8.513 Problem Set # 4

Problems:

1. (15 pts) **Casimir effect of photons**

   Consider a 1D quantum boson gas between two hard walls separated by a distance L. The energy-momentum relation of the bosons is

   $$\epsilon(p) = c|p|.$$

   (a) (b)

   We can view the 1D quantum boson gas as a quantum string of length L with fixed boundary condition. The frequency-wave-vector relation of the wave on the string is given by

   $$\omega_k = c|k|.$$

   The quantum string can be viewed as a collection of quantum oscillators labeled by wave vectors $\kappa_n = n\frac{\pi}{L}$, $n = 1, 2, 3, 4, \cdots$.

   The ground state energy of the quantum string is given by

   $$E_{\text{vac}}(L) = \sum_{\kappa_n} \frac{1}{2}\epsilon(\hbar\kappa_n) = \sum_{\kappa_n} \frac{1}{2}c\hbar\kappa_n$$

   which diverges.

   (a) To fix the infinite problem, we may use the so called **heat kernel regularization**, by rewriting the ground state energy as

   $$E^{\text{heat}}_{\text{vac}}(L) = \sum_{\kappa_n} \frac{1}{2}c\hbar\kappa_n \mathrm{e}^{-\kappa_n a}$$

   where $a \ll L$ is a small length scale (or cut-off length scale). Find $E^{\text{heat}}_{\text{vac}}(L)$. Find $E^{\text{heat}}_{\text{vac}}(L)$ in small a limit, by dropping the linear a-term and the higher order terms.

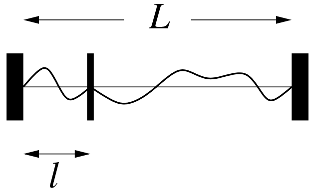
   (b) To fix the infinite problem, we may also use the so called **lattice regularization**, by rewriting the ground state energy as

   $$E^{\text{latt}}_{\text{vac}}(L) = \sum_{n=1}^{N} c\hbar a^{-1}\sin(\frac{1}{2}\kappa_n a), \quad N = L/a,$$

   where $a \ll L$ is a small length scale (or cut-off length scale). Find $E^{\text{latt}}_{\text{vac}}(L)$. Find $E^{\text{latt}}_{\text{vac}}(L)$ in small a limit, by dropping the linear a-term and the higher order terms.

   Remark: from the above two calculations, you will see a sign of a infamous identity $\sum_{n=1}^{\infty} n = -1/12$.

(c) Consider the following boson systems with three hard walls.



Find the new ground state energies $E_{\text{vac}}^{\text{heat}}(L, l)$ and $E_{\text{vac}}^{\text{latt}}(L, l)$, assuming $a \ll l, L$. Find the force between the two hard walls separated by l, assuming $l \ll L$. This is the Casimir force.

2. (25pts) **Quantum coherent states and classical phase-space Lagrangian:**

Many-body Hamiltonian for 1D interacting lattice bosons in real space is given by

$$\hat{H} = \sum_i \Big[t(\hat{\varphi}_i^\dagger \hat{\varphi}_i + \hat{\varphi}_i^\dagger \hat{\varphi}_i) - t(\hat{\varphi}_{i+1}^\dagger \hat{\varphi}_i + \hat{\varphi}_i^\dagger \hat{\varphi}_{i+1})\Big] - \sum_i \mu \hat{\varphi}_i^\dagger \hat{\varphi}_i + \sum_i U \hat{\varphi}_i^\dagger \hat{\varphi}_i^\dagger \hat{\varphi}_i \hat{\varphi}_i,$$

where i labels the lattice sites, and

$$[\hat{\varphi}_i, \hat{\varphi}_j^\dagger] = \delta_{i,j}.$$

We want to find the low energy spectrum of the above interacting system via the semi classical approach.

(a) Show that

$$\langle 0| \mathrm{e}^{\sum_i \tilde{\phi}_i^* \hat{\varphi}_i} \mathrm{e}^{\sum_j \phi_j \hat{\varphi}_j^\dagger} |0\rangle = \mathrm{e}^{\sum_i \tilde{\phi}_i^* \phi_i},$$

where $|0\rangle$ is defined as state satisfying

$$\hat{\varphi}_i |0\rangle = 0.$$

(Hint: $\mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}} = \mathrm{e}^{[\hat{A},\hat{B}]}\mathrm{e}^{\hat{B}}\mathrm{e}^{\hat{A}}$ provided that $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.)

(b) Show that

$$|\{\phi_i\}\rangle \equiv \frac{1}{\sqrt{N(\phi_i)}} \mathrm{e}^{\sum_i \phi_i \hat{\varphi}_i^\dagger} |0\rangle$$

is a normalized common eigenstate of $\hat{\varphi}_i$'s with eigenvalues ϕ_i. Here

$$N(\phi_i) = \mathrm{e}^{\sum_i |\phi_i|^2}.$$

(c) Show that classical Lagrangian (in the semi classical approximation)

$$L(\partial_t \phi_i(t), \phi_i(t)) = \langle \phi_i(t)| \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \hat{H} |\phi_i(t)\rangle$$

is given by

$$L = \sum_i \mathrm{i}\phi_i^* \partial_t \phi_i - \bar{H}$$

where the average energy is given by

$$\bar{H} = \sum_i \big[t(\phi_i^* \phi_i + \phi_i^* \phi_i) - t(\phi_i^* \phi_{i+1} + \phi_{i+1}^* \phi_i) - \mu|\phi_i|^2 + U|\phi_i|^4\big]$$

(d) For trial wave function $|\{\phi_i\}\rangle$ with $\phi_i = \phi$ (*ie* independent of i), find the complex variational parameter ϕ that minimize $\bar{H}$, assuming $t, U > 0$ and $\mu \geq 0$. We denote such a value of ϕ as $\bar{\phi}$.

(e) Let $\phi_i = \bar{\phi} + \delta\phi_i$. Find the equation of motion for $\delta\phi_i$, keeping only terms linear in $\delta\phi_i$. Find the frequency ω_k of the plane wave solution $\delta\phi_i = A\mathrm{e}^{\mathrm{i}(ki-\omega_k t)}$. What is ω_k for small k at the transition point $\mu = 0$ and in superfluid phase $\mu > 0$? What is ω_k for small k in the insulating phase $\mu < 0$?

Remark: you will see that the dynamical exponent $z = 2$ at the critical point. There is no emergence of Lorentz symmetry at low energies. But the dynamical exponent is not an arbitrary number. In the superfuild phase the gapless excutations have $\omega_k \sim |k|$ (ie $z = 1$). There is an emergence of Lorentz symmetry at low energies.

As a quantum theory (instead of as a classical field theory), the low energy spectrum of the interacting bosons is given by a collection of decoupled oscillators labeled by k with frequencies ω_k.

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