

16 Discrete Groups

16.1 Review

Last time, we looked at discrete⁵⁷ subgroups $G \leq M_2$. Then, we looked at a projection π :

$$\ker(\pi) = (\mathbb{R}^2, +) \subset M_2 \xrightarrow{\pi} O_2,$$

$$t_b \circ A \mapsto A;$$

essentially, it gets rid of the translation part of an isometry.

We can restrict π to G to get a mapping

$$G \xrightarrow{\pi|_G} O_2,$$

and we call the kernel,

$$L := \ker(\pi|_G),$$

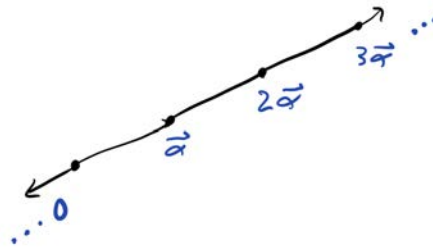
and it consists of all the translations in G .

The image of G in O_2 , denoted $\overline{G} := \pi(G)$, is called the *point group* of G . For some element $g \in G$, its image $\overline{g} := \pi(g) \in \overline{G}$ only "remembers" the angle of rotation or the slope of the reflection line.

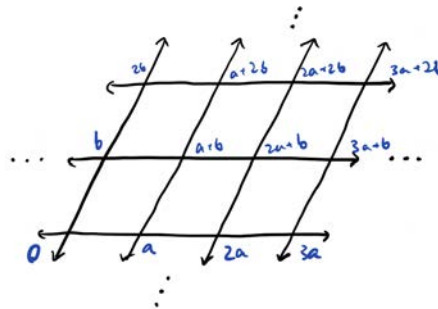
If \overline{G} is discrete, it is either C_n or D_n , which we proved earlier.

If $L \subseteq \mathbb{R}^2$ is discrete, then we obtained three possible cases.

- (i) $L = \{0\}$;



- (ii) $L = \mathbb{Z}\alpha$ where $\alpha \neq 0$;



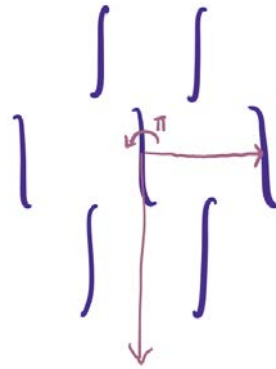
- (iii) $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$, where α, β are linearly independent.⁵⁸

16.2 Examples for L and \overline{G}

For a given plane figure, it is actually not difficult to see what L and \overline{G} are! For the translation subgroup L , since it must either be the identity translation, $\mathbb{Z}\alpha$, or a lattice, it is possible to simply eyeball which translations preserve the figure. Let's consider the following plane figures. Later in this lecture, we will discuss the possibilities for \overline{G} ; it consists of the (untranslated) rotations and reflections preserving a figure.

⁵⁷The translations and rotations that cannot be arbitrarily small

⁵⁸When you look at two vectors and everything you generate from them, it's called a lattice.



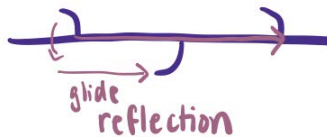
Example 16.1 (A)

For this first figure, say figure A, the translation subgroup L is a rectangular lattice generated by two translation vectors, to the right and upward. Also, \overline{G} is D_2 , since it contains a reflection as well as rotation by π .



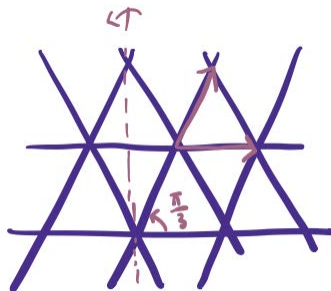
Example 16.2 (B)

For figure B, the translation subgroup is trivial, consisting of 0. Also, \overline{G} is C_3 , since there cannot be any reflections but rotation by $2\pi/3$ or $4\pi/3$ around the center both preserve the figure.



Example 16.3 (C)

For figure C, the translation subgroup is generated by one vector, so $L = \mathbb{Z}\alpha$ where $\alpha = (1, 0)$. Also, \overline{G} is D_1 , since there is a reflection (corresponding to a glide reflection in G) and no rotations possible.



Example 16.4 (D)

For figure D, the translation subgroup is a triangular lattice generated by two vectors at an angle of $\pi/3$ to each other.^a The point group is $\bar{G} = D_6$, since rotation at a lattice point by any multiple of $\pi/3$ preserves the figure, as well as reflection.

^aOr two vectors at an angle of $2\pi/3$.

16.3 Crystallographic Restriction

Now that we have decomposed studying G into studying groups we understand better, L , a subgroup of translations, and $\bar{G} \subseteq O_2$, the point group, we can actually constrain G further!

Recall that

- The translation subgroup $L \subseteq (\mathbb{R}^2, +)$ ⁵⁹ must be one of three possibilities, which we get from studying discrete subgroups of \mathbb{R}^2 ;
- \bar{G} must be C_n or D_n , which we get from studying discrete subgroups of O_2 .

Now that we understand the components L and \bar{G} separately, we want to use this knowledge to understand G better.

Guiding Question

How do L and \bar{G} interact with each other?

Example 16.5

Consider our earlier example 16.4. In this case, any element of the point group D_6 preserved the triangular lattice.

In fact, \bar{G} acts on L for any discrete group $G \subseteq M_2$; this is a very strong constraint on how \bar{G} and L interact.

Theorem 16.6

For the point group $\bar{G} \leq O_2$ of some discrete subgroup G of M_2 , and the translation subgroup $L \subset \mathbb{R}^2$, the group \bar{G} must map L to itself.

For any element $A \in \bar{G}$ and $b \in L$, the image of b under the action of A is

$$b \mapsto Ab \in L.$$

We already know that O_2 and thus \bar{G} acts on the plane \mathbb{R}^2 and therefore L . The surprising part is that under the action of any element of \bar{G} , an element of L is actually mapped to another element in L !

Proof. Since $A \in \bar{G}$, it is the image of an element of G , say $t_{\vec{c}} \circ A \in G$ for some $\vec{c} \in \mathbb{R}^2$. Then, $\vec{b} \in L$, so $t_{\vec{b}} \in G$. The key observation in this proof is that $L = \ker(\pi|_G)$ is the kernel of a homomorphism! Thus, the subgroup $L \trianglelefteq G$ is actually normal, so conjugating an element of L by anything in G stays in L .

Then for $t_{\vec{b}} \in L$,

$$(t_{\vec{c}} \circ A) \cdot t_{\vec{b}} \cdot (t_{\vec{c}} \circ A)^{-1} \in L$$

also. As isometries in M_2 , we know how to manipulate these products, and so expanding out this expression gives us

$$\begin{aligned} t_{\vec{c}} \cdot A \cdot t_{\vec{b}} \cdot A^{-1} \cdot t_{\vec{c}}^{-1} &= t_{\vec{c}} t_{A\vec{b}} \cdot A \cdot A^{-1} \cdot t_{-\vec{c}} \\ &= t_{\vec{c}} t_{A\vec{b}} t_{-\vec{c}} \\ &= t_{A\vec{b}} \in L. \end{aligned}$$

⁵⁹The translation subgroup L is sometimes written ambiguously in one of two equivalent ways; an element of L can either be the translation $t_{\vec{b}} \in L$ considered as an element in G , or simply the vector $\vec{b} \in L$ considered as an element in \mathbb{R}^2 . So L could be considered either as a subgroup of G or of \mathbb{R}^2 .

Thus, conjugating $t_{\vec{b}} \in L$ by $t_{\vec{c}} \circ A$ gives $t_{A\vec{b}} \in L$. Using the identification of L with \mathbb{R}^2 , $A\vec{b} \in L \subset \mathbb{R}^2$, and so every $A \in \overline{G}$ takes vectors \vec{b} in L to other vectors in L , preserving the translation subgroup. \square

Student Question. *We're studying discrete groups, which are groups with the requirement that the translations or rotations can't be arbitrarily small. Are we also requiring that they have to be groups preserving a given diagram, or can they be any discrete groups of isometries?*

Answer. *Earlier on in this lecture, we saw some **examples** of discrete groups G that came from the symmetry groups of certain diagrams, but what we are actually doing is simply looking at groups G with the condition that the rotations and translations must be arbitrarily small⁶⁰, and classifying them; mathematically, there is no requirement that they come from pictures.*

However, the way that these discrete groups actually show up and the way that we find them is by drawing these kinds of pictures; this is one of the main reasons why we care about them! In fact, for every discrete subgroup $G \subseteq M_2$, there will be some picture that produces the group G as its symmetry group. The pictures in this lecture are mainly so that there are concrete examples to look at and think about.

In Section 16.2, each of the examples has a symmetry group G consisting of the isometries of the plane sending the picture to itself.⁶¹ For example, in Example 16.2, rotation by 120 degrees preserves the "triangle," while 5 degrees does not, so $\rho_{2\pi/3} \in G$, whereas $\rho_{\pi/36} \notin G$.

Theorem 16.6 states that the point group \overline{G} , which is a different group from G , actually preserves $L \subseteq \mathbb{R}^2$, the translation group.

In Example 16.4, L is generated by

$$\mathbb{Z}(1, 0)^t + \mathbb{Z}(1/2, 3/2)^t,$$

the two sides of an equilateral triangle, and the point group is D_6 . Any element of D_6 will send an element of L to a different element in L .

In fact, when L is a lattice, preservation by some point group \overline{G} is a strong constraint on the possible angles that show up in the lattice; only certain angles are allowed. Given \overline{G} , most lattices are not preserved by every element. Thus, the theorem constrains \overline{G} and L together — not on each of them separately, but on how they interact.

The groups that show up this way are often called crystallographic groups.⁶² They are well-studied; in fact, there are only finitely many.

Theorem 16.7 (Crystallographic Restriction)

^a Let $L \neq \{0\}$. Then $\overline{G} = C_n$ or D_n , where $n = 1, 2, 3, 4$, or 6 .

^aThe theorem name comes from the fact that it restricts the possible crystallographic groups.

Although we could imagine that there are lots of possibilities for \overline{G} and L , the fact that \overline{G} preserves L constrains the possible point groups to finitely many, and there are also only certain choices of L allowed for a given n .

Proof. The group \overline{G} is a discrete subgroup of O_2 , and so it is C_n or D_n for some integer n .

Since L is discrete, there is a (non-unique) shortest nonzero vector $\alpha \neq 0$. Consider a rotation $\rho = \rho_\theta \in G$. The result of rotating α by θ is another vector in L , and since rotations are length-preserving, $\rho_\theta\alpha$ is also a vector of shortest length. Since both vectors are in L , $\rho\alpha - \alpha$ is also in L .⁶³ If θ is too small, $\rho\alpha - \alpha$ will have a shorter length, and there will be a contradiction.

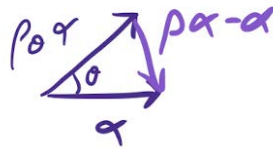
In particular, if $\theta < 2\pi/6$, $\rho\alpha - \alpha$ is shorter than α , so $\theta \geq 2\pi/6$. Since C_n and D_n contain $\rho_{2\pi/n}$, it must be the case that $n \leq 6$.

⁶⁰These are called discrete groups

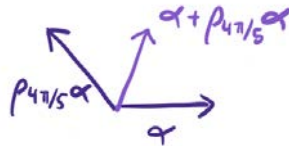
⁶¹Not each point individually is sent to itself; the picture as a whole is sent to an identical copy of itself.

⁶²Especially when L is a lattice, and there are two different directions to translate.

⁶³Since L is a subgroup, it is closed under addition/subtraction.



A similar argument holds to rule out $n = 5$. The vector $\alpha + \rho_{4\pi/5}\alpha$ will be shorter than any α , which is also a contradiction.⁶⁴



So $n = 1, 2, 3, 4$, or 6 . □

Actually, for C_n or D_n where $n = 1, 2, 3, 4$, or 6 , it is possible to constrain the translation subgroups L that can simultaneously show up.

For instance, when L is a lattice,⁶⁵ there are only 17 possible symmetry groups G that can occur. When L is 0 , \bar{G} can be C_n or D_n for any arbitrary n , but allowing nontrivial translations constrains \bar{G} significantly.

Student Question. *How much does constraining \bar{G} and L constrain the actual symmetry group G itself?*

Answer. *Finding G from \bar{G} and L is precisely the same as figuring out the 17 plane symmetry groups,⁶⁶ and is precisely the last step! We will do one example now.*

Let's consider a specific group \bar{G} and try to figure out what the actual symmetry group G can be!

⁶⁴This question is equivalent to the feasibility of tiling the plane with a regular pentagon, and in fact that is not possible!

⁶⁵When L is a lattice, it is two-dimensional, and it is $\mathbb{Z}\vec{a} + \mathbb{Z}\vec{b}$ for generating vectors \vec{a} and \vec{b} . It is also possible for L to be $\mathbb{Z}\vec{a}$, which is one-dimensional.

⁶⁶These are called wallpaper groups, since wallpapers are 2-dimensional patterns that usually have nontrivial symmetry groups.

Example 16.8 (C_4)

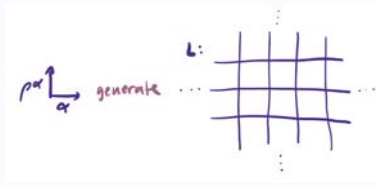
Suppose $\bar{G} = C_4$.^a Then $L \subset G \xrightarrow{\pi|_G} C_4$, and the index $[G : L] = 4$.^b

Also, $\bar{\rho} = \rho_{\pi/2} \in \bar{G}$ is a generator of \bar{G} . Where α is some shortest-length vector in L , it's possible to show^c that $\rho\alpha$ and α do generate L . Thus,

$$L = \mathbb{Z}\alpha + \mathbb{Z}(\rho\alpha),$$

a square lattice.

Also, there exists some rotation $\rho \in G$ giving $\pi(\rho) = \bar{\rho}$. Then ρ is in fact a rotation by $\pi/2$ around some other point, which we will call the origin.^d The group G contains L , of index 4, as well as some rotation by $\pi/2$, ρ .^e



Thus, G is "generated" by L and ρ , and must consist of everything of the form

$$G = \{t_v \circ \rho^i : v \in L, i = 0, 1, 2, 3\}.$$

Also, $\rho t_v = t_{\rho v} \circ \rho$, so the group multiplication can be written down, and G is completely determined by knowing that \bar{G} was C_4 ; this is 1 out of the 17 wallpaper groups!

^aRotations by 90 degrees, but no reflections.

^bThe index $[G : \ker(\pi|_G)] = [G : L]$ is equal to the size of the image under $\pi|_G$, which is $\bar{G} = C_4$.

^cThere is a more involved argument there, but it is not super relevant here.

^dIn the discussion of the four kinds of isometries in M_2 , the elements which were mapped to rotations were in fact rotations around some point.

^eThe rotation ρ is $\bar{\rho}$, lifted to be in G , and it is an element of G not in L which generates the quotient, C_4 .

Student Question. Can you explain where ρ came from? Why is it a rotation?

Answer. By definition, \bar{G} is the image of G under $\pi : M_2 \rightarrow O_2$ taking $t_b \circ A \mapsto A$. Then there are four possibilities for elements in M_2 : translation, rotations, reflections, and glide reflections. The first two are orientation-preserving, and the last two are orientation-reversing. Reflections and glide reflections map to reflections in O_2 ⁶⁷ under π , translations will map to the identity, and rotations will map to rotations (around the origin). So ρ has an image of $\bar{\rho}$, which is a rotation, and thus ρ is a rotation around some point.

If $\bar{\rho}$, the element in \bar{G} , were a reflection instead of a rotation, the preimage in G could have been either a reflection or a glide reflection, so when the point group $\bar{G} = D_n$, one of the dihedral groups, instead of C_n , the analysis is more subtle. In fact, there might not be any reflections in G at all. (In Example 16.3, there were no reflections, only glide reflections.)

Example 16.9

If $\bar{r} = \pi(r)$ where r , then $\bar{r} = \pi(t_b \circ r_\ell)$, where b is some zero^a or nonzero^b vector parallel to the line ℓ . Does this mean there are uncountably many possibilities for b and therefore r ? In fact, b is constrained a bit more: $t_b r_\ell t_b r_\ell = t_{2b}$, so $2b \in L$. Thus, there are two possible situations:

- The vector is in the lattice: $b \in L$;
- The vector b is halfway between two lattice points, as in Example 16.1.

^areflection

^bglide reflection

From these two examples, we see that given some \bar{G} , of which there are finitely many, and working through the information that is present, there aren't too many possibilities for G , and in fact there are finitely many — 17

⁶⁷Reflections across lines through the origin

in total.

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