Highly entangled quantum many-body systems – Topological order

Xiao-Gang Wen

Our world is very rich with all kinds of materials

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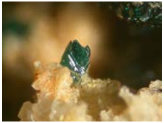

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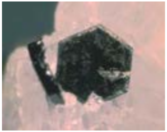

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In middle school, we learned ...

there are four states of matter:

Solid

Liquid

Gas

Plasma

In university, we learned ... ...

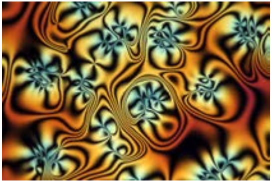

Image courtesy of Oleg D. Lavrentovich. Used with permission.

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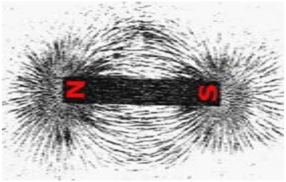

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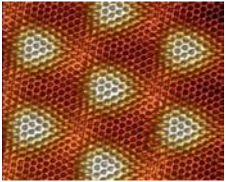

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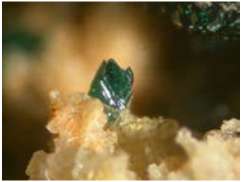

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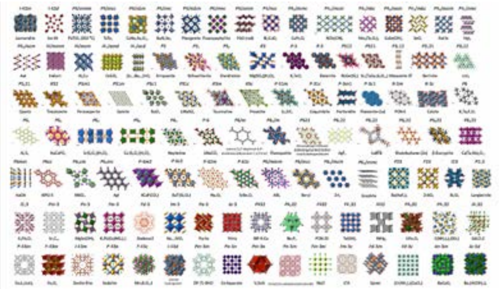
- Rich forms of matter $\leftarrow$ rich types of **order**
- A deep insight from Landau: **different orders come from different symmetry breaking**.
- A corner stone of condensed matter physics

Classify phases of quantum matter ($T = 0$ phases)

For a long time, we thought that Landau symmetry breaking classify all phases of matter

- **Symm. breaking phases are classified by a pair** $G_\Psi \subset G_H$
  G_H = symmetry group of the Hamiltonian H.
  G_Ψ = symmetry group of the ground states Ψ.
- **230 crystals** from group theory

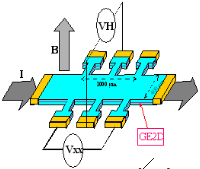


Can symmetry breaking describes all phases of matter?

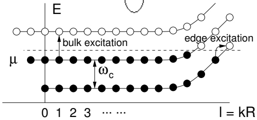
A **spin-liquid theory** of high T_c superconductors:

- It was proposed that a 2d spin liquid can have **spin-charge separation**:
  An electron can change into two topological quasi particles:
  electron $=$ *holon* $\otimes$ *spinon*,
  holon: charge-1 spin-0 boson,
  spinon: charge-0 spin-1/2 fermion.
  Holon condensation $\to$ high T_c superconductivity.
- **Does such a strnge spin liquid exist? How to characterize it?**
  A spin liquid was explicitly constructed Kalmeyer-Laughlin, PRL **59** 2095 (87), and we found that it is a state that break time reversal and parity symmetry, but not spin rotation symmetry, with order parameter
  $\boldsymbol{S}_1 \cdot (\boldsymbol{S}_2 \times \boldsymbol{S}_3) \neq 0 \to$ **Chiral spin liquid** Wen, Wilczek, Zee, PRB **39** 11413 (89)
- However, we also discovered several different chiral spin states with identical symmetry breaking pattern.
  How distinguish those chiral spin states with the same symmetry breaking?

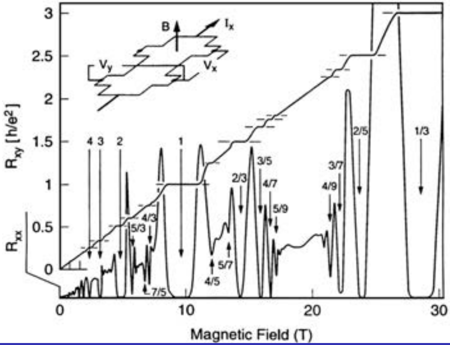
Topological orders in quantum Hall effect



- Quantum Hall (QH) states $R_{xy} = V_y/I_x = \frac{m}{n}\frac{2\pi\hbar}{e^2}$
  vonKlitzing Dorda Pepper, PRL **45** 494 (1980)
  Tsui Stormer Gossard, PRL **48** 1559 (1982)



- Fractional quantum Hall (FQH) states have different phases even when the only $U(1)$ symmetry is not broken for those states.
- Chiral spin and FQH liquids must contain a new kind of order, which was named as **topological order**
  Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)



What is topological order?

- Three kinds of quantum matter:

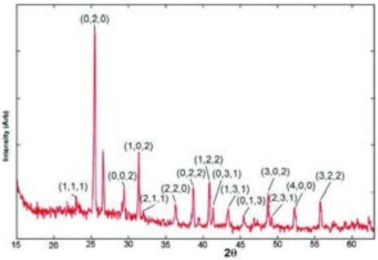
(1) no low energy excitations (Insulator) $\to$ **trivial**
(2) some low energy excitations (Superfluid) $\to$ **interesting**
(3) a lot of low energy excitations (Metal) $\to$ **messy**

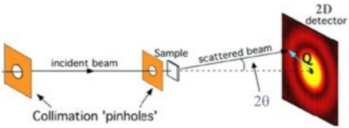
Topological orders belong to the "trivial" class
(*ie* have an energy gap and no low energy excitations)

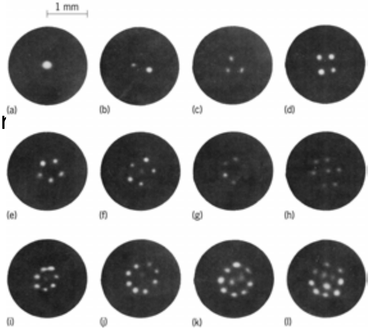
Topological orders are trivial state of matter?!

Every physical concept is defined by experiment

- The concept of crystal order is defined via X-ray scattering





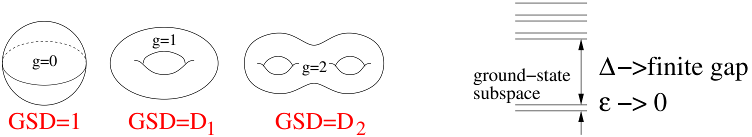


- The concept of superfuild order
  no low energy excitations is defined via zer
  quantization of vorticity

What is **topological order**? How to characterize it?

- How to extract universal information (topological invariants) from complicated many-body wave function $\Psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{10^{20}})$
  Put the gapped system on space with various topologies, and measure the ground state degeneracy.
  (The dynamics of a quantum many-body system is controlled by a hermitian operator, Hamiltonian H, acting on the many-body wave functions. The spectrum of the Hamiltonian has a gap)
  $\to$ The notion of **topological order**

Wen PRB **40** 7387 (89)

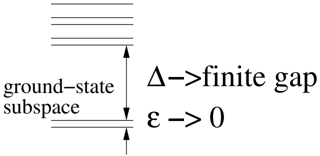


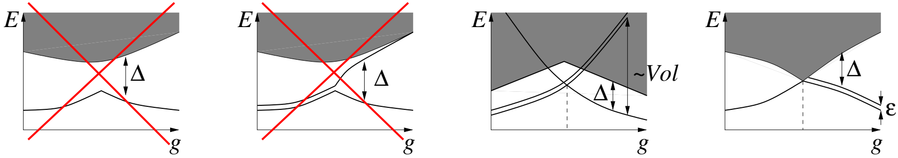
- The name **topological order** was motivated by Witten's **topological quantum field theory** (field theories that do not depend on spacetime metrics), such as **Chern-Simons** theories which happen to be the low energy effective theories for both chiral spin states and QH states.

Witten CMP **121** 351 (1989)

The ground state degeneracy is a topological invariant

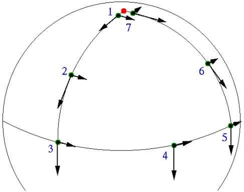
- At first, some people objected that the ground state degeneracies are finite-size effects or symmetry effect, not reflecting the intrinsic order of a phase of matter.
- The ground state degeneracies are robust against any local perturbations that can **break any symmetries**.
  $\rightarrow$ **topological degeneracy** (another motivation for the name **topological**) Wen Int. J. Mod. Phys. B **04** 239 (90); Wen Niu PRB **41** 9377 (90)
- The ground state degeneracies can only vary by some large changes of Hamiltonian
  $\rightarrow$ gap-closing phase transition.

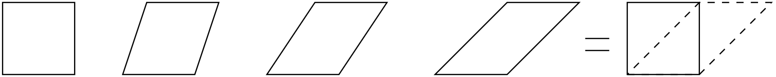




How to fully characterize topological order?

Deform the space and measure the **non-Abelian geometric phase** of the deg. ground states. Wilczek & Zee PRL **52** 2111 (84)





- For 2d torus $\Sigma_2 = S^1 \times S^1$:
  Dehn twist: $|\Psi_i\rangle \to |\Psi_i'\rangle = T_{ij}|\Psi_j\rangle$
  90° rotation $|\Psi_i\rangle \to |\Psi_i'\rangle = S_{ij}|\Psi_j\rangle$

  S, T generate a representation of modular group: $S^2 = (ST)^3 = C,\ C^2 = 1$

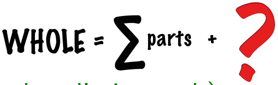
How to fully characterize topological order?

Conjecture: The non-Abelian geometric phases of the degenerate ground states for closed spaces with all kinds of topologies can fully characterize topological orders. Wen, IJMPB **4** 239 (1990); KeskiVakkuri & Wen, IJMPB **7** 4227 (1993)

- Non-Abelian geometric phases = Projective representations of the mapping class group of closed spaces with all kinds of topologies

An modern understanding of topological degeneracy

- In 2005, we discovered that topological order has **topological entanglement entropy** Kitaev-Preskill hep-th/0510092 Levin-Wen cond-mat/0510613 and **long range quantum entanglement** Chen-Gu-Wen arXiv:1004.3835
- For a long-range entangled many-body quantum system, **knowing every overlapping local parts still cannot determine the whole.**

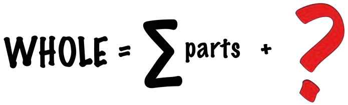


\- In other words, there are different "wholes", that their every local parts are identical (Like fiber bundle in math).
\- Local interactions/impurities can only see the local parts → those different "wholes" (the whole quantum states) have the same energy.

Topological degeneracy comes from long range entanglement.

The pseudo-gauge transformations → different "wholes" with identical local "parts". **Long-range entanglement → Chern-Simons theory**

Why knowing every part does not imply knowing whole?



- What is a "whole"?, what is "part"?
  whole = many-body wave function $|\Psi\rangle = \Psi(m_1, m_2, \cdots, m_N)$
  where m_i label states on site-i
  part = local entanglement density matrix:

$$\rho_{\text{site-1,2,3}} = \mathrm{Tr}_{\text{site-3},\cdots,N} |\Psi\rangle\langle\Psi|,$$

$$\rho_{m_1,m_2,m_3;m'_1,m'_2,m'_3}$$

$$= \sum_{m_4,\cdots,m_N} \Psi^*(m_1, m_2, m_3, m_4, \cdots, m_N)\Psi(m'_1, m'_2, m'_3, m_4, \cdots, m_N)$$

- The energy only depends on the local parts $\rho_{\text{site-1,2,3}}$ due to the local interaction $H_{1,2,3}$

$$\langle H_{1,2,3}\rangle = \mathrm{Tr}(H_{1,2,3}\rho_{\text{site-1,2,3}})$$

Entanglement through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle$ = direct-product state $\to$ unentangled (classical)

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- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$ $\to$ more entangled

Entanglement through examples

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  $= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle$ → unentangled

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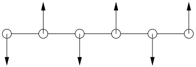
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$ unentangled

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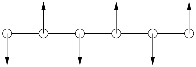
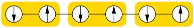
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ → short-range entangled (SRE) entangled

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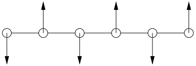
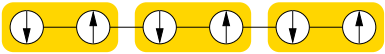
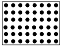
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \dots$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \dots$ → short-range entangled (SRE) entangled
- Crystal order: $|\Phi_{\text{crystal}}\rangle =$ |  ⟩ $= |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \dots$
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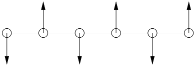
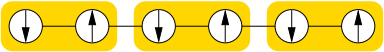
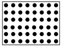
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ → short-range entangled (SRE) entangled
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} ...$
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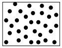
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

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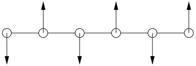
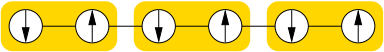
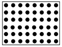
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ → short-range entangled (SRE) entangled
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} ...$
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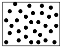
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..) ...$
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What is long-range entanglement?

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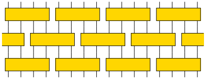
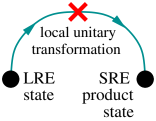
$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



Chen-Gu-Wen arXiv:1004.3835

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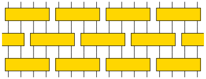
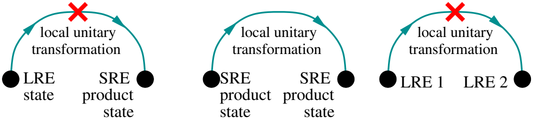
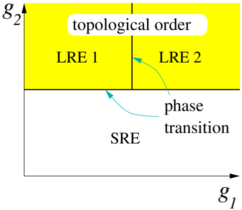
$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements**
  = different **topological orders** Wen PRB **40** 7387 (89)

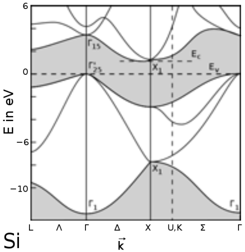
Macroscopic characterization → microscopic origin

- From macroscopic characterization of **topological order** (1989) (topological ground state degeneracies, mapping class group representations)
  → microscopic origin (**long range entanglement** 2010)
  took 20+ years

Macroscopic characterization $\rightarrow$ microscopic origin

- From macroscopic characterization of **topological order** (1989) (topological ground state degeneracies, mapping class group representations)
  $\rightarrow$ microscopic origin (**long range entanglement** 2010)
  took 20+ years

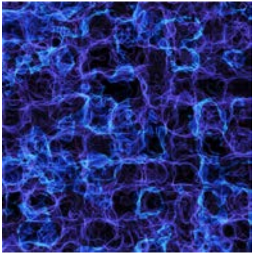
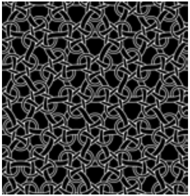
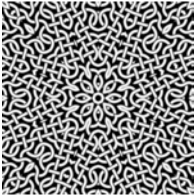
- From macroscopic characterization of **superconductivity** (1911) (zero-resistivity, quantized vorticity)
  $\rightarrow$ microscopic origin (**BSC electron-pairing** 1957)
  took 46 years



Topology in topological insulator/superconductor (2005) corresponds to the twist in the band structure of orbitals, which is similar to the topological structure that distinguishes a sphere from a torus. This kind of topology is *classical topology*.

Kane-Mele cond-mat/0506581

This **topology** is not that *topology*

Topology in topological order (1989) corresponds to pattern of many-body entanglement in many-body wave function $\Psi(m_1, m_2, \cdots, m_N)$, that is robust against any local perturbations that can break any symmetry. Such robustness is the meaning of **topological** in topological order. This kind of topology is **quantum topology**.

Wen PRB **40** 7387 (1989)

How to make long range entanglement? (Mechanism of topological order)

A mechanism of superconductivity: electron pairing
$\to$ boson condensation $\to$ superconductivity

To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\to\to ..\rangle$

How to make long range entanglement? (Mechanism of topological order)

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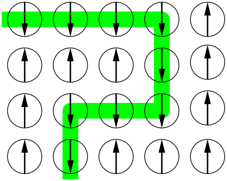


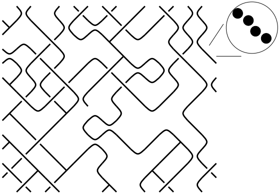
$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\to\to ..\rangle$

- **A mechanism**:
  Sum over a subset of spin configurations:

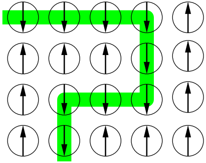
$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum \Big|$  $\Big\rangle$

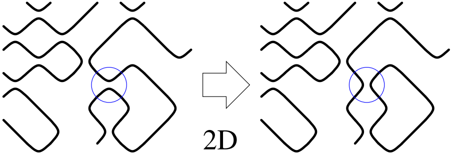
$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}} \Big|$  $\Big\rangle$

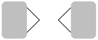
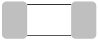
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum (e^{i\theta})^{\#\text{ of loops}} \Big|$  $\Big\rangle$



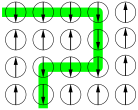
- Can the above wavefunctions be the ground states of local Hamiltonians?

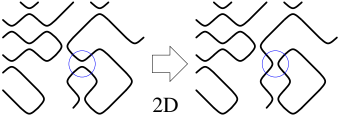


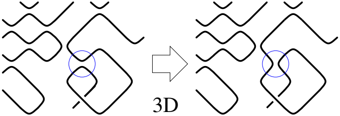


- Local rules of a string liquid (for ground state):
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $= \Phi_{str}$ (), Φ_{str} () $= \Phi_{str}$ ()
  → Global wave function of loops Φ_{str} () $= 1$
- There is a local Hamiltonian H:
  (1) Open ends cost energy
  (2) string can hop and reconnect freely. *ie* H contains terms causing
   → ,   →  with negative coefficient.

  The ground state of H gives rise to the above string lqiuid wave function. (For the explicite H, see page 33).







- Local rules of another string liquid (ground state):
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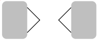
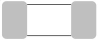
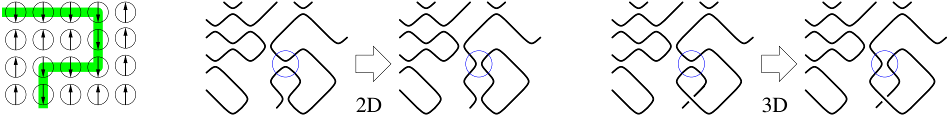
  (2) Φ_{str} () $=\Phi_{str}$ (), Φ_{str} () $=-\Phi_{str}$ ()
  → Global wave function of loops Φ_{str} () $=(-)^{\#\text{ of loops}}$



- Local rules of another string liquid (ground state):
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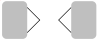
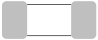
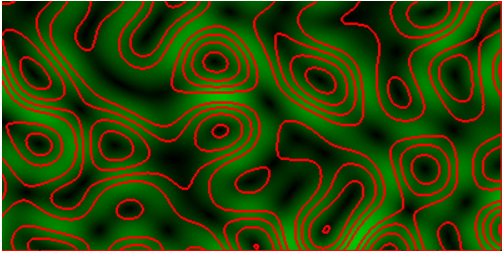
  (2) Φ_{str} () $=$ Φ_{str} (), Φ_{str} () $= -\Phi_{\text{str}}$ ()
  → Global wave function of loops Φ_{str} () $= (-)^{\#\text{ of loops}}$

- The second string liquid Φ_{str} () $= (-)^{\#\text{ of loops}}$ can exist only in 2-dimensions.
- The first string liquid Φ_{str} () $= 1$ can exist in both 2- and 3-dimensions.
- The thirsd string liquid Φ_{str} () $= (e^{i\theta})^{\#\text{ of loops}}$ can exist in neither 2- nor 3-dimensions.

Knowing all the parts $\neq$ knowing the whole

- Do those two string liquids really have topological order? Do they have topological ground state degenercy?



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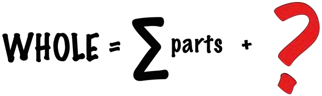
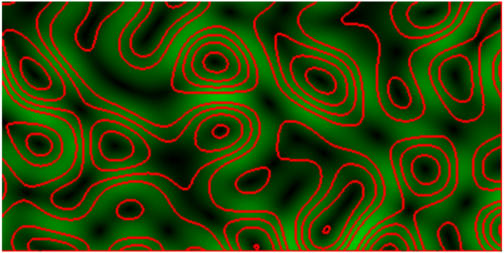
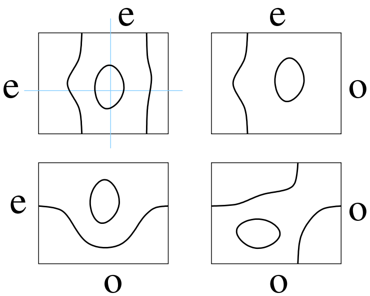




- 4 locally indistinguishable states on torus for both liquids $\to$ **topological order**
- Ground state degeneracy cannot distinguish them.

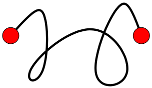


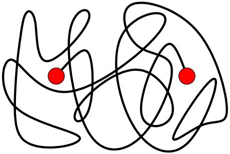
$D^{\text{tor}} = 4$

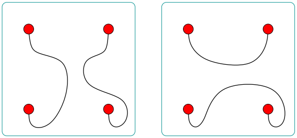
Topological excitations

- **Ends of strings** behave like point objects.
- They cannot be created alone → **topological**
- Let us fix 4 ends of string on a sphere S^2. *How many locally indistinguishable states are there?*

- There are 2 sectors → 2 states.







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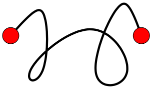

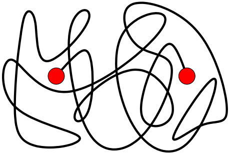

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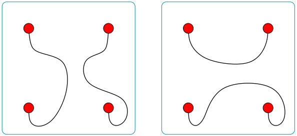
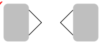
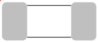


- ~~There are 2 sectors → 2 states.~~
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations $\Phi_{\text{str}}\Big($  $\Big) = \pm\Phi_{\text{str}}\Big($  $\Big)$.

- **For our string liquids**, in general, fixing $2N$ ends of string → 1 state. Each end of string has no degeneracy → no internal degrees of freedom.

Topological excitations

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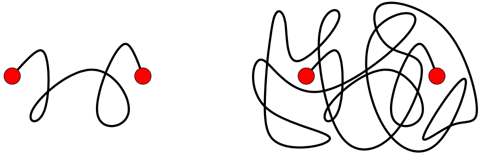
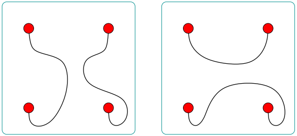
- They cannot be created alone → **topological**





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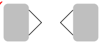
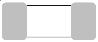
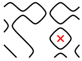
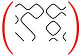
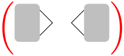
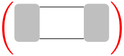
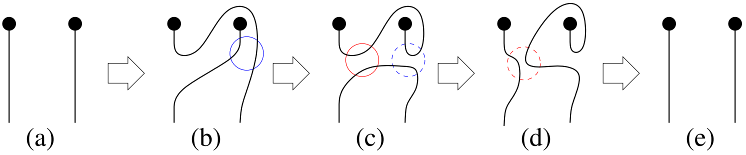
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations $\Phi_{str}\Big($  $\Big) = \pm\Phi_{str}\Big($  $\Big)$.

- **For our string liquids**, in general, fixing $2N$ ends of string → 1 state. Each end of string has no degeneracy → no internal degrees of freedom.
- Another type of topological excitation **vortex** at × (by modifying the string wave function): $|m\rangle = \sum(-)^{\#\text{ of loops around }\times}\Big|$  $\Big\rangle$

- Ends of strings are point-like. Are they bosons or fermions?
  Two ends = a small string = a boson, but each end can still be a fermion. Fidkowski-Freedman-Nayak-Walker-Wang cond-mat/0610583
- Φ_{str}  $= 1$ string liquid Φ_{str}  $= \Phi_{\text{str}}$ 
- End of string wave function: $|\text{end}\rangle =$  $+ c$  $+ c$  $+ \cdots$
  The string near the end is totally fixed, since the end is determined by a trapping Hamiltonian δH which can be chosen to fix the string. The string alway from the end is not fixed, since they are determined by the bluk Hamiltonian H which gives rise to a string liquid.
- 360° rotation:  $\to$  and  $=$  $\to$ : $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- We find four types of topological exitations
  (1) $|e\rangle =$  $+$  spin 0. (2) $|f\rangle =$  $-$  spin $1/2$.

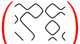
Spin-statistics theorem: Emergence of Fermi statistics



- (a) $\to$ (b) $=$ exchange two string-ends.
- (d) $\to$ (e) $= 360^\circ$ rotation of a string-end.
- Amplitude (a) $=$ Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

$\to$ **Spin-statistics theorem**

Z_2 topological order and its physical properties

Φ_{str} () $= 1$ string liquid has Z_2-topological order.

- 4 **types** of topological excitations: *(f is a fermion)*

  (1) $|e\rangle =$  $+$  spin 0. (2) $|f = e \otimes m\rangle =$  $-$  spin $1/2$.

  (3) $|m = e \otimes f\rangle = \times -$  spin 0. (4) $|1\rangle = \times +$  spin 0.

- The type-**1** excitation is the tirivial excitation, that can be created by local operators.
  The type-e, type-m, and type-f excitations are non-tirivial excitation, that cannot be created by local operators.

- **1**, e, m are bosons and f is a fermion. e,m, and f have π mutual statistics between them.

- **Fusion rule**:
  $e \otimes e = \mathbf{1}$; $f \otimes f = \mathbf{1}$; $m \otimes m = \mathbf{1}$;
  $e \otimes m = f$; $f \otimes e = m$; $m \otimes f = e$;
  $1 \otimes e = e$; $1 \otimes m = m$; $1 \otimes f = f$;

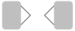
Z_2 topological order is described by Z_2 gauge theory

Physical properties of Z_2 gauge theory
= Physical properties of Z_2 topological order

- Z_2-charge (a representatiosn of Z_2) and Z_2-vortex (π-flux) as two bosonic point-like excitations.
- Z_2-charge and Z_2-vortex bound state $\to$ a fermion (f), since Z_2-charge and Z_2-vortex has a π mutual statistics between them (charge-1 around flux-π).
- Z_2-charge, Z_2-vortex, and their bound state has a π mutual statistics between them.
- Z_2-charge $\to e$, Z_2-vortex $\to m$, bound state $\to f$.
- Z_2 gauge theory on torus also has 4 degenerate ground states

Emergence of fractional spin and semion statistics

Consider another string wave function:

Φ_{str} () $= (-)^{\#\text{ of loops}}$ string liquid. Φ_{str} () $= -\Phi_{\text{str}}$ ()

- End of string wave function: $|\text{end}\rangle = \ + c \ - c \ + \cdots$
- 360° rotation: $\rightarrow$ and $= -$ $\rightarrow -$: $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- **Types** of topological excitations: *($s_\pm$ are semions)*

(1) $|s_+\rangle = \ + \mathrm{i}$ spin $\frac{1}{4}$. (2) $|s_-\rangle = \ - \mathrm{i}$ spin $-\frac{1}{4}$

(3) $|m = s_- \otimes s_+\rangle = \times -$ spin 0. (4) $|1\rangle = \times +$ spin 0.

- **double-semion topological order** $= U^2(1)$ Chern-Simon gauge theory

$L(a_\mu) = \frac{2}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{2}{4\pi} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \epsilon^{\mu\nu\lambda}$

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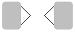
Φ_{str} () $= (-)^{\#\text{ of loops}}$ string liquid. Φ_{str} () $= -\Phi_{str}$ ()

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- Two string lqiuids $\to$ Two topological orders:

Z_2 **topological order** Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner-Sondhi PRL 86 1881 (01) and **double-semion topo. order** Freedman etal cond-mat/0307511, Levin-Wen cond-mat/0404617

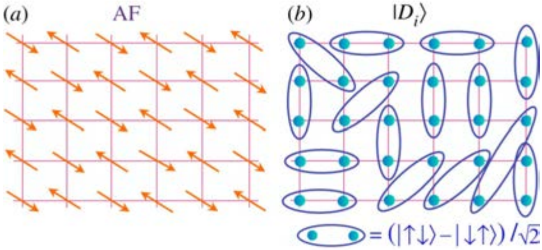
Lattice Hamiltonians to realize Z_2 topological order

- Frustrated spin-1/2 model on square lattice (slave-particle meanfield theory) Read Sachdev, PRL **66** 1773 (91); Wen, PRB **44** 2664 (91).

$$H = J \sum_{nn} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + J' \sum_{nnn} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

- Dimer model on triangular lattice (Mont Carlo numerics)

Moessner Sondhi, PRL **86** 1881 (01)



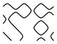
Why dimmer liquid has topological order

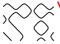
- Dimmer liquid $\sim$ string liquid:

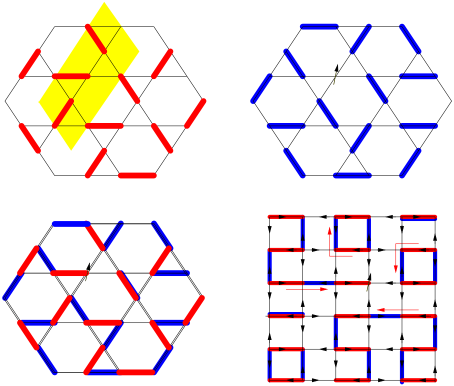
- Non-bipartite lattice: unoritaded string $\to$ Z_2 topological order $=Z_2$ gauge theory

- Bipartite lattice: oriented string $\to$ $U(1)$ gauge theory

- Which local Hamiltonians can realize the following string wavefunctions:

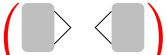
$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum \Big|$  $\Big\rangle$

$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}} \Big|$  $\Big\rangle$

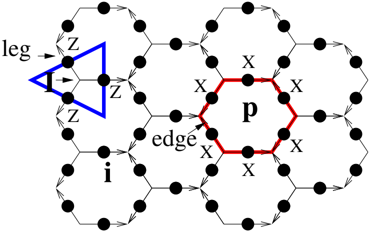


Toric-code model: Z_2 topological order, Z_2 gauge theory

Local Hamiltonian enforces local rules on any lattice: $\hat{P}\Phi_{\text{str}} = 0$

Φ_{str} () $-$ Φ_{str} () $=$ Φ_{str} () $-$ Φ_{str} () $= 0$

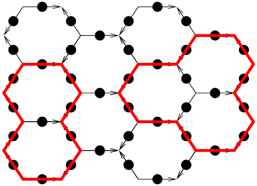
- The Hamiltonian to enforce the local rules: Kitaev quant-ph/9707021

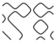


$$H = -U\sum_{\boldsymbol{I}} \hat{Q}_{\boldsymbol{I}} - g\sum_{\boldsymbol{p}} \hat{F}_{\boldsymbol{p}}, \quad \hat{Q}_{\boldsymbol{I}} = \prod_{\text{legs of } \boldsymbol{I}} \sigma^z_{\boldsymbol{i}}, \quad \hat{F}_{\boldsymbol{p}} = \prod_{\text{edges of } \boldsymbol{p}} \sigma^x_{\boldsymbol{i}}$$

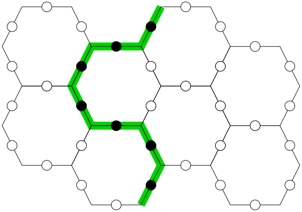
- The Hamiltonian is a sum of commuting operators
  $[\hat{F}_{\boldsymbol{p}}, \hat{F}_{\boldsymbol{p}'}] = 0$, $[\hat{Q}_{\boldsymbol{I}}, \hat{Q}_{\boldsymbol{I}'}] = 0$, $[\hat{F}_{\boldsymbol{p}}, \hat{Q}_{\boldsymbol{I}}] = 0$. $\hat{F}^2_{\boldsymbol{p}} = \hat{Q}^2_{\boldsymbol{I}} = 1$
- Ground state $|\Psi_{\text{grnd}}\rangle$: $\hat{F}_{\boldsymbol{p}}|\Psi_{\text{grnd}}\rangle = \hat{Q}_{\boldsymbol{I}}|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$
  $\rightarrow (1 - \hat{Q}_{\boldsymbol{I}})\Phi_{\text{grnd}} = (1 - \hat{F}_{\boldsymbol{p}})\Phi_{\text{grnd}} = 0$.

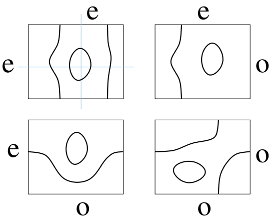
A string picture



- The $-U\sum_I \hat{Q}_I$ term enforces closed-string ground state.
- $\hat{F}_p$ adds a small loop and deform the strings →
  permutes among the loop states $\Big|$$\Big\rangle$ → Ground states

  $|\Psi_{\text{grnd}}\rangle = \sum_{\text{loops}}$ $\Big|$$\Big\rangle$ → highly entangled
- There are four degenerate ground states $\alpha = ee, eo, oe, oo$





$D^{\text{tor}}=4$

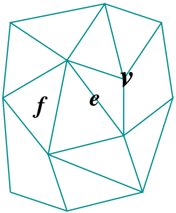
- On genus g surface, ground state degeneracy $D_g = 4^g$

Exactly soluble model on any graph

- On every link $\boldsymbol{i}$, we degrees of freedom $\uparrow, \downarrow$.

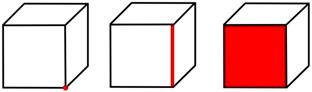
$$H = -U \sum_{\boldsymbol{v}} \hat{Q}_{\boldsymbol{v}} - g \sum_{\boldsymbol{f}} \hat{F}_{\boldsymbol{f}},$$

$$\hat{Q}_{\boldsymbol{v}} = \prod_{\text{legs of } \boldsymbol{v}} \sigma^z_{\boldsymbol{e}}, \quad \hat{F}_{\boldsymbol{f}} = \prod_{\text{edges of } \boldsymbol{f}} \sigma^x_{\boldsymbol{e}}$$



The Hamiltonian is a sum of commuting operators
$[\hat{F}_{\boldsymbol{f}}, \hat{F}_{\boldsymbol{p}'}] = 0$, $[\hat{Q}_{\boldsymbol{v}}, \hat{Q}_{\boldsymbol{v}'}] = 0$, $[\hat{F}_{\boldsymbol{f}}, \hat{Q}_{\boldsymbol{v}}] = 0$. $\hat{F}^2_{\boldsymbol{f}} = \hat{Q}^2_{\boldsymbol{v}} = 1$

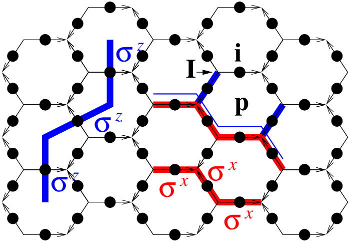
- Identities $\otimes_{\boldsymbol{v}} \hat{Q}_{\boldsymbol{v}} = 1$, $\otimes_{\boldsymbol{f}} \hat{F}_{\boldsymbol{f}} = 1$.



- Ground state degeneracy (GSD)
  Number of degrees of freedom $= E$.
  Number of constraints $= V + F - 2$.
  $GSD = 2^E / 2^{V+F-2} = 2^{2-\chi}$, $\chi = V - E + F$ – Euler characteristic.
- GSD on genus g Riemann surface Σ_g: from $\chi(\Sigma_g) = 2 - 2g$ we obtain $GSD = 2^{2g}$. In fact, the degeneracy of any eigenstates is 2^g.

The string operators and topological excitations

- Topological excitations:
  e-type: $\hat{Q}_{\boldsymbol{I}} = 1 \to \hat{Q}_{\boldsymbol{I}} = -1$
  m-type: $\hat{F}_{\boldsymbol{p}} = 1 \to \hat{F}_{\boldsymbol{p}} = -1$
- e-type and m-type excitations cannot be created alone due to identiy: $\prod_{\boldsymbol{I}} \hat{Q}_{\boldsymbol{I}} = \prod_{\boldsymbol{p}} \hat{F}_{\boldsymbol{p}} = 1$



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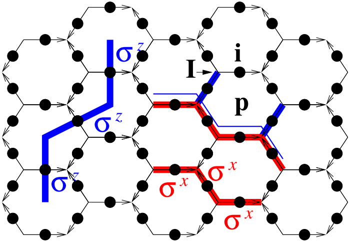
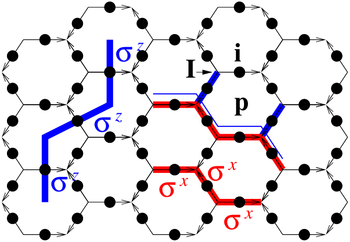


- Type-e string operator: $W_e = \prod_{\text{string}} \sigma_i^x$
- Type-m string operator: $W_m = \prod_{\text{string*}} \sigma_i^z$
- Type-f string operator: $W_f = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$
- $[H, W_e^{\text{close}}] = [H, W_m^{\text{close}}] = [H, W_f^{\text{closed}}] = 0$.
  $\to$ Closed strings cost no energy ($\to$ higher symmetry)
- $[\hat{Q}_{\boldsymbol{I}}, W_e^{\text{open}}] \neq 0 \to W_e^{\text{open}}$ flip $\hat{Q}_{\boldsymbol{I}} \to -\hat{Q}_{\boldsymbol{I}}$,
  $[\hat{F}_{\boldsymbol{p}}, W_m^{\text{open}}] \neq 0 \to W_m^{\text{open}}$ flip $\hat{F}_{\boldsymbol{p}} \to -\hat{F}_{\boldsymbol{p}}$
  An open-string creates a pair of topo. excitations at its ends

Three types of topological excitations and their fusion

- Type-e string operator $W_e = \prod_{\text{string}} \sigma_i^x$
- Type-m string operator $W_m = \prod_{\text{string*}} \sigma_i^z$
- Type-f string operator $W_f = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$
- **Fusion algebra** of string operators $W_e^2 = W_m^2 = W_\epsilon^2 = W_e W_m W_\epsilon = 1$ when strings are parallel
- Fusion of topo. excitations:
  e-type. $e \times e = 1$
  m-type. $m \times m = 1$
  f-type $= e \times m$
- 4 types of excitations:
  $1, e, m, f$



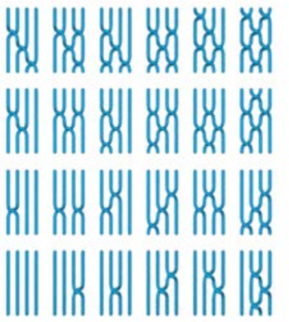
What are bosons? What are fermions?

- **Statistical distribution**
  Boson: $n_b = \frac{1}{e^{\epsilon/k_B T} - 1}$ Fermion: $n_f = \frac{1}{e^{\epsilon/k_B T} + 1}$
  They are just properties of non-interacting bosons or fermions
- **Pauli exclusion principle**
  Only works for non-interacting bosons or fermions
- **Symmtric/anti-symmetric wave function**.
  For identical particles $|x, y\rangle$ and $|y, x\rangle$ are just differnt names of same state. A generic state $\sum_{x,y} \psi(x, y)|x, y\rangle$ is always described symmetric wave function $\psi(x, y) = \psi(y, x)$ regardless the statistics of the identical particles.
- **Commuting/anti-commuting operators**
  Boson: $[a_x, a_y] = 0$ Fermiion: $\{c_x, c_y\} = 0$
- **C-number-field/Grassmann-field**
  Boson: $\phi(x)$ Fermion: $\psi(x)$

"Exchange" statistics and Braid group

- Quantum statistics is defined via phases induced by exchanging identical particles.
- *Quantum statistics is not defined via exchange, but via braiding.*

  Yong-Shi Wu, PRL **52** 2103 (84)
- **Braid group:**



- Representations of braid group (not permutation group) define quantum statistics:

- Trivial representation of braid group → Bose statistics.

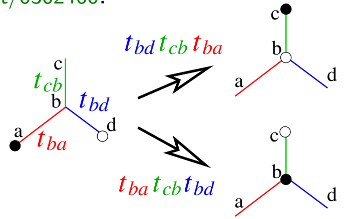
- 1-dimensional representation of braid group → Fermi/fractional statistics → **anyon**. Leinaas-Myrheim 77; Wilczek 82

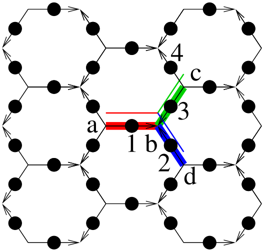
- higher dimensional representation of braid group → non-Abelian statistics → **non-Abelian anyon**. Wen 91; More-Read 91

Statistics of ends of strings

- The statistics is determined by particle hopping operators

Levin-Wen cond-mat/0302460:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string op. determines the statistics.
- For type-e string: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$

  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$

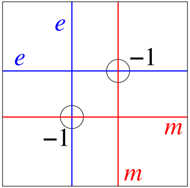
  The ends of type-e string are bosons
- For type-f strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x}\sigma_4^z$, $t_{bd} = \sigma_2^x\underline{\sigma_3^z}$

  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$

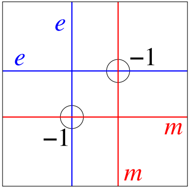
  The ends of type-f strings are fermions

Topological ground state degeneracy and code distance

- When strings cross, $W_e W_m = (-)^{\#\text{ of cross}} W_m W_e$
  $\to$ 4^g degeneracy on genus g surface
  $\to$ **Topological degneracy**
  Degeneracy remain exact for any perturbations localized in a finite region.



Topological ground state degeneracy and code distance



- When strings cross, $W_e W_m = (-)^{\# \text{ of cross}} W_m W_e$
  → 4^g degeneracy on genus g surface
  → **Topological degneracy**
  Degeneracy remain exact for any perturbations localized in a finite region.
- The above degenerate ground states form a "code", which has a large **code distance** of order L (the size of the system).
- Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by first-order local perturbation δH: $\langle\psi'|\delta H|\psi\rangle > O(|\delta H|), \quad L \to \infty$
  → code distance $= 1$.

  Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by n^{th}-order local perturbation → code distance $= n$.
- Symmetry breaking ground states in d-dim have code distance $\sim L^d$ respected to symmetry preserving perturbation. code distance ~ 1 respected to symmetry breaking perturbation.

Higher symmetry

- The toric code model has higher symmetry (1-symmetry), whose symmetry transformation is generated the loop operators W_e^{loop} and W_m^{loop}:

$$HW_e(S^1) = W_e(S^1)H, \qquad HW_m(S^1) = W_m(S^1)H.$$

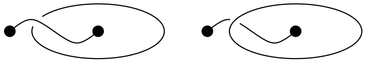
for any loops S^1. If the transformation is n-dimensional, the symmetry is $(d-n)$-symmetry, in d-dimensional space. The transformation is d-dimensional for the usual global symmetry, which is a 0-symmetry.

- **Charged operator** (for Abelian symmetry):

$$WO_{\text{charged}} = e^{i\varphi} O_{\text{charged}} W$$

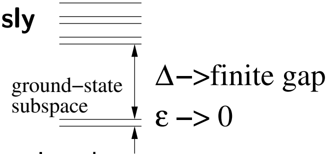
For $U(1)$ symmetry, $\varphi = q\theta$ if W generate θ-rotation. For Z_2 symmetry, $\varphi = \pi$ if W is the generator.

- W_m(open-string) is the charged operators for the $W_e(S^1)$ 1-symmetry:

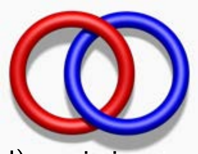


$$W_e(S^1)W_m(\text{open-string}) = \pm W_m(\text{open-string})W_e(S^1).$$

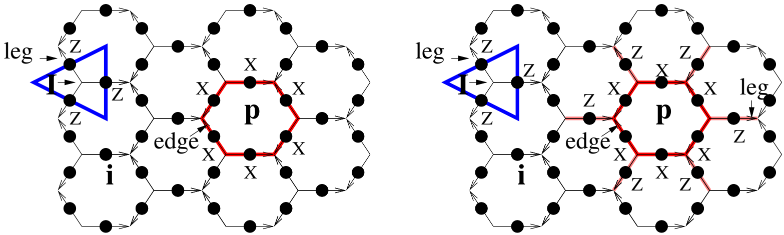
Spontaneous breaking of higher symmetry



- **Defintion**: A (higher) symmetry is **spontaneously broken** if the symmetry transformations have non-trivial actions on the ground states, *ie* is not proportional to an identity operator $W \neq \mathrm{e}^{\mathrm{i}\varphi}\mathrm{id}$ in the ground state subspace, for any closed space.
- The toric code model has a W_e 1-symmetry (Z_2^e 1-symmetry). Its ground states spontaneously breaks the Z_2^e 1-symmetry.
- The toric code model has a W_m 1-symmetry (Z_2^m 1-symmetry). Its ground states spontaneously breaks the Z_2^m 1-symmetry.
- **Spondtaneous breaking of higher symmetry $\to$ topological order**
  But, topological order $\neq$ Spondtaneous breaking of higher symmetry
- The toric code model has a $Z_2^e \vee Z_2^m$ 1-symmetry. Its ground states must spontaneously break the $Z_2^e \vee Z_2^m$ 1-symmetry $\to$ **Enforaced spontaneous symmetry breaking** when ends of the symmetry transformation operators (*ie* the strings W_e, W_m) have non-trivial (mutual) statistics.



Toric-code model in terms of closed string operators



- Toric-code Hmailtonian

$$H = -U \sum_{I} W_m^{\text{closed}} - g \sum_{p} W_e^{\text{closed}}$$

- A new Hamitonian

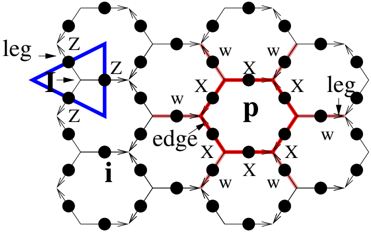
$$H = -U \sum_{I} W_m^{\text{closed}} - g \sum_{p} W_f^{\text{closed}}$$

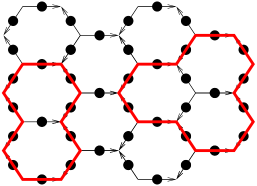
which realizes the same Z_2 topological order.

Double-semion model: taking square root of fermion string

Local rules: Levin-Wen cond-mat/0404617

- The Hamiltonian to enforce the local rules:





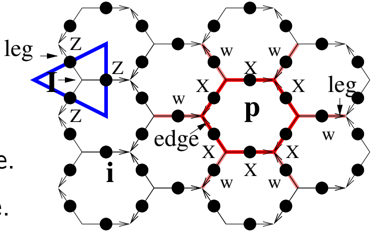
$$H = -U\sum_I \hat{Q}_I - \frac{g}{2}\sum_p (\hat{F}_p + h.c.), \qquad \mathrm{i}^{\frac{1-\sigma_i^z}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} = w_i \sim \sqrt{\sigma_i^z}$$

$$\hat{Q}_I = \prod_{\text{legs of } I} \sigma_i^z, \quad \hat{F}_p = \Big(\prod_{\text{edges of } p} \sigma_j^x\Big)\Big(-\prod_{\text{legs of } p} \mathrm{i}^{\frac{1-\sigma_i^z}{2}}\Big)$$

Double-semion model

- The action of operator $\hat{F}_{\boldsymbol{p}} = (\prod_{\text{edges of } \boldsymbol{p}} \sigma_{\boldsymbol{j}}^x)(-\prod_{\text{legs of } \boldsymbol{p}} \mathrm{i}^{\frac{1-\sigma_{\boldsymbol{i}}^z}{2}})$:
  (1) flip string around the loop;
  (2) add a phase $-(\mathrm{i}^{\#\text{ of strings attatched to the loop}})$,
  which is ± 1 in the closed-string subspace.

  Combine the above two in the closed-string subspace:
  $\hat{F}_{\boldsymbol{p}}$ adds a loop and a sign $(-)^{\text{change in }\#\text{ of loops}}$

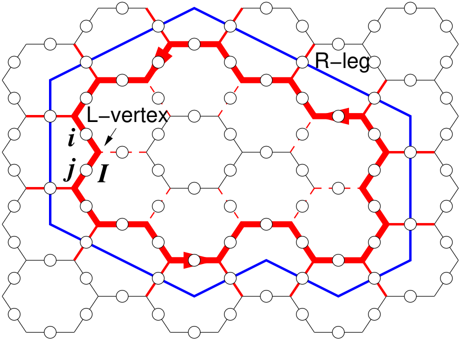


This allows us to conclude:

- $\hat{F}_{\boldsymbol{p}}$ is hermitian in the closed-string subspace.
- $\hat{F}_{\boldsymbol{p}}\hat{F}_{\boldsymbol{p}'} = \hat{F}_{\boldsymbol{p}'}\hat{F}_{\boldsymbol{p}}$ in the closed-string subspace.
- Ground state wave function $\Phi(X) = (-)^{\#\text{ of loops}}$.

Dressed string operators and topological excitations

- To create a pair of topological excitations, we need find closed string operators that commute with $\hat{Q}_I$ and $\hat{F}_p$ terms in the Hamiltonian.
- We find 4 types of string operators



$$W_{\mathbf{1}} = \mathrm{id},$$

$$W_{s_1} = \prod_{i\in\text{str}} \sigma_i^x \prod_{\text{R-legs of str}} \mathrm{i}^{\frac{1-\sigma_j^z}{2}} \prod_{\text{L-vertices of str}} (-)^{s_I}$$

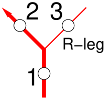
$$W_{s_2} = \prod_{i\in\text{str}} \sigma_i^x \prod_{\text{R-legs of str}} (-\mathrm{i})^{\frac{1-\sigma_j^z}{2}} \prod_{\text{L-vertices of str}} (-)^{s_I} = W_{s_1} W_b$$

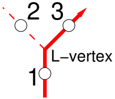
$$W_b = \prod_{\text{R-legs of str}} \sigma_j^z = W_m, \qquad \text{where } s_I = \frac{1}{4}(1-\sigma_{I-}^z)(1+\sigma_{I+}^z)$$

Levin-Wen cond-mat/0404617

Commutators of dressed string operators W_{s_1}

Overlapped strings are in the same direction:



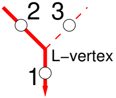


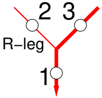
$$\Big[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\Big]\Big[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\Big]$$

$$=\Big[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1+\sigma_3^z}{2}}\,\mathrm{i}^{-\sigma_3^z}\Big]\Big[\sigma_1^x\sigma_3^x(-)^{\frac{(1+\sigma_1^z)(1+\sigma_3^z)}{4}}(-)^{-\frac{\sigma_1^z(1+\sigma_3^z)}{2}}\Big]$$

$$=\Big[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\Big]\Big[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\Big]\ \mathrm{i}^{\sigma_3^z}(-)^{-\frac{(1+\sigma_3^z)}{2}}$$

$$=\Big[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\Big]\Big[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\Big]\ \mathrm{i}$$





$$\Big[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\Big]\Big[\sigma_3^x\sigma_1^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\Big]$$

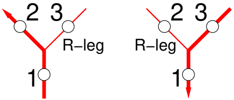
$$=\Big[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1-\sigma_1^z)}{4}}(-)^{\frac{\sigma_1^z(1-\sigma_2^z)}{2}}\Big]\Big[\sigma_3^x\sigma_1^x\,\mathrm{i}^{\frac{1+\sigma_2^z}{2}}\,\mathrm{i}^{-\sigma_2^z}\Big]$$

$$=\Big[\sigma_3^x\sigma_1^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\Big]\Big[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\Big]\ (-)^{-\frac{\sigma_1^z(1-\sigma_2^z)}{2}}\,\mathrm{i}^{-\sigma_2^z}$$

$$=\Big[\sigma_3^x\sigma_1^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\Big]\Big[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\Big]\ (-\mathrm{i})$$

Commutators of dressed string operators W_{s_1}

Overlapped strings are in opposite direction:

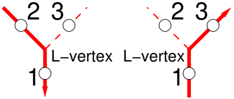


$$\left[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\right]\left[\sigma_1^x\sigma_3^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\right]$$

$$=\left[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1+\sigma_3^z}{2}}\,\mathrm{i}^{-\sigma_3^z}\right]\left[\sigma_1^x\sigma_3^x\,\mathrm{i}^{\frac{1+\sigma_2^z}{2}}\,\mathrm{i}^{-\sigma_2^z}\right]$$

$$=\left[\sigma_1^x\sigma_3^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\right]\left[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\right]\;\mathrm{i}^{\sigma_3^z}\,\mathrm{i}^{-\sigma_2^z}$$

$$=\left[\sigma_1^x\sigma_3^x\,\mathrm{i}^{\frac{1-\sigma_2^z}{2}}\right]\left[\sigma_1^x\sigma_2^x\,\mathrm{i}^{\frac{1-\sigma_3^z}{2}}\right]\;\sigma_3^z\sigma_2^z$$



$$\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\right]\left[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\right]$$

$$=\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1-\sigma_1^z)}{4}}(-)^{\frac{\sigma_1^z(1-\sigma_2^z)}{2}}\right]\left[\sigma_1^x\sigma_3^x(-)^{\frac{(1+\sigma_1^z)(1+\sigma_3^z)}{4}}(-)^{-\frac{\sigma_1^z(1+\sigma_3^z)}{2}}\right]$$

$$=\left[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\right]\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\right]\;(-)^{-\frac{\sigma_1^z(1-\sigma_2^z)}{2}}(-)^{-\frac{\sigma_1^z(1+\sigma_3^z)}{2}}$$

$$=\left[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\right]\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\right]\;\sigma_2^z\sigma_3^z$$

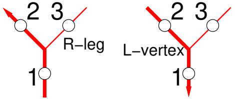
Commutators of dressed string operators W_{s_1}

$$\left[\sigma_1^x\sigma_2^x\, i^{\frac{1-\sigma_3^z}{2}}\right]\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\right]$$

$$=\left[\sigma_1^x\sigma_2^x\, i^{\frac{1-\sigma_3^z}{2}}\right]\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1+\sigma_2^z)(1-\sigma_1^z)}{4}}(-)^{\frac{\sigma_1^z-\sigma_2^z}{2}}\right]$$

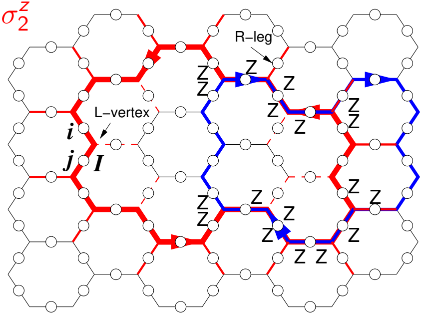
$$=\left[\sigma_2^x\sigma_1^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_1^z)}{4}}\right]\left[\sigma_1^x\sigma_2^x\, i^{\frac{1-\sigma_3^z}{2}}\right](-)^{\frac{\sigma_1^z-\sigma_2^z}{2}}$$

$$=\left[\sigma_1^x\sigma_3^x(-)^{\frac{(1-\sigma_1^z)(1+\sigma_3^z)}{4}}\right]\left[\sigma_1^x\sigma_2^x\, i^{\frac{1-\sigma_3^z}{2}}\right]\ \sigma_1^z\sigma_2^z$$



Overlapped strings are in opposite direction

- Different loops of W_{s_1}-string operators commute in the closed string subspace, shown by collecting the "phase factors" $\sigma_i^z = Z_i$.
- Loops of W_{s_1}-string operators commute with $\hat{Q}_I$.

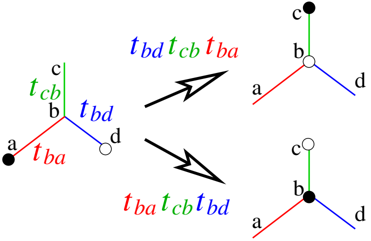


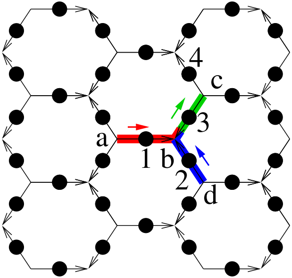
We can use $\hat{Q}_I$ and loops of W_{s_1}-string operators to construct a soluble Hamiltonian, and which is what we have before.

Statistics of ends of dressed strings

- The statistics is determined by particle hopping operators

  Levin-Wen cond-mat/0302460:





- For dressed strings: $t_{ba} = \sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x (-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}$

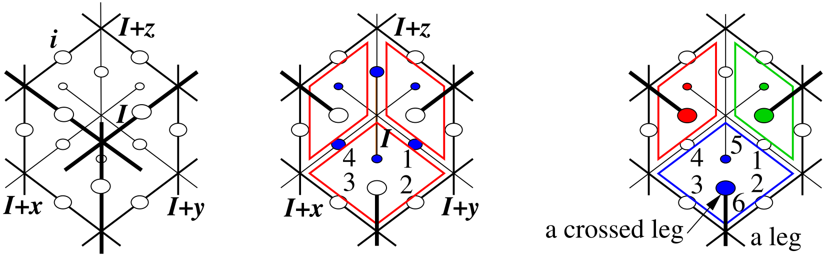
  We find $t_{bd}t_{cb}t_{ba} = -\mathrm{i}\, t_{ba}t_{cb}t_{bd}$ via

$$[\sigma_2^x (-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}][\sigma_3^x][\sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}] = [\sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}][\sigma_3^x][\sigma_2^x (-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}](-\mathrm{i})$$

The end of string is a semion.

The computation

$$[\sigma_2^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}][\sigma_3^x][\sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}]$$
$$= [\sigma_2^x(-)^{\frac{(1-\sigma_2^z)(1-\sigma_3^z)}{4}}(-)^{\frac{\sigma_3^z(1-\sigma_2^z)}{2}}][\sigma_3^x][\sigma_1^x \mathrm{i}^{\frac{1+\sigma_2^z}{2}} \mathrm{i}^{-\sigma_2^z}]$$
$$= [\sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}][\sigma_3^x][\sigma_2^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}]\,(-)^{-\frac{\sigma_3^z(1-\sigma_2^z)}{2}}\mathrm{i}^{-\sigma_2^z}$$
$$= [\sigma_1^x \mathrm{i}^{\frac{1-\sigma_2^z}{2}}][\sigma_3^x][\sigma_2^x(-)^{\frac{(1-\sigma_2^z)(1+\sigma_3^z)}{4}}]\,(-\mathrm{i})$$



- Untwisted-string model: $H = -U \sum_I Q_I - g \sum_p F_p$

$$Q_I = \prod_{i \text{ next to } I} \sigma_i^z, \quad F_p = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$$

Can get 3D fermions for free (almost) Levin-Wen cond-mat/0302460

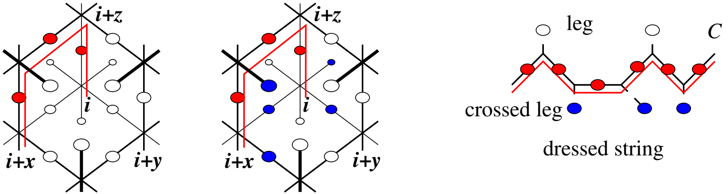
Just add a little twist

- Twisted-string model: $H = U \sum_I Q_I - g \sum_p F_p$

$$F_p = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^z \sigma_6^z$$

String operators and Z_2 charges Levin-Wen cond-mat/0302460

- A pair of Z_2 charges is created by an open string operator which commute with the Hamiltonian except at its two ends.
Strings cost no energy and is unobservable.



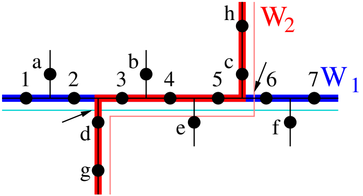
- In untwisted-string model – untwisted-string operator

$$\sigma^x_{\mathbf{i}_1}\sigma^x_{\mathbf{i}_2}\sigma^x_{\mathbf{i}_3}\sigma^x_{\mathbf{i}_4}\cdots$$

- In twisted-string model – twisted-string operator

$$(\sigma^x_{\mathbf{i}_1}\sigma^x_{\mathbf{i}_2}\sigma^x_{\mathbf{i}_3}\sigma^x_{\mathbf{i}_4}\dots)\prod_{\mathbf{i}\text{ on crossed legs of } C}\sigma^z_{\mathbf{i}}$$

Twisted string operators commute $[W_1, W_2] = 0$



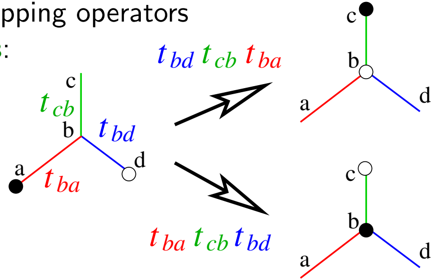
$$W_1 = (\sigma_1^x\sigma_2^x\sigma_3^x\sigma_4^x\sigma_5^x\boldsymbol{\sigma}_6^x\sigma_7^x)\ [\boldsymbol{\sigma}_d^z\sigma_e^z\sigma_f^z]$$
$$W_2 = (\sigma_h^x\sigma_c^x\sigma_5^x\sigma_4^x\sigma_3^x\boldsymbol{\sigma}_d^x\sigma_g^x)\ [\boldsymbol{\sigma}_6^z\sigma_e^z]$$

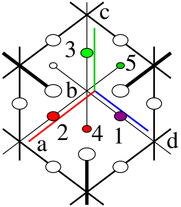
- We also have $[W, Q_I] = 0$ for closed string operators W, since W only create closed strings.

Statistics of ends of twisted strings

- The statistics is determined by particle hopping operators

Levin-Wen 03:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string op. determine the statistics.
- For untwisted-string model: $t_{ba} = \sigma^x_2$, $t_{cb} = \sigma^x_3$, $t_{bd} = \sigma^x_1$

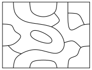
  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$

  The ends of untwisted-string are bosons
- For twisted-string model: $t_{ba} = \sigma^z_4\sigma^z_1\sigma^x_2$, $t_{cb} = \sigma^z_5\sigma^x_3$, $t_{bd} = \sigma^x_1$

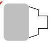
  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$

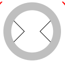
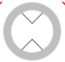
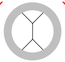
  The ends of twisted-string are fermions

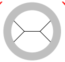
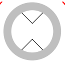
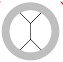
Ground state:

- String-net liquid: allow three strings to join, but do not allow a string to end $\Phi_{str}\Big($$\Big)$

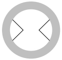
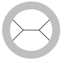
Levin-Wen cond-mat/0404617

- The dancing rule : $\Phi_{str}\Big($$\Big) = \Phi_{str}\Big($$\Big)$

$\Phi_{str}\Big($$\Big) = a\,\Phi_{str}\Big($$\Big) + b\,\Phi_{str}\Big($$\Big)$

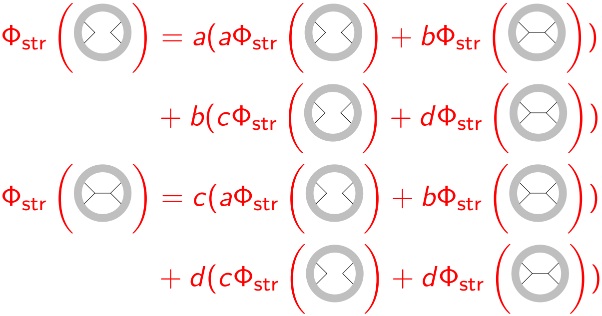
$\Phi_{str}\Big($$\Big) = c\,\Phi_{str}\Big($$\Big) + d\,\Phi_{str}\Big($$\Big)$

\- The above is a relation between two orthogonal basis: two local resolutions of how four strings join (quantum geometry)

,  and , , $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ = orthogonal matrix

$$a^2 + b^2 = 1, \quad ac + bd = 0, \quad ca + db = 0, \quad c^2 + d^2 = 1$$

Apply reconnection rule twice:

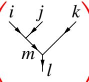


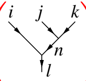
We find

$$a^2 + bc = 1, \quad ab + bd = 0, \quad ac + dc = 0, \quad bc + d^2 = 1$$

$$\to d = -a, \quad b = c, \quad a^2 + b^2 = 1.$$

- Rewrite the string reconnection rule (0→no-string, 1→string)

$$\Phi\left(\right.$$ 

$$\left. \right) = \sum_{n=0}^{1} F^{ijm}_{kln} \Phi \left(\right.$$ 

$$\left. \right), \qquad i,j,k,l,m,n = 0,1$$

The 2-by-2 matrix $F^{ij}_{kl} \to (F^{ij}_{kl})^m_n$ is unitary. We have

F^{000}_{000}  $= 1$

F^{000}_{111}  $= (F^{011}_{100}$  $)^* = (F^{101}_{010}$  $)^* = F^{110}_{001}$  $= 1$

F^{011}_{011}  $= (F^{101}_{101}$  $)^* = 1$

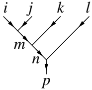
F^{011}_{111}  $= (F^{101}_{111}$  $)^* = F^{111}_{011}$  $= (F^{111}_{101}$  $)^* = 1$

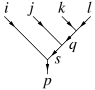
F^{110}_{110}  $= a$

F^{110}_{111}  $= b = (F^{111}_{110}$  $)^* = c^*$

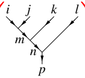
F^{111}_{111}  $= d = -a,$

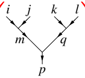
More self consistency condition

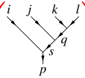
- 

can be trans. to 

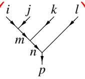
through two different paths:

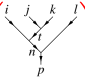
$\Phi\Big($ 

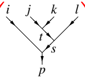
$\Big) = \sum_q F^{mkn}_{lpq} \Phi\Big($ 

$\Big) = \sum_{q,s} F^{mkn}_{lpq} F^{ijm}_{qps} \Phi\Big($ 

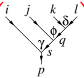
$\Big),$

$\Phi\Big($ 

$\Big) = \sum_t F^{ijm}_{knt} \Phi\Big($ 

$\Big) = \sum_{t,s} F^{ijm}_{knt} F^{itn}_{lps} \Phi\Big($ 

$\Big)$

$= \sum_{t,s,q} F^{ijm}_{knt} F^{itn}_{lps} F^{jkt}_{lsq} \Phi\Big($ 

$\Big).$

- The two paths should lead to the same relation

$$\sum_t F^{ijm}_{knt} F^{itn}_{lps} F^{jkt}_{lsq} = F^{mkn}_{lpq} F^{ijm}_{qps}$$

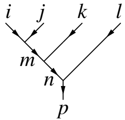
Such a set of non-linear algebraic equations is the famous pentagon identity.

The pentagon identity

- $i, j, k, l, p, m, n, q, s = 0, 1 \rightarrow$
  $2^9 = 512+$ non-linear equations with $2^6 = 64$ unknowns.

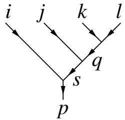
- Solving the pentagon identity: choose $i, j, k, l, p = 1$

$$\sum_{t=0,1} F^{11m}_{1nt} F^{1tn}_{11s} F^{11t}_{1sq} = F^{m1n}_{11q} F^{11m}_{q1s}$$



choose $n, q, s = 1, m = 0$

$$\sum_{t=0,1} F^{110}_{11t} F^{1t1}_{111} F^{11t}_{111} = F^{011}_{111} F^{110}_{111}$$



$$\rightarrow a \times 1 \times b + b \times (-a) \times (-a) = 1 \times b$$

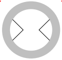
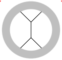
$$\rightarrow a + a^2 = 1, \qquad \rightarrow a = (\pm\sqrt{5} - 1)/2$$

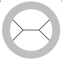
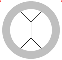
Since $a^2 + b^2 = 1$, we find

$$a = (\sqrt{5} - 1)/2 \equiv \gamma, \quad b = \sqrt{a} = \sqrt{\gamma}$$

String-net dancing rule

- The dancing rule : $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = \gamma\Phi_{\text{str}}\Big($  $\Big) + \sqrt{\gamma}\,\Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = \sqrt{\gamma}\,\Phi_{\text{str}}\Big($  $\Big) - \gamma\Phi_{\text{str}}\Big($  $\Big)$

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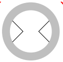
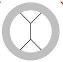
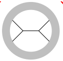
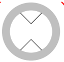
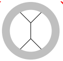
$\Phi_{\text{str}}\Big($  $\Big) = \gamma\Phi_{\text{str}}\Big($  $\Big) + \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big) - \gamma\Phi_{\text{str}}\Big($  $\Big)$

- **Topological excitations**:
For fixed 4 ends of string-net on a sphere S^2, how many locally indistinguishable states are there?

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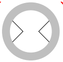
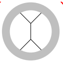
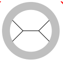
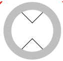
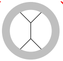
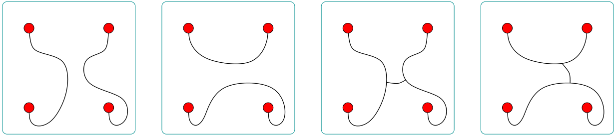
$\Phi_{str}\Big($  $\Big) = \gamma\Phi_{str}\Big($  $\Big) + \sqrt{\gamma}\Phi_{str}\Big($  $\Big)$

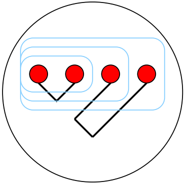
$\Phi_{str}\Big($  $\Big) = \sqrt{\gamma}\Phi_{str}\Big($  $\Big) - \gamma\Phi_{str}\Big($  $\Big)$

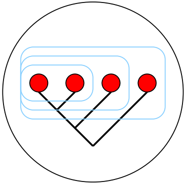
- **Topological excitations**:
  For fixed 4 ends of string-net on a sphere S^2, how many locally indistinguishable states are there? **four states?**



Topological degeneracy with 4 fixed ends of string-net

To get linearly independent states, we fuse the end of the string-net in a particular order:





→ There are only **two** locally indistinguishable states
= a qubit

This is a quantum memory that is robust angainst any environmental noise.

→ The defining character of topological order:
a material with robust quantum memory.

Direct sum $\oplus$ = accidental degeneracy

- Consider two spin-$\frac{1}{2}$ particles.
  If we view the two particle as one particle spin-$\frac{1}{2} \otimes$ spin-$\frac{1}{2}$ =?
  What is the spin of the bound state?

- The bound state is a degeneracy of spin-0 particle and spin-1 particle:

$$\text{spin-}\frac{1}{2} \otimes \text{spin-}\frac{1}{2} = \text{spin-}0 \oplus \text{spin-}1, \quad 2 \times 2 = 1 + 3.$$

$\oplus$ is the **direct sum** of Hilbert space in mathematics
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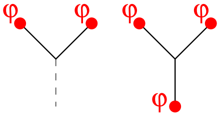

- Fusion of the ends of string-net φ:

$$\varphi\otimes\varphi=\mathbf{1}\oplus\varphi,\quad \varphi\otimes\varphi\otimes\varphi=(\mathbf{1}\oplus\varphi)\otimes\varphi=\mathbf{1}+2\varphi.$$

A bound state of 2 φ's = an accidentical degeneracy of an $\mathbf{1}$ and a φ.
A bound state of 3 φ's = an accidentical degeneracy of an $\mathbf{1}$, a φ, and a φ.

Compute the degeneracy of excitations on S^2

Consider n topological excitations (string ends) on a sphere. What is the ground state degeneracy? ($GSD = 0$ means not allowed)

- Consider the loop liquid (*ie* the Z_2 topological order).

- Trivial particle $\mathbf{1} \to$ a state with no string ends, allowed $GSD = 1$.
- One e particles $\to$ a state with 1 string ends, not allowed $GSD = 0$.
- Two e particles $\to$ a state with 2 string ends, allowed $GSD = 1$.
- Three e particles $\to$ a state w/ 3 string ends, not allowed $GSD = 0$.
- Fusion $e \otimes e = \mathbf{1}$, $e \otimes e \otimes e = e \to$ GSD $= \#$ of $\mathbf{1}$'s.

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- Consider the string-net liquid.

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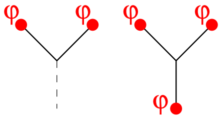

Internal degrees of freedom – quantum dimension

- Let D_n be the number of locally indistinguishable states for n φ-particles on a sphere. The internal degrees of freedom of φ – **quantum dimension** – $d = \lim_{n\to\infty} D_n^{1/n}$

$$\underbrace{\varphi \otimes \cdots \otimes \varphi}_{n} = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{D_n} \oplus \underbrace{\varphi \oplus \cdots \oplus \varphi}_{F_n}$$

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$$\underbrace{\varphi \otimes \cdots \otimes \varphi}_{n} \otimes \varphi = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{F_n} \oplus \underbrace{\varphi \oplus \cdots \oplus \varphi}_{F_n + D_n}$$

$$D_{n+1} = F_n, \quad F_{n+1} = F_n + D_n = F_n + F_{n-1}, \qquad D_1 = 0, \ F_1 = 1.$$

The internal degrees of freedom of φ is (spin-$\frac{1}{2}$ electron $d = 2$)

$$d = \lim_{n\to\infty} F_{n-1}^{1/n} = \frac{1+\sqrt{5}}{2} = 1.61803398874989\cdots .$$

Double-Fibonacci topological order = double G_2 Chern-Simon theory at level 1

$$
\begin{aligned}
L(a_\mu, \tilde{a}_\mu) = {} & \frac{1}{4\pi}\mathrm{Tr}(a_\mu \partial_\nu a_\lambda + \frac{\mathrm{i}}{3} a_\mu a_\nu a_\lambda)\epsilon^{\mu\nu\lambda} \\
& - \frac{1}{4\pi}\mathrm{Tr}(\tilde{a}_\mu \partial_\nu \tilde{a}_\lambda + \frac{\mathrm{i}}{3} \tilde{a}_\mu \tilde{a}_\nu \tilde{a}_\lambda)\epsilon^{\mu\nu\lambda}
\end{aligned}
$$

a_μ and $\tilde{a}_\mu$ are G_2 gauge fields.

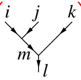
String-net liquid can also realize a gauge theory of a finite group G

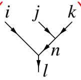
- Trivial type-0 string → trivial represental of G

- Type-i string → irreducible represental R_i of G

- Triple-string join rule If $R_i \otimes R_j \otimes R_k$ contain trivial representation → type-i type-j type-k strings can join.

- String reconnection rule:

$\Phi\Big($ 

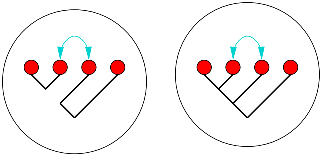
$\Big) = \sum_{n=0}^{1} F^{ijm}_{kln} \Phi \Big($ 

$\Big), \qquad i, j, k, l, m, n = 0, 1$

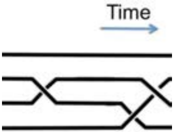
with F^{ijm}_{kln} given by the 6-j simple of G.

Topo. qubits and topo. quantum computation

- Four fixed Fibonacci anyons on S^2 has 2-fold **topological degeneracy** (two locally indistinguishable states) → **topological qubit**



- Exchange two Fibonacci anyons induce a 2×2 unitary matrix acting on the topological qubit → **non-Abelian statistics**
  also appear in $\chi^3_{\nu=2}(z_i)$ FQH state, and the non-Abelian statistics is described by $SU_2(3)$ CS theory Wen PRL **66** 802 (91)
  → universal **Topo. quantum computation** (via CS theory)



Freedman-Kitaev-Wang quant-ph/0001071; Freedman-Larsen-Wang quant-ph/0001108

Topological order is the natural medium (the "silicon") to do topological quantum computation