by (14.4.1) and (14.4.2) are written in terms of currents rather than voltages.

$$I = I_+ + I_-; \qquad V = \frac{1}{Y_o}(I_+ - I_-) \tag{1}$$

where $Y_o \equiv 1/Z_o$. When $t = 0$, the initial conditions are zero, so on characteristic lines originating on the $t = 0$ axis, I_+ and I_- are zero. At $z = l$, it follows from (1b) that $I_+ = I_-$. At $z = 0$, summation of currents at the terminal gives

$$i_g = (G_g/Y_o)(I_+ - I_-) + (I_+ + I_-) \tag{2}$$

which, solved for the reflected wave in terms of the incident wave gives

$$I_+ = I_g + I_- \Gamma_g \tag{3}$$

where

$$I_g \equiv \frac{I_o}{1+(G_g/Y_o)}; \qquad \Gamma_g \equiv [(G_g/Y_o)-1]/[(G_g/Y_o)+1] \tag{4}$$

From these relations, the wave components I_+ and I_- are constructed as summarized in the figure. The voltage at the terminals of the line is

$$V(0,t) = \frac{I_o}{G_g[1+(Y_o/G_g)]}\Gamma_g^{N-1}; \quad 2(N-1)\frac{l}{c} < t < 2N\frac{l}{c} \tag{5}$$

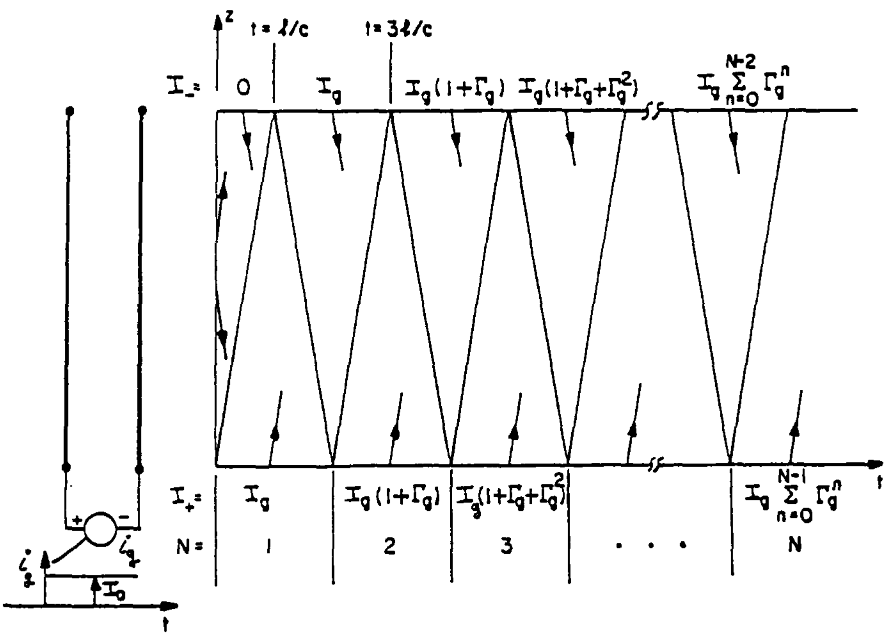


Figure S14.4.7

It follows that during this same interval, the terminal current is

$$I(0,t) = I_o\left(1 - \frac{\Gamma_g^{N-1}}{(1+Y_o/G_g)}\right) \tag{6}$$

(b) In terms of the terminal current I, the circuit equation for the line in the limit where it behaves as an inductor is

$$i_g = lLG_g\frac{dI}{dt} + I \tag{7}$$

Solution of this expression with $i_g = I_o$ and $I(0) = 0$ is

$$I(0,t) = I_o(1 - e^{-t/\tau}); \quad \tau \equiv lLG_g \tag{8}$$

(c) In the limit where G_g/Y_o is very large

$$\Gamma_g \rightarrow 1 - 2(Y_o/R_g) \tag{9}$$

Thus,

$$I(0,t) \rightarrow I_o\Big[1 - \big(1 - \frac{2Y_o}{G_g}\big)^{N-1}\Big]; \quad 2(N-1)\frac{l}{c} < t < \frac{2Nl}{c} \tag{10}$$

Following the same arguments as given by (14.4.28)–(14.4.31), gives

$$I(0,t) \rightarrow I_o[1 - e^{-(N-1)(2Y_o/G_g)}]; \quad 2(N-1)\frac{l}{c} < t < 2N\frac{l}{c} \tag{11}$$

which in the limit here, (14.4.31) holds the same as (8) where $(Y_o/G_g)(c/l) = \sqrt{C/L}/l\sqrt{LC}G_g = 1/lLG_g$. Thus, the current reponse (which has the same stair-step dependence on time as for the analogous example represented by Fig. 14.4.8) becomes the exponential response of the circuit in the limit where the inductor takes a long time to "charge" compared to the transit-time of an electromagnetic wave.

14.4.8

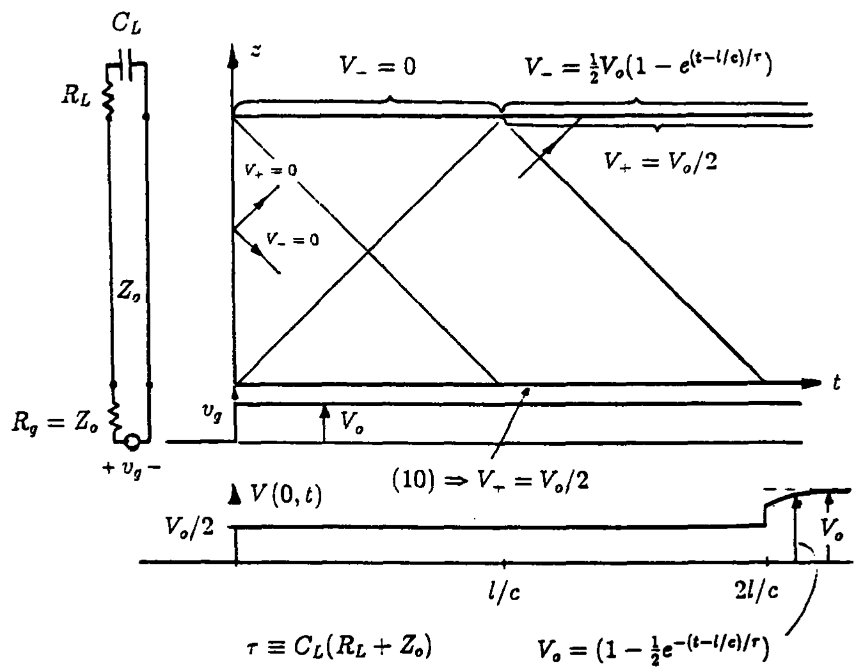


Figure S14.4.8

The initial conditions on the voltage and current are zero and it follows from (14.4.4) and (14.4.5) that V_+ and V_- on characteristics originating on the $t = 0$

axis are zero. It follows from (10) that on the lines originating on the $z = 0$ axis, $V_+ = V_o/2$. Then, for $0 < t < l/c$, the incident V_+ at $z = l$ is zero and hence from the differential equation representing the load resistor and capacitor, it follows that $V_- = 0$ during this time as well. For $l/c < t, V_+ = V_o/2$ at $z = l$. In view of the steady state established while $t < 0$, the initial capacitor voltage is zero. Thus, the initial value of $V_-(l, 0)$ is zero and the reflected wave is predicted by

$$C_L(R_L + Z_o)\frac{dV_-}{dt} + V_- = \frac{V_o}{2}u_{-1}(t - l/c)$$

The appropriate solution is

$$V_-(l, t) = \frac{1}{2}V_o(1 - e^{-(t-l/c)/\tau}); \quad \tau \equiv C_L(R_L + Z_o)$$

This establishes the wave incident at $z = 0$. The solution is summarized in the figure.

14.5 TRANSMISSION LINES IN THE SINUSOIDAL STEADY STATE

14.5.1 From (14.5.20), for the load capacitor where $Z_L = 1/j\omega C_L$,

$$\frac{Y(\beta l = -\pi/2)}{Y_o} = \frac{Y_o}{Y_L} = \frac{Y_o}{j\omega C_L}$$

Thus, the impedance is inductive.

For the load inductor where $Z_L = j\omega L_L$, (14.5.20) gives

$$\frac{Z(\beta l = -\pi/2)}{Z_o} = \frac{Z_o}{j\omega L_L}$$

and the impedance is capacitive.

14.5.2 For the open circuit, $Z_L = \infty$ and from (14.5.13), $\Gamma_L = 1$. The admittance at any other location is given by (14.5.10).

$$\frac{Y(-l)}{Y_o} = \frac{1 - \Gamma_L e^{-2j\beta l}}{1 + \Gamma_L e^{-2j\beta l}} = \frac{1 - e^{-2j\beta l}}{1 + e^{-2j\beta l}}$$

where characteristic admittance $Y_o = 1/Z_o$. This expression reduces to

$$\frac{Y(-l)}{Y_o} = j\tan\beta l$$

which is the same as the impedance for the shorted line, (14.5.17). Thus, with the vertical axis the admittance normalized to the characteristic admittance, the frequency or length dependence is as shown by Fig. 14.5.2.

14.5.3 The matched line requires that $\hat{V}_- = 0$. Thus, from (14.5.5) and (14.5.6),

$$\hat{V} = \hat{V}_+ \exp(-j\beta z); \qquad \hat{I} = \frac{\hat{V}_+}{Z_o} \exp(-j\beta z)$$

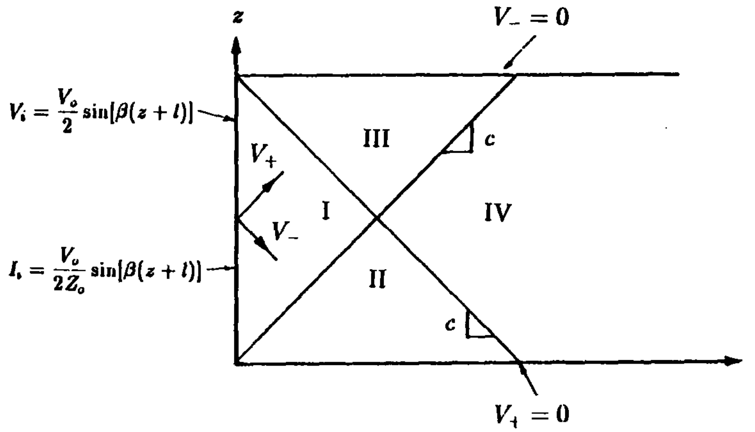


Figure S14.5.3

At $z = -l$, the circuitis described by

$$\hat{V}_g = \hat{I}(-l)R_g + \hat{V}(-l)$$

where, in complex notation, $V_g = \operatorname{Re}\hat{V}_g \exp(j\omega t)$, $\hat{V}_g \equiv -jV_o$. Thus, for $R_g = Z_o$,

$$\hat{V}_g = \frac{R_g}{Z_o}\hat{V}_+ e^{j\beta l} + \hat{V}_+ e^{j\beta l} \Rightarrow \hat{V}_+ = \frac{1}{2}\hat{V}_g e^{-j\beta l}$$

and the given sinusoidal steady state solutions follow.

$$V = \operatorname{Re}\frac{-jV_o}{2}e^{-j\beta(z+l)}e^{j\omega t}; \quad I = \operatorname{Re}\frac{-jV_o}{2Z_o}e^{-j\beta(z+l)}e^{j\omega t}$$

14.5.4 Initially, both the current and voltage are sero. With the solution written as the sum of the sinusoidal steady state solution found in Prob. 14.5.3 and a transient solution,

$$V = V_s(z,t) + V_t(z,t); \qquad I = I_s(z,t) + I_t(z,t)$$

the initial conditions on the transient part are therefore,

$$V_t(z,0) = -V_s(z,0) = \frac{V_o}{2}\sin[\beta(z+l)]$$

$$I_t(z,0) = -I_s(z,0) = \frac{V_o}{2Z_o}\sin[\beta(z+l)]$$

The boundary conditions for $0 < t$ and with the given driving source are satisfied by V_s. Thus, V_t must satisfy the boundary conditions that result if $V_g = 0$. In terms of a transient solution written as 14.3.9 and 14.3.10, these are that $V_- = 0$ at $z = 0$ and [from (14.5.10) with $V_g = 0$ and $R_g = Z_o$] that $V_+ = 0$ at $z = -l$. Thus, the initial and boundary conditions for the transient part of the solution are as summarized in the figure. With the regions in the $x-t$ plane denoted as shown in the figure, the voltage and current are therefore,

$$V = V_s + V_t; \qquad I = I_s + I_t$$

where V_s and I_s are as given in Prob. 14.5.3 and

$$V_t = V_+ + V_-; \qquad I_t = \frac{1}{Z_o}(V_+ - V_-)$$

with (from 14.3.18-19)

$$V_+ = \frac{V_o}{2}\sin[\beta(z+l)]; \qquad V_- = 0$$

in regions I and III, and

$$V_+ = 0; \qquad V_- = 0$$

in regions II and IV.

14.6 REFLECTION COEFFICIENT REPRESENTATION OF TRANSMISSION LINES

14.6.1 The Smith chart solution is like the case of the Quarter-Wave Section exemplified using Fig. 14.6.3. The load is at $r = 2, x = 2$ on the chart. The impedance a quarater-wave toward the generator amounts to a constant radius clockwise rotation of 180° to the point where $r = 0.25$ and $x = -j0.25$. Evaluation of (14.6.20) checks this result, because it shows that

$$r + jx\Big|_{z=-l} = \frac{1}{r_L + jx_L} = \frac{1}{2+j2} = \frac{2-j2}{8}$$

14.6.2 From (14.6.3), $\Gamma = 0.538 + j0.308$ and $|\Gamma| = 0.620$. It follows from (14.6.10) that the VSWR is 4.26. These values also follow from drawing a circle through $r + jx = 2 + j2$, using the radius of the circle to obtain $|\Gamma|$ and the construction of Fig. 14.6.4a to evaluate (14.6.10).

14.6.3 The angular distance on the Smith charge from the point $y = 2 + j0$ to the circle where y has a real part of 1 is $l = 0.0975\lambda$. To cancel the reactance, where $y = 1 + j0.7$ at this point, the distance from the shorted end of the stub to the point where it is attached to the line must be $l_s = 0.347\lambda$.

14.6.4 Adjustment of the length of the first stub makes it possible to be anywhere on the circle $g = 2$ of the admittance chart at the terminals of the parallel stub and load. If this admittance can be transferred onto the circle $g = 1$ by moving a distance l toward the generator (clockwise), the second stub can be used to match the line by compensating for the reactive part of the impedance. Thus, determination of the stub lengths amounts to finding a pair of points on these circles that are at the same radius and separated by the angle 0.042λ. This then gives both the combined stub (1) and load impedance (for the case given, $y = 2 + j1.3$) and combined stub (2) and line impedance at $z = -l$ (for the case given, $y = 1 + j1.16$). To create the needed susceptance at the load, $l_1 = 0.04\lambda$. To cancel the resulting susceptance at the second stub, $l_2 = 0.38\lambda$.

14.6.5 The impedance at the left end of the quarter wave section is 0.5. Thus, normalized to the impedance of the line to the left, the impedance there is $Z/Z_o^a = 0.25$. It follows from the Smith chart and (14.6.10) that the VSWR = 4.0.

14.7 DISTRIBUTED PARAMETER EQUIVALENTS AND MODELS WITH DISSIPATION

14.7.1 The currents must sum to zero at the node. With those through the conductance and capacitance on the right,

$$I(z) - I(z + \Delta z) = G\Delta z V + C\Delta z \frac{\partial V}{\partial t}$$

The voltage drop around a loop comprised of the terminals and the series resistance and inductance must sum of zero. With the voltage drops across the resistor and inductor on the right,

$$V(z) - V(z + \Delta z) = R\Delta z I + L\Delta z \frac{\partial I}{\partial t}$$

In the limit where $\Delta z \to 0$, these expressions become the transmission line equations, (14.7.1) and (14.7.2).

14.7.2 (a) If the voltage is given by (14.7.12), as a special case of (14.7.9), then it follows that $I(z, t)$ is the special case of (14.7.10)

$$I = \mathrm{Re}\, \frac{\hat{V}_g}{Z_o} \frac{\left(e^{-j\beta z} - e^{j\beta z}\right)}{\left(e^{j\beta l} + e^{-j\beta l}\right)} e^{j\omega t}$$

(b) The desired impedance is the ratio of the voltage, (14.7.12), to this current, evaluated at $z = -l$.

$$Z = Z_o \frac{(e^{j\beta l} + e^{-j\beta l})}{(e^{j\beta l} - e^{-j\beta l})}$$

(c) In the long-wave limit, $|\beta l| \ll 1$, $\exp(j\beta l) \to 1 + j\beta l$ and this expression becomes

$$Z = \frac{Z_o}{j\beta l} = \frac{(R + j\omega L)}{-\beta^2 l} = \frac{1}{[G + j\omega C]l}$$

where (14.7.8) and (14.7.11) have been used to write the latter equality. (Note that (14.7.8) is best left in the form suggested by (14.7.7) to obtain this result.) The circuit having this impedance is a conductance lG shunted by a capacitance lC.

14.7.3 The short requires that $V(0,t) = 0$ gives $V_+ = V_-$. With the magnitude adjusted to match the condition that $V(-l,t) = V_g(t)$, (14.7.9) and (14.7.10) become

$$\hat{V} = \hat{V}_g \frac{(e^{-j\beta z} - e^{j\beta z})}{(e^{j\beta l} - e^{-j\beta l})}; \quad I = \mathrm{Re}\, \frac{\hat{V}_g}{Z_o} \frac{(e^{-j\beta z} + e^{j\beta z})}{(e^{j\beta l} - e^{-j\beta l})}$$

Thus, the impedance at $z = -l$ is

$$Z = Z_o(e^{j\beta l} - e^{-j\beta l})/(e^{j\beta l} + e^{-j\beta l})$$

In the limit where $|\beta l| \ll 1$, it follows from this expression and (14.7.8) and (14.7.11) that because $\exp j\beta l \to 1 + j\beta l$

$$Z \to Z_o j\beta l = l(R + j\omega L)$$

which is the impedance of a resistance lR in series with an inductor lL.

14.7.4 (a) The theorem is obtained by adding the negative of V times (1) to the negative of I times (2).

(b) The identity follows from

$$\begin{aligned}
\mathrm{Re}\, \hat{A}e^{j\omega t}\mathrm{Re}\, \hat{B}e^{j\omega t} &= \frac{1}{2}[\hat{A}e^{j\omega t} + \hat{A}^*e^{-j\omega t}]\frac{1}{2}[\hat{B}e^{j\omega t} + \hat{B}^*e^{-j\omega t}] \\
&= \frac{1}{4}[\hat{A}\hat{B}e^{2j\omega t} + \hat{A}^*\hat{B}^*e^{-2j\omega t}] + \frac{1}{4}[\hat{A}\hat{B}^* + \hat{A}^*\hat{B}] \\
&= \mathrm{Re}\, \frac{1}{2}\hat{A}\hat{B}e^{2j\omega t} + \mathrm{Re}\, \frac{1}{2}\hat{A}\hat{B}^*
\end{aligned}$$

(c) Each of the quadratic terms in the power theorem take the form of (1), a time independent part and a part that varies sinusoidally at twice the driving frequency. The periodic part time-averages to zero in the power flux term on the left and in the dissipation terms (the last two terms) on the right. The only contribution to the energy storage term is due to the second harmonic, and that time-averages to zero. Thus, on the time-average there is no contribution from the energy storage terms.

The integral theorem, (d) follows from the integration of (c) over the length of the system. Integration of the derivative on the left results in the integrand evaluated at the end points. Because the current is zero where $z = 0$, the only contribution is the time-average input power on the left in (d).

(d) The left hand side is evaluated using (14.7.12) and (14.7.6). First, using (14.7.11), (14.7.6) becomes

$$\hat{I} = -Y_o \hat{V}_g \tan \beta z \tag{2}$$

Thus,

$$\frac{1}{2}\mathrm{Re}\,\hat{V}\hat{I}^*\big|_{z=-l} = \frac{1}{2}\mathrm{Re}\,\hat{V}_g^*\hat{I}\big|_{z=-l} = \frac{1}{2}\mathrm{Re}\,|\hat{V}_g|^2 Y_o \tan\beta l \tag{3}$$

That the right hand side must give the same thing follows from using (14.7.3) and (14.7.4) to write

$$G\hat{V}^* = j\omega C\hat{V}^* - \frac{d\hat{I}^*}{dz} \tag{4}$$

$$R\hat{I} = -\frac{d\hat{V}}{dz} - j\omega L\hat{I} \tag{5}$$

Thus,

$$\begin{aligned}
\int_{-l}^{0}\frac{1}{2}\mathrm{Re}\,[\hat{I}^* R I + \hat{V}\hat{V}^* G]dz &= \int_{-l}^{0}\frac{1}{2}\mathrm{Re}\,\Big[-\hat{I}\frac{d\hat{V}^*}{dz} - j\omega L\hat{I}\hat{I}^* \\
&\quad + j\omega C\hat{V}\hat{V}^* - \hat{V}\frac{d\hat{I}^*}{dz}\Big]dz \\
&= -\int_{-l}^{0}\frac{1}{2}\mathrm{Re}\,\Big[\hat{I}\frac{d\hat{V}^*}{dz} + \hat{V}\frac{d\hat{I}^*}{dz}\Big]dz \\
&= -\frac{1}{2}\mathrm{Re}\int_{-l}^{0}\frac{d(\hat{I}\hat{V}^*)}{dz}dz \\
&= \frac{1}{2}\mathrm{Re}\,\hat{I}\hat{V}^*\Big|_{z=-l}
\end{aligned} \tag{6}$$

which is the same as (3).

14.8 UNIFORM AND TEM WAVES IN OHMIC CONDUCTORS

14.8.1 In Ampère's law, represented by (12.1.4), $\mathbf{J}_u = \sigma\mathbf{E}$. Hence, (12.1.6) becomes

$$\nabla(\nabla \cdot \mathbf{A} + \mu\sigma\Phi + \mu\epsilon\frac{\partial \Phi}{\partial t}) - \nabla^2\mathbf{A} = -\mu\sigma\frac{\partial \mathbf{A}}{\partial t} - \mu\epsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{1}$$

Hence, the gauge condition, (14.8.3), becomes

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = -\mu\sigma\Phi - \mu\epsilon\frac{\partial \Phi}{\partial t} \tag{2}$$

Evaluation of this expression on the conductor surface with (14.8.9) and (14.8.11) gives

$$L\frac{\partial I}{\partial z} = -\mu\sigma V - LC\frac{\partial V}{\partial t} \tag{3}$$

From (8.6.14) and (7.6.4)

$$\frac{\mu\sigma}{L} = \frac{\mu\sigma C}{\mu\epsilon} = \frac{G}{C}C = G \tag{4}$$

Thus,

$$\frac{\partial I}{\partial z} = -GV - C\frac{\partial V}{\partial t} \tag{5}$$

This and (14.8.12) are the desired transmission line equations including the losses represented by the shunt conductance G. Note that, provided the conductors are "perfect", the TEM wave represented by these equations is exact and not quasi-one-dimensional.

14.8.2 The transverse dependence of the electric and magnetic fields are respectively the same as for the two-dimensional EQS capacitor-resistor and MQS inductor. The axial dependence of the fields is as given by (14.8.10) and (14.8.11). Thus, with (Prob. 14.2.1)

$$C = 2\pi\epsilon/ln(a/b); \quad L = \frac{\mu_o}{2\pi}ln(a/b); \quad G = \frac{\sigma}{\epsilon}C = 2\pi\sigma/ln(a/b)$$

and hence β and Z_o given by (14.7.8) and (14.7.11) with $R = 0$, the desired fields are

$$\mathbf{E} = \mathrm{Re}\,\frac{\hat{V}_g(e^{-j\beta z} + e^{j\beta z})}{r\,ln(a/b)(e^{j\beta l} + e^{-j\beta l})}e^{j\omega t}\mathbf{i}_r$$

$$\mathbf{H} = \mathrm{Re}\,\frac{\hat{V}_g(e^{-j\beta z} - e^{j\beta z})}{2\pi r Z_o(e^{j\beta l} + e^{-j\beta l})}e^{j\omega t}\mathbf{i}_\phi$$

14.8.3 The transverse dependence of the potential follows from (4.6.18)–(4.6.19), (4.6.25) and (4.6.27). Thus, with the axial dependence given by (14.8.10),

$$\mathbf{E} = -\frac{\partial\Phi}{\partial x}\mathbf{i_x} - \frac{\partial\Phi}{\partial y}\mathbf{i_y}$$

where

$$\Phi = -\mathrm{Re}\,\frac{\hat{V}_g}{2}\,\frac{ln\left[\frac{\sqrt{(\sqrt{l^2-R^2}-x)^2+y^2}}{\sqrt{(\sqrt{l^2-R^2}+x)^2+y^2}}\right]}{ln\left[\frac{l}{R}+\sqrt{(l/R)^2-1}\right]}\,\frac{(e^{-j\beta z}+e^{j\beta z})}{(e^{j\beta l}+e^{j\beta l})}e^{j\omega t}$$

Using (14.2.2) A_z follows from this potential.

$$\mathbf{H} = \frac{1}{\mu_o}\left(\frac{\partial A_z}{\partial y}\mathbf{i_x} - \frac{\partial A_z}{\partial x}\mathbf{i_y}\right)$$

where

$$A_z = -\mathrm{Re}\,\frac{\mu_o\hat{V}_g}{2\pi Z_o}ln\frac{\sqrt{(\sqrt{l^2-R^2}-x)^2+y^2}}{\sqrt{(\sqrt{l^2-R^2}+x)^2+y^2}}\,\frac{(e^{-j\beta z}-e^{j\beta z})}{(e^{j\beta l}+e^{-j\beta l})}e^{j\omega t}$$

In these expressions, β and Z_o are evaluated from (14.7.8) and (14.7.11) using the values of C and L given by (4.6.27) and (4.6.12) with $R = 0$ and $G = (\sigma/\epsilon)C$.

14.8.4 (a) The integral of $\mathbf{E}$ around the given contour is equal to the negative rate of change of the magnetic flux linked. Thus,

$$aE^a(z+\Delta z) - aE^a(z) + bE^b(z+\Delta z) - bE^b(z) = -(a+b)\Delta z\frac{\partial}{\partial t}(\mu_o H_y) \quad (1)$$

and in the limit $\Delta z \to 0$,

$$a\frac{\partial E^a}{\partial z} + b\frac{\partial E^b}{\partial z} = -\mu_o(a+b)\frac{\partial H_y}{\partial t} \quad (2)$$

Because $\epsilon_a E^a = \epsilon_b E^b$, this expression becomes

$$\left(a + \frac{\epsilon_a}{\epsilon_b}b\right)\frac{\partial E^a}{\partial z} = -\mu_o(a+b)\frac{\partial H_y}{\partial t} \quad (3)$$

If E^a and H_y were to be respectively written in terms of V and I, this would be the transmission line equation representing the law of induction (see Prob. 14.1.1).

(b) A similar derivation using the contour closing at the interface gives

$$aE^a(z+\Delta z) - aE^a(z) + \Delta z E_z = -a\mu_o\Delta z\frac{\partial H_y}{\partial t} \quad (4)$$

and in the limit $\Delta z \to 0$,

$$E_z = -a\mu_o \frac{\partial H_y}{\partial t} - a\frac{\partial E^a}{\partial z} \tag{5}$$

With the use of (3), this expression becomes

$$E_z = -\left[a\mu_o\Big(\frac{\epsilon_a}{\epsilon_b} - 1\Big)b/\Big(a + \frac{\epsilon_a}{\epsilon_b}b\Big)\right]\frac{\partial H_y}{\partial t} \tag{6}$$

Finally, for a wave having a z dependence $\exp(-j\beta z)$, the desired ratio follows from (6) and (3).

$$\frac{|E_z|}{|E_a|} = \frac{b(\beta a)}{a+b}\Big|1 - \frac{\epsilon_a}{\epsilon_b}\Big| \tag{7}$$

Thus, the approximation is good provided the wavelength is large compared to a and b and is exact in the limit where the dielectric is uniform.

14.9 QUASI-ONE-DIMENSIONAL MODELS

14.9.1 From (14.9.11)

$$R = \frac{2}{\pi R^2 \sigma}$$

while, from (4.7.2) and (8.6.12) respectively

$$C = \frac{2\pi\epsilon}{ln\big[(l/a) + \sqrt{(l/a) - 1}\big]}; \qquad L = \frac{\mu}{\pi}ln\big[(l/a) + \sqrt{(l/a)^2 - 1}\big]$$

To make the skin depth small compared to the wire radius

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \gg R \Rightarrow \omega \ll 2/a^2\mu\sigma$$

For the frequency to be high enough that the inductive reactance dominates

$$\frac{\omega L}{R} = \frac{\omega\mu\sigma a^2}{2} ln\big[(l/a) + \sqrt{(l/a)^2 - 1}\big]$$

Thus, the frequency range over which the inductive reactance dominates but the constant resistance model is still appropriate is

$$\frac{2}{\mu\sigma R^2 ln[(l/a) + \sqrt{(l/a)^2 - 1}]} < \omega < \frac{2}{a^2\mu\sigma}$$

For this range to exist, the conductor spacing must be large enough compared to their radii that

$$1 \ll ln\Big[\Big(\frac{l}{a}\Big) + \sqrt{(l/a)^2 - 1}\Big]$$

Because of the logarithmic dependence, the quantity on the right is not likely to be very large.

14.9.2 From (14.9.11),

$$R = \frac{1}{\sigma 2\pi a\Delta} + \frac{1}{\sigma\pi b^2} = \frac{1}{\pi\sigma}\left[\frac{1}{a\Delta} + \frac{1}{b^2}\right]$$

while, from Prob. 14.2.1

$$L = \frac{\mu_o}{2\pi} ln(a/b); \qquad C = \frac{2\pi\epsilon}{ln(a/b)}$$

For the skin depth to be large compared to the transverse dimensions of the conductors

$$\delta \equiv \sqrt{\frac{2}{\omega\mu\sigma}} \gg \Delta \quad \text{or} \quad b \Rightarrow \omega \ll 2/b^2\mu\sigma \quad \text{and} \quad 2/\Delta^2\mu\sigma$$

This puts an upper limit on the frequency for which the model is valid. To be useful, the model should be valid at sufficiently high frequencies that the inductive reactance can dominate the resistance. Thus, it should extend to

$$\frac{\omega L}{R} = \frac{\omega\mu_o\sigma}{2} ln(a/b)/\left[\frac{1}{a\Delta} + \frac{1}{b^2}\right] > 1$$

For the frequency range to include this value but not exceed the skin depth limit,

$$\frac{2\left[\frac{1}{a\Delta} + \frac{1}{b^2}\right]}{\mu\sigma ln(a/b)} \ll \omega \ll \frac{2}{b^2\mu\sigma} \quad \text{and} \quad \frac{2}{\Delta^2\mu\sigma}$$

which is possible only if

$$1 \ll \frac{ln(a/b)}{\left[\frac{1}{a\Delta} + \frac{1}{b^2}\right](b^2 \text{and} \Delta^2)}$$

Because of the logarithms dependence of L, this is not a very large range.

14.9.3 Comparison of (14.9.18) and (10.6.1) shows the mathematical analogy between the charge diffusion line and one-dimensional magnetic diffusion. The analogous electric and magnetic variables and parameters are

$$H_z \leftrightarrow V, \quad K_p \leftrightarrow V_p, \quad \mu\sigma \leftrightarrow RC, \quad b \leftrightarrow l, \quad x \leftrightarrow z$$

Because the boundary condition on V at $z = 0$ is the same as that on H_z at $x = 0$, the solution is found by following the steps of Example 10.6.1. From 10.6.21, it follows that the desired distribution of V is

$$V = -V_p\frac{z}{l} - \sum_{n=1}^{\infty} 2V_p\frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi z}{l}\right)e^{-t/\tau_n}; \quad \tau_n \equiv \frac{RCl^2}{(n\pi)^2}$$

This transient response is represented by Fig. 10.6.3a where $H_z/K_p \rightarrow V/V_p$ and $x/b \rightarrow z/l$.

14.9.4 See solution to Prob. 10.6.2 using analogy described in solution to Prob. 14.9.3.

SOLUTIONS TO CHAPTER 15

15.1 SOURCE AND MATERIAL CONFIGURATIONS

15.1.1

TABLE P15.1.1. Modal Field Representation

	Physical Constraints	Example/Prob.
Cartesian		
Laplace's Eq.	EQS Potential	Sec. 5.5, Demo. 5.5.1 Probs. 5.5.1-7
	EQS 3-Dimensional	Examp. 5.10.1 Probs. 5.10.1,3
	Polarization	Examp. 6.6.3, 6.7.1 Prob. 6.3.10, 6.6.9 Prob. 6.7.1
	Conduction	Examp. 7.4.1
	Charge Relax.	Prob. 7.9.12
	MQS, Equi-A	Examp. 8.6.3 Demo. 8.6.2 Prob. 8.6.10
	Magnetization	Prob. 9.6.9
	MQS E	Prob. 10.1.2
	MQS Eddy Current	Prob. 10.1.5
Poisson's Eq.	EQS Potential	Probs. 5.6.7-9, 13
	MQS Equa-A	Prob. 8.6.7
Polar		
Laplace's Eq.	EQS Potential	Examp. 5.8.2-3 Probs. 5.8.3-9
	Conduction	Prob. 7.4.4, 7.5.6
	MQS, Constrained Current	Prob. 8.5.2
	MQS, Equi-A	Prob. 8.6.5
	MQS E	Examp. 10.12 Prob. 10.1.3
Initial Value	Diffusion Eq.	Examp. 10.6 Prob. 10.6.1-2
Helmholtz Eq.	TM Modes	Examp. 13.3.1 Prob. 13.3.1-6
	3-Dimensional	Probs. 13.4.3-4 Demo. 13.3.1
	TE Modes	Examp. 13.3.2 Demo. 13.3.2

15.2 MACROSCOPIC MEDIA

✓

15.2.1 In each case, the excitation is an imposed uniform field at infinity. For (a), the field is tangential to the spherical surface everywhere except at the singular points at the poles. Thus, i) the system could be EQS with the regions insulating dielectrics and $\epsilon_a \gg \epsilon_b$, ii) the system could be a stationary conductor with the field lines either **J** or **E** and $\sigma_a \gg \sigma_b$, iii) it could be MQS with the lines **B** or **H**, the materials insulating and $\mu_a \gg \mu_b$ and iv) it could be a perfectly conducting sphere in an insulating media with the lines either **B** or **H** changing in time rapidly enough to induce the currents in the sphere required to exclude the field.

For (b), the field is perpendicular to the surface. Thus, i) it could be EQS and a perfect conductor in an insulating medium with the lines representing **E**, ii) it could be EQS **E** with the materials perfect insulators (the field changing rapidly compared to the charge relaxation time in either material) with $\epsilon_b \gg \epsilon_a$, iii) it could be **J** or **E** in stationary conduction with $\sigma_b \gg \sigma_a$, iv) and it could be MQS **H** or **B** with the materials insulating and $\mu_b \gg \mu_a$.

✓

15.2.2 The excitation is inside the sphere. In (a), the field in that region is perpendicular to the interface. Thus, i) the lines could be EQS **E** with the inside an insulator and the outside a perfect conductor, ii) the system could again be EQS and the lines could be **E** with both materials perfect insulators and $\epsilon_a \gg \epsilon_b$, iii) it could be stationary conduction with the lines either **E** or **J** and a dipole current source with $\sigma_a \gg \sigma_b$ and iv) the lines could be MQS **H** or **B** with a magnetic dipole and the regions magnetizable insulators with $\mu_a \gg \mu_b$.

In (b), the interior field lines are tangential to the surface. Thus, i) the dipole could be electric and the materials perfect insulators having $\epsilon_b \gg \epsilon_a$, ii) the dipole could be a current source for stationary conduction with the lines **E** or **J** and $\sigma_b \gg \sigma_a$, iii) the system could be MQS with the dipole magnetic and the materials magnetizable insulators having $\mu_b \gg \mu_a$, and iv) the system could be MQS with a magnetic dipole varying rapidly enough with time to make the outer material a perfect conductor while the interior one remains a perfect insulator.

15.3 CHARACTERIC TIMES, PHYSICAL PROCESSES, AND APPROXIMATIONS

15.3.1 Because it does not involve σ, ω is normalized to τ_{em}. Thus, the horizontal axis is

$$\log(\omega\tau_{em}) = \log(\omega l \sqrt{\mu\epsilon})$$

Then

$$\omega\tau_e = \frac{\omega\epsilon}{\sigma} = \omega\tau_{em}\frac{\epsilon}{\sigma l\sqrt{\mu\epsilon}} = 1 \Rightarrow \omega\tau_{em} = \frac{\sigma}{(\sqrt{\epsilon/\mu}/l)}$$

Thus, with the *characteristic conductivity* defined as

$$\sigma^* \equiv \sqrt{\epsilon/\mu}/l$$

the critical line indicating charge relaxation, $\omega\tau_e = 1$, is written in terms of the independent variables of normalized frequency and conductivity as

$$\log\omega\tau_{em} = \log\left(\frac{\sigma}{\sigma^*}\right)$$

Similarly,

$$\omega\tau_m = 1 \Rightarrow \omega\tau_{em} = \frac{\tau_{em}}{\tau_m} = \frac{l\sqrt{\mu\epsilon}}{\mu\sigma l^2} = \left(\frac{\sigma}{\sigma^*}\right)^{-1} \Rightarrow \log\omega\tau_{em} = -\log\left(\frac{\sigma}{\sigma^*}\right)$$

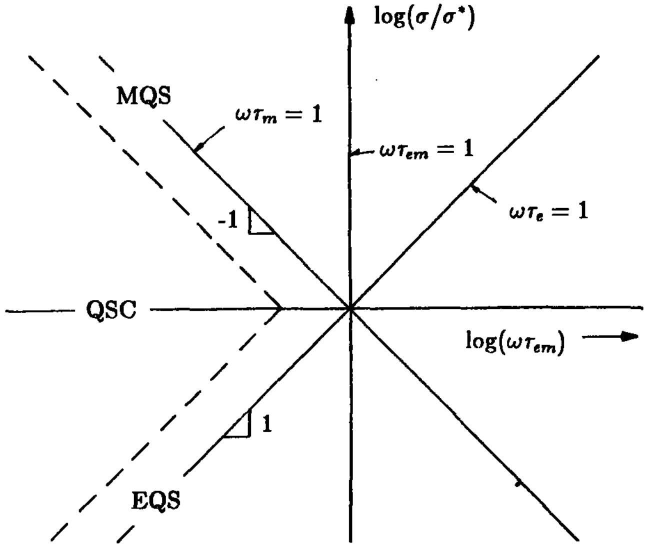


Figure S15.3.1

Thus, the plot is as shown in Fig. S15.3.1. If $\sigma > \sigma^*$, raising the frequency results in a transition from stationary conduction to the MQS regime while if $\sigma < \sigma^*$, the transition is to the EQS regime.

15.3.2 (a) In the limit of zero frequency, the electric and magnetic fields are as summarized by (7.5.7) and (7.5.11) and by (11.3.10) and (11.2.12). With (a) and (b) respectively designating the nonconducting annulus and the rod,

$$\mathbf{E}^b = \frac{v}{L}\mathbf{i}_z \tag{1}$$

$$\mathbf{E}^a = -\frac{v}{ln(a/b)}\left[\frac{z}{rL}\mathbf{i}_r + \frac{ln(r/a)}{L}\mathbf{i}_z\right] \tag{2}$$

$$\mathbf{H}^b = \frac{\sigma v}{L}\frac{r}{2}\mathbf{i}_\phi \tag{3}$$

$$\mathbf{H}^a = \frac{\sigma v}{L}\frac{b^2}{2r}\mathbf{i}_\phi \tag{4}$$

The magnetic field is induced by the uniform current density

$$\mathbf{J} = \frac{\sigma v}{L}\mathbf{i}_z; \qquad 0 < r < b \tag{5}$$

which is returned as the surface current density $\mathbf{K} = -[\sigma v b^2/2La]$ in the perfectly conducting wall. There is no volume charge density in the interior of the rod. On its surface and on the inner surface of the outer wall, the surface charge densities are

$$\sigma_s(r = a) = \frac{\epsilon_o v}{ln(a/b)}\frac{z}{aL}; \qquad \sigma_s(r = b) = -\frac{\epsilon_o v}{ln(a/b)}\frac{z}{Lb} \tag{6}$$

These fields and sources are sketched in Fig. S15.3.2a.

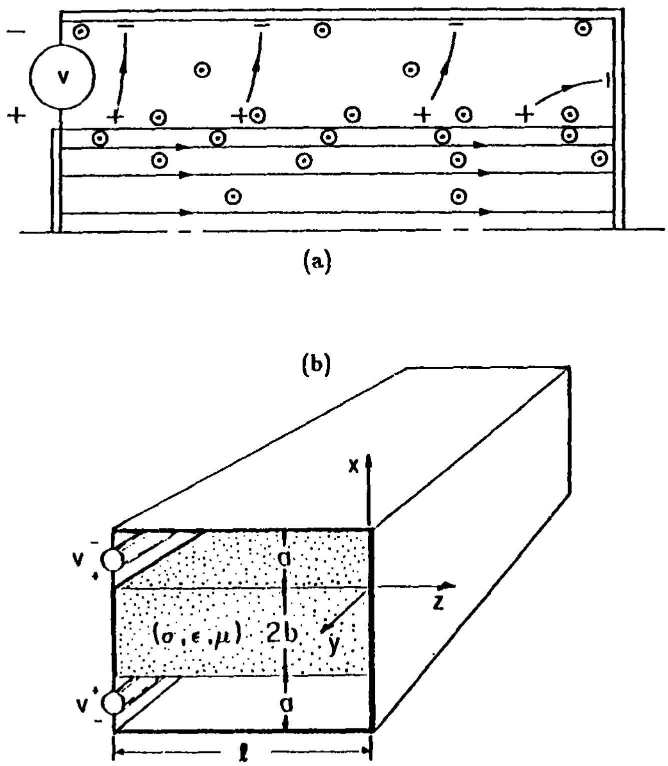


Figure S15.3.2a,b

(b) With all dimensions on the same order, the argument is as given in this section. Any one of the dimensions, a, b or L is the typical dimension. The ratio of that dimension to either of the other two is presumed to be perhaps 2 or 3.

The permittivity and permeability can similarly be taken as that of either region with the respective ratios of these quantities again presumed to be less than an order of magnitude. Thus, the system is first EQS as the frequency is raised if the characteristic dimension, a, b or L, is small compared to l^*, where the latter is based on the conductivity of the rod and the permittivity and permeability of either region. In the case where the charge relaxation time is the longest of the characteristic times, the EQS case, the magnetic induction is not important as the frequency is raised to the point where the sources begin to alter their distribution. In this case, the dominant source is the charge density, specifically the surface charge density. With each half-cycle, the surface charge density on the surface of the rod undergoes a sign reversal. To change this charge, the current density of (5) must be revised so that there is a component normal to the interface. In the "distributed circuit" picture of Fig. P15.3.2a, this is the current required to charge the capacitors. (In the next problem, the energy stored in the capacitors is used as a means of establishing the equivalent capacitance needed to account for the charging of the surface.)

In the case where the characteristic length is large compared to l^*, the system is MQS. The displacement current is negligible. This is equivalent to saying that the accumulation of charge has essentially no effect on the current density, which is itself solenoidal. Thus, the conductivity of the rod is large enough that the current that enters at one end is negligibly diverted by supplying surface charge, essentially all reaching the far end. However, because the magnetic induction is important, these currents try to link as little magnetic flux as possible. As suggested by the distributed circuit picture of Fig. P15.3.2b, the current distribution tends to crowd to the outer surface of the rod. The inductive reactance for a current circulating through the interior of the rod is less than that of a current nearer the surface. Thus, as the frequency is raised, the dominant field source, the current density, displays skin effect.

In Cartesian rather than cylindrical geometry, Example 10.7.1 illustrates the distribution of magnetic field and current density. The radial direction in this problem plays the role of the x direction in the example. In both cases, the field and current density are independent of the axial direction (y in the example and z in this problem). One dimensional magnetic diffusion was pictured in Sec. 14.8 in terms of an $L-G$ transmission line (negligible capacitance). Note that this is equivalent to the $R-L$ distributed circuit used to schematically portray the MQS behavior in Fig. P15.3.2b. The transmission line would be an exact representation if the rod were replaced by a "slab" conductor and the return conductors were planar rather than circular cylindrical. Such a configuration is shown in Fig. S15.3.2b.

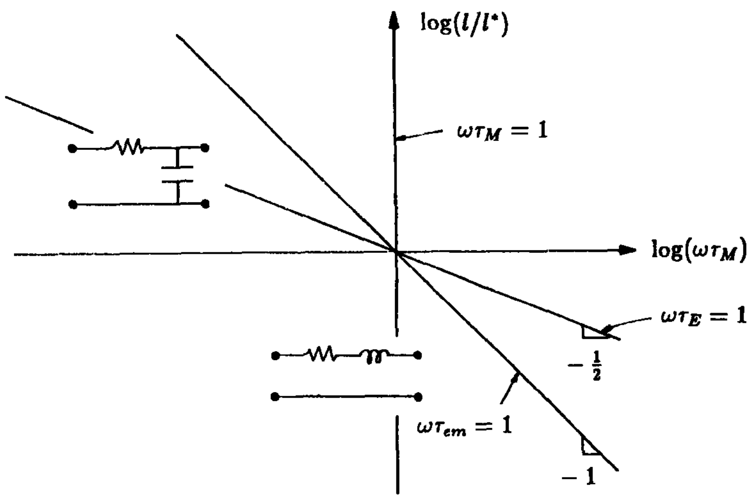
Demonsatration 10.7.1 makes use of a transformer rather than a current source to drive the currents through the conductor. In the limit where the probed conductor is very long compared to its depth, it gives rise to the same current distribution as obtained in the slab conductor of Fig. S15.3.2b. In the problem, the current distribution is somewhat different from that in the slab when the skin depth is on the order of the rod radius because of the cylindrical geometry.

(c) The conditions are as discussed in Sec. 14.9. So that the skin depth is large compared to the rod radius, the frequency must be low enough that the current distribution in the center conductor is essentially uniform. The inductance will nevertheless be self-consistently retained in the model provided that the conditions found in Prob. 14.9.2 are satisfied.

$$1 \ll ln(a/b) \tag{1}$$

(Here, the outer conductor has been effectively made to have an infinite conductivity by setting $\Delta \to \infty$ in the solution to Prob. 14.9.2.). Once we have decided to consider systems that are long in the axial direction, z, compared to the transverse dimensions and taken the quasi-one-dimensional model as representing the dynamics, it is interesting to see how the length, l, in the z direction determines the order of the characteristic times

$$\tau_M = \frac{L}{R}; \quad \tau_{em} = \frac{l}{c} = l\sqrt{LC}; \quad \tau_E = l^2 RC \tag{2}$$



(c)

Figure S15.3.2c

In the limit where the inductance is not important, the system is a charge diffusion line as discussed in Sec. 14.9. Interestingly, the characteristic time associated with this EQS limiting model depends on the square of the length. Again, by contrast with a system having a single typical length, the interaction between the inductance and the resistance is independent of length (magnetic relaxation rather than diffusion). Thus, in constructing a length-frequency plane for sorting out the physical possibilities, it is the time L/R that can be selected for normalizing the frequency. Thus, in this plane the critical lines are

$$\omega\tau_M = 1; \quad \omega\tau_{em} = 1 \Rightarrow \frac{l}{l^*} \equiv (\omega\tau_M)^{-1}; \quad \omega\tau_E = 1 \Rightarrow \frac{l}{l^*} = (\omega\tau_M)^{-1/2} \tag{3}$$

and it follows (see Fig. S15.3.2c) that for the system to first be EQS as the frequency is raised, $l > l^* \equiv \sqrt{L/C}/R$.

15.4 ENERGY, POWER, AND FORCE

15.4.1 The electric field intensity in the three regions follows from Example 7.5.2. From (7.5.7) and (7.5.11), respectively,

$$\mathbf{E}^b = (v/L)\mathbf{i}_z \tag{1}$$

$$\mathbf{E}^a = -\frac{v}{ln(a/b)}\left[\frac{z}{rL}\mathbf{i}_r + \frac{ln(r/a)}{L}\mathbf{i}_z\right] \tag{2}$$

The magnetic field intensity is summarized in Example 11.3.1. From (11.3.10) and (11.2.12), respectively,

$$\mathbf{H}^b = \frac{\sigma v}{2L} r\mathbf{i}_\phi \tag{3}$$

$$\mathbf{H}^a = \frac{\sigma v}{L}\frac{b^2}{2r}\mathbf{i}_\phi \tag{4}$$

The required electric energy, magnetic energy, and dissipation follow by carrying out the piece-wise volume integrations.

$$w_e = \int_{-L}^{0}\int_{b}^{a}\frac{1}{2}\epsilon_a\mathbf{E}^a\cdot\mathbf{E}^a 2\pi r dr dz + \int_{-L}^{0}\int_{0}^{b}\frac{1}{2}\epsilon_b\mathbf{E}^b\cdot\mathbf{E}^b 2\pi r dr dz \tag{5}$$

$$w_m = \int_{-L}^{0}\int_{b}^{a}\frac{1}{2}\mu_a\mathbf{H}^a\cdot\mathbf{H}^a 2\pi r dr + \int_{-L}^{0}\int_{0}^{b}\frac{1}{2}\mu_b\mathbf{H}^b\cdot\mathbf{H}^b 2\pi r dr dz \tag{6}$$

and

$$p_d = \int_{-L}^{0}\int_{0}^{b}\sigma\mathbf{E}^b\cdot\mathbf{E}^b 2\pi r dr dz \tag{7}$$

Note that this last integral is essentially one of the two carried out in (5). Evaluation of these expressions, using (1)-(6), gives

$$\begin{aligned} w_e = {} & \frac{1}{2}v^2\left\{\frac{2\pi\epsilon_a}{Lln^2(a/b)}\left[\frac{1}{3}L^2 ln(a/b) + \frac{a^2}{2}\left\{\frac{1}{2} + (b/a)^2[ln(b/a) - ln^2(b/a) - \frac{1}{2}]\right\}\right]\right\} \\ & + \frac{1}{2}v^2\left[\epsilon_b\frac{\pi b^2}{L}\right] \end{aligned} \tag{8}$$

$$w_m = \frac{\pi\mu_a b^4 v^2\sigma^2}{4L} ln(a/b) + \frac{\pi\mu_b\sigma^2}{16L}v^2 b^4 \tag{9}$$

$$p_d = v^2\left[\frac{\sigma\pi b^2}{L}\right] \tag{10}$$

Written with the voltage replaced by the total current,

$$i = v\left(\frac{\sigma\pi b^2}{L}\right) \tag{11}$$

the magnetic energy, (9), becomes

$$w_m = \frac{1}{2}\left[\frac{\mu_a L ln(a/b)}{2\pi} + \frac{\mu_b L}{\pi 8}\right] i^2 \tag{12}$$

From a comparison of (8), (12), and (10), respectively, to

$$w_e = \frac{1}{2}Cv^2; \quad w_m = \frac{1}{2}Li^2; \quad p_d = i^2 R = v^2 G \tag{13}$$

it follows that the quasi-stationary parameters that model the system at frequencies that are low compared to either R/L or $1/RC$, whichever is the lower, are

$$C = \frac{2\pi\epsilon_a}{L ln^2(a/b)}\left\{\frac{1}{3}L^2 ln(a/b) + \frac{a^2}{2}\left[\frac{1}{2} + (b/a)^2\left[ln(b/a) - ln^2(b/a) - \frac{1}{2}\right]\right]\right\} + \epsilon_b\frac{\pi b^2}{L} \tag{14}$$

$$L = \frac{1}{2}\left[\mu_a\frac{L ln(a/b)}{2\pi} + \frac{\mu_b L}{8\pi}\right] \tag{15}$$

$$G = \frac{\sigma_b \pi b^2}{L} \tag{16}$$

(Note that L on the right is the length L of the device, to be distinguished from the inductance L on the left in (15).) Written in the form of (15.2.8), the ratio of the total magnetic to the total electric energy is, from (9) and (8)

$$\frac{w_m}{w_e} = K\left(\frac{b}{l^*}\right)^2; \qquad l^* \equiv \sqrt{\frac{\epsilon_a}{\mu_a}\frac{1}{\sigma}} \tag{17}$$

where

$$K \equiv \left(ln(a/b) + \frac{\mu_b}{4\mu_a}\right)/\left\{\frac{4(a/b)^2}{ln^2(a/b)}\left[\frac{1}{3}(L/a)^2 ln(a/b) + \frac{1}{2}\left[\frac{1}{2} + (b/a)^2\left[ln(b/a) - ln^2(b/a) - \frac{1}{2}\right]\right]\right] + \frac{2\epsilon_b}{\epsilon_a}\right\} \tag{17}$$

Provided the ratio of all dimensions and of the permittivities and permeabilities are on the same order, the coefficient K is "of the order of unity."

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