Highly entangled quantum many-body systems
SPT order in free fermion systems

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Understand (classify) Chern insulators systematically

First, we try to systematically understand (classify) gapped 0+1D free fermion system with $U(1)$ symmetry (fermion number conservation).

- 0+1D free fermion system with $U(1)$ symmetry is described by the following many-body Hamiltonian

$$\hat{H} = \sum_{ab} M_{ab} \hat{c}_a^\dagger \hat{c}_b$$

It is fully characterized by a $N \times N$ hermitian matrix $M = M^\dagger$. So we will concentrate on the matrix M. Eigenvalues of M are called the single-body energy level.

- The many-body ground state has all the negative single-body energy levels filled.
- Gapped $\to$ M has no zero eigenvalue. Space of 0+1D gapped free fermion system with $U(1)$ symmetry $\tilde{C}_0 =$ space of hermitian matrices with no zero eigenvalue.

Classify gapped phases of 0+1D free fermions with $U(1)$

- Gapped phases of 0+1D free fermions with $U(1)$ symmetry are labeled by $\pi_0(\tilde{\mathcal{C}}_0)$ = disconnected parts of the space of hermitian matrices with no zero eigenvalue.
- Let $\mathcal{C}_0$ = the space of hermitian matrices with eigenvalue ± 1. $\tilde{\mathcal{C}}_0$ and $\mathcal{C}_0$ are homotopic equivalent (one can deform into the other without closing gap, like "a point $\sim$ a ball"): $\pi_n(\tilde{\mathcal{C}}_0) = \pi_n(\mathcal{C}_0)$
  Gapped phases of 0+1D free fermions with $U(1)$ symmetry are labeled by $\pi_0(\mathcal{C}_0)$ = disconnected parts of the space of hermitian matrices with eigenvalues ± 1.

- Hermitian matrices with eigenvalues ± 1 has a form
  $U_{n+m} \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix} U_{n+m}^\dagger$. $\mathcal{C}_0 = \frac{U(m+n)}{U(m)\times U(n)} \times \{(m,n)\}$ where m = the number of -1 eigenvalues and n = the number of $+1$ eigenvalues.
- For $N = \infty$, $\pi_0(\mathcal{C}_0) = \mathbb{Z}$ is labeled an integer.
  Gapped phases of 0+1D free fermions with $U(1)$ symmetry are classified by integer $\mathbb{Z}$. The number of the fermions in the ground state. The result is also valid for interacting fermions.

Classify gapped phases of $1+1$D free fermions with $U(1)$

- Start with a large (universal) gapless system, such that other gapless systems can be viewed as partially gapped systems.
- Find all different disconnected ways to gap the universal gapless system. Kitaev arXiv:0901.2686
- Consider a gapless 1D free fermion $\epsilon(k) = -\sin k$, which is gapless at $k = 0$ (right movers) and $k = \pi$ (left movers).
  Double unit cell (half the Brillouin zone) $\to$ right movers and left movers are both a $k = 0$.

- Continuum limit: $M_{\text{one-body}} = \mathrm{i}\sigma^3\partial_x$ (acting on $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$)
  or $\hat{H}_{\text{many-body}} = \int \mathrm{d}x\ \psi^\dagger(x)\mathrm{i}\sigma^3\partial_x\psi(x)$ $\to$ 1D Dirac fermion
- Can be gapped by adding the mass term $M_{\text{one-body}} = \mathrm{i}\sigma^3\partial_x + m\sigma^1$.

- Universal gapless system $M_{\text{one-body}} = \mathrm{i}\sigma^3 \otimes I_n\partial_x$ acting on $\psi(x)$, a $2n$-component wave function.

- Gap by mass term $M_{\text{one-body}} = \mathrm{i}\sigma^3 \otimes I_n\partial_x + M$, where $M^\dagger = M$, $\sigma^3 \otimes I_n M = -M\sigma^3 \otimes I_n$ and M has no zero eigenvalue

The space of gapped $1+1$D free fermions w/ $U(1)$ symm.

is the space of the mass matrices that satisfy

$$M^\dagger = M, \quad M^2 = 1, \quad \gamma^1 M = -\gamma^1 M, \quad \gamma^1 = \sigma^3 \otimes I_n$$

If $\mathrm{i}\gamma_1\partial_x + M_{\text{gen}}$ has no zero eigenvalue, then we can deform $M_{\text{gen}} = M_A + fM_C$ from $f = 1$ to $f = 0$, without encounter zero eigenvalue.

- M must have n eigenvalues $+1$ and n eigenvalues -1.
  The space of such M is $\frac{U(2n)}{U(n)\times U(n)}$:
  $$M = U_{2n}^\dagger (U_n^\dagger \oplus \tilde{U}_n^\dagger)(\sigma^1 \otimes I_n)(U_n \oplus \tilde{U}_n) U_{2n}$$
- M also must satisfy $\gamma^1 M = -\gamma^1 M$, the unitary rotations $U(2n)$ and $U(n)\times U(n)$ must also keep γ^1 invariant.

- $U_{2n} = U_n \oplus \tilde{U}_n$: $U(2n) \to U(n)\times U(n)$.
- $U(n)\times U(n) = \sigma^0 \otimes U_n$: $U(n)\times U(n) \to U(n)$

- The space of gapped $1+1$D free fermion systems with $U(1)$ symmetry
  $$\mathcal{C}_1 = \frac{U(n)\times U(n)}{U(n)} = U(n), \quad n\to\infty.$$
- $\pi_0[U(n)] = 0 \to$ **There is only one trivial phase for gapped $1+1$D free fermion systems with $U(1)$ symmetry.**

Gapped $d+1$D free fermion systems with $U(1)$ symmetry

- $d+1$D gapless system $H_{\text{one-body}} = \mathrm{i}\gamma^i\partial_i + M$ $(i=1,\cdots,d)$
- The gapping mass matrix satisfies

$$M^\dagger = M,\ M^2=1,\ \gamma^i M = -\gamma^i M,\ \ (\gamma^i)^2=1,\ (\gamma^i)=(\gamma^i)^\dagger,\ \gamma^i\gamma^j=-\gamma^j\gamma^i$$

- $d=1$: $\quad M^\dagger = M,\ M^2=1,\ \gamma^1 M = -\gamma^1 M, \quad \gamma^1=\sigma^3\otimes I_n.$
- $d=2$: $\quad M^\dagger = M,\ M^2=1,\ \gamma^i M = -\gamma^i M,$
  $\gamma^1=\sigma^3\otimes I_n,\ \gamma^2=\sigma^1\otimes I_n.$
- $d=3$: $\quad M^\dagger = M,\ M^2=1,\ \gamma^i M = -\gamma^i M,$
  $\gamma^1=\sigma^3\otimes\sigma^0\otimes I_n,\ \gamma^2=\sigma^1\otimes\sigma^0\otimes I_n,\ \gamma^3=\sigma^2\otimes\sigma^3\otimes I_n.$

- For $d=3$, M has a form $M=\sigma^2\otimes\tilde{M}$, and $\tilde{M}$ satisfy
  $\tilde{M}^\dagger = M,\ \tilde{M}^2=1,\ \gamma^3\tilde{M}=-\gamma^3\tilde{M},\ \gamma^3=\sigma^3\otimes I_n.$
  The space of $d=3$ gapped sys. $=$ the space of $d=1$ gapped sys.

  The d-dimensional gapped phases $=$ the $d+2$-dimensional gapped phases, for free fermions with $U(1)$ symmetry: $\mathcal{C}_d=\mathcal{C}_{d+2}$

Symmetry	class	$d=0$	1	2	3	4	5	6	7
$U(1)$	A	$\mathbb{Z}$	0	$\mathbb{Z}$ IQH states	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Edge excitations

- 2d bulk has even number of 2-component Direc fermions (R-L pairs)

$$\hat{H}_{\text{many-body}} = \int \mathrm{d}^2\boldsymbol{x}\ \psi^\dagger(x)(\mathrm{i}\sigma^3\partial_x + \mathrm{i}\sigma^1\partial_y + m\sigma^2)\psi(x) + \int \mathrm{d}^2\boldsymbol{x}\ \Psi^\dagger(x)(\mathrm{i}\sigma^3\partial_x - \mathrm{i}\sigma^1\partial_y + M\sigma^2)\Psi(x)$$

- The Edge excitations are described by the low energy part $H = \mathrm{i}\sigma^i\partial_i + m\sigma^2$ (assuming $M \gg |m|$)
  Two different ways of gapping $m > 0$ and $m < 0$
  $\rightarrow$ $n = 1$ state and $n = 0$ state. Edge is where m change sign.
- For one edge $(\mathrm{i}\sigma^3\partial_x + \mathrm{i}\sigma^1\partial_y + y\sigma^2)\psi_2 = \mathrm{i}\partial_t\psi_2$
  Can be solved by $\psi_2(x, y, t) = c(x, t)\tilde{\psi}_2(y)$, and
  $(\mathrm{i}\sigma^1\partial_y + y\sigma^2)\tilde{\psi}_2(y) = \begin{pmatrix} 0 & \mathrm{i}(\partial_y - y) \\ \mathrm{i}(\partial_y + y) & 0 \end{pmatrix} \tilde{\psi}_2(y) = 0.$
  We find $\tilde{\psi}_2^\top = (\mathrm{e}^{-\frac{y^2}{2}}, 0) \rightarrow \mathrm{i}\partial_x c = \mathrm{i}\partial_t c$ ($k = -\omega$ left mover).
- For the other edge $(\mathrm{i}\sigma^3\partial_x + \mathrm{i}\sigma^1\partial_y - y\sigma^2)\psi_2 = \mathrm{i}\partial_t\psi_2$
  $\rightarrow$ right mover.

The gapped phases of 4+1D free fermions with $U(1)$ symm

Those phases are classified by $\mathbb{Z}$ (*ie* labeled by an integer $n \in \mathbb{Z}$)

Edge excitations for $n = 1$ phase

The bulk low-energy Hamiltonian: $H = \mathrm{i}\gamma^i \partial_i + m\gamma^5, \quad i = 1, \cdots, 4$

$\gamma^1 = \sigma^1 \otimes \sigma^3, \gamma^2 = \sigma^2 \otimes \sigma^3, \gamma^3 = \sigma^3 \otimes \sigma^3, \gamma^4 = \sigma^0 \otimes \sigma^1, \gamma^5 = \sigma^0 \otimes \sigma^2.$

Two different ways of gapping $m > 0$ and $m < 0 \to n = 0, 1$.

Edge is where m change sign.

- +Edge: $[(\sum_{i=1,2,3} \mathrm{i}\gamma^i \partial_{x^i}) + \sigma^0 \otimes \sigma^1 \partial_{x^4} + x^4 \sigma^0 \otimes \sigma^2]\psi_4 = \mathrm{i}\partial_t \psi_4.$
  Let $\psi_4(x^i, x^4) = \psi_2(x^i) \otimes \tilde\psi_2(x^4)$ and $(\mathrm{i}\sigma^1 \partial_{x^4} + x^4 \sigma^2)\tilde\psi_2(x^4) = 0.$
  We find $\tilde\psi_2^\top = (\mathrm{e}^{-\frac{(x^4)^2}{2}}, 0) \to \mathrm{i}\sigma^i \partial_{x^i} \psi_2(x^i) = \mathrm{i}\partial_t \psi_2(x^i)$
  $\to$ right-hand massless Weyl fermion
- −Edge: $[(\sum_{i=1,2,3} \mathrm{i}\gamma^i \partial_{x^i}) + \sigma^0 \otimes \sigma^1 \partial_{x^4} - x^4 \sigma^0 \otimes \sigma^2]\psi_4 = \mathrm{i}\partial_t \psi_4.$
  Let $\psi_4(x^i, x^4) = \psi_2(x^i) \otimes \tilde\psi_2(x^4)$ and $(\mathrm{i}\sigma^1 \partial_{x^4} - x^4 \sigma^2)\tilde\psi_2(x^4) = 0.$
  We find $\tilde\psi_2^\top = (0, \mathrm{e}^{-\frac{(x^4)^2}{2}}) \to -\mathrm{i}\sigma^i \partial_{x^i} \psi_2(x^i) = \mathrm{i}\partial_t \psi_2(x^i)$
  $\to$ left-hand massless Weyl fermion

Is the handness of 3+1D Weyl fermion absolute?

- Right-hand Weyl fermion: $\mathrm{i}\sigma^i \partial_{x^i} \psi_2^R = \mathrm{i}\partial_t \psi_2^R$
- Left-hand Weyl fermion: $-\mathrm{i}\sigma^i \partial_{x^i} \psi_2^L = \mathrm{i}\partial_t \psi_2^L$

  To give Weyl fermion a mass $\to$

- Massive Dirac fermion = Right-hand Weyl $\oplus$ Left-hand Weyl:

$$\mathrm{i}\sigma^i \otimes \sigma^3 \partial_{x^i} \psi_4 + m\sigma^0 \otimes \sigma^2 \psi_4 = \mathrm{i}\partial_t \psi_4$$

In the standard model, each family $(e, \mu, q_r, q_g, q_b, \nu)$ *has*
7 right-hand Weyl fermions and 8 left-hand Weyl fermions, or
8 right-hand Weyl fermions and 7 left-hand Weyl fermions, or
15 right-hand Weyl fermions and 0 left-hand Weyl fermions.

- The transformation $\psi_2^L = \mathrm{i}\sigma^2(\psi_2^R)^*$ changes $\mathrm{i}\sigma^i \partial_{x^i} \psi_2^R = \mathrm{i}\partial_t \psi_2^R$ to $-\mathrm{i}\sigma^i \partial_{x^i} \psi_2^L = \mathrm{i}\partial_t \psi_2^L$.

$$-\mathrm{i}(\sigma^i)^* \partial_{x^i} (\psi_2^R)^* = \mathrm{i}\partial_t (\psi_2^R)^* \ \to\ -\mathrm{i}\sigma^i \partial_{x^i} \mathrm{i}\sigma^2 (\psi_2^R)^* = \mathrm{i}\partial_t \mathrm{i}\sigma^2 (\psi_2^R)^*$$

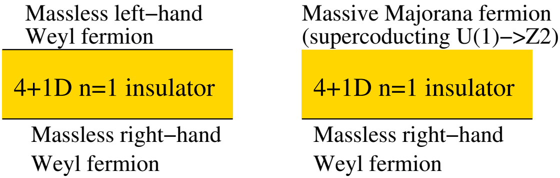
Charge conjugation of right-hand Weyl fermion
= left-hand Weyl fermion

3+1D massive Majorana fermion

- $\bar{\psi}_4 = \sigma^2 \otimes \sigma^2 (\psi_4)^*$ and ψ_4 satisfy the same massive Dirac equation
$$\mathrm{i}\sigma^i \otimes \sigma^3 \partial_{x^i}\psi_4 + m\sigma^0 \otimes \sigma^2 \psi_4 = \mathrm{i}\partial_t \psi_4$$
$$\mathrm{i}(\sigma^i)^* \otimes \sigma^3 \partial_{x^i}\psi_4^* - m\sigma^0 \otimes (\sigma^2)^* \psi_4^* = \mathrm{i}\partial_t \psi_4^*$$
$$\mathrm{i}\sigma^i \otimes \sigma^3 \partial_{x^i}\bar{\psi}_4 + m\sigma^0 \otimes \sigma^2 \bar{\psi}_4 = \mathrm{i}\partial_t \bar{\psi}_4$$
If we requires that $\bar{\psi}_4 = \psi_4 \to$ massive 3+1D Majorana fermion.
- 3+1D massless Weyl fermion: 2 complex components
3+1D massive Dirac fermion: 4 complex components
3+1D massive Majorana fermion: 4 real = 2 complex components
- Rewrite the EOM of massive 3+1D Majorana fermion
$\psi_4 = (\psi_2^R, \psi_2^L), \quad \psi_2^L = \mathrm{i}\sigma^2(\psi_2^R)^*, \quad \psi_2^R = -\mathrm{i}\sigma^2(\psi_2^L)^*$
$$\mathrm{i}\sigma^i \partial_{x^i}\psi_2^R - \mathrm{i}m\psi_2^L = \mathrm{i}\partial_t \psi_2^R$$
$$-\mathrm{i}\sigma^i \partial_{x^i}\psi_2^L + \mathrm{i}m\psi_2^R = \mathrm{i}\partial_t \psi_2^L$$
which is $\mathrm{i}\sigma^i \partial_{x^i}\psi_2^R + m\sigma^2(\psi_2^R)^* = \mathrm{i}\partial_t \psi_2^R$.
The right-hand Weyl fermion gains a mass at the cost of $U(1)$ symm. breaking down to Z_2 (EOM not inv. under $\psi_2^R \to \mathrm{e}^{\mathrm{i}\theta}\psi_2^R$). The electrons in superconductor are Majorana ferions.

$U(1)$ anomaly: realize 3D massless Weyl fermion in 3D

- We can give a massless right-hand Weyl fermion a mass if we break the $U(1)$ symmetry down to Z_2. $\to$
- Non-interacting 4+1D $n = 1$ insulator is trivial without the $U(1)$ symmetry, but non-trivial with the $U(1)$ symmetry.
- For two gapped states of non-interating fermions, existance of a gapped boundary $\leftrightarrow$ existance of a deformation path without closing gap.
- A single 3+1D massless right-hand Weyl fermion with $U(1)$ symmetry is anomalous $\to$ cannot be realized on a 3+1D lattice if we preserve the $U(1)$ symmetry.
- Can realize 3+1D massless right-hand Weyl fermion on a 3D lattice if we break the $U(1)$ symm. down to Z_2



U(1) symmetry anomaly, but no gravitational anomaly

Put the chiral $SO(10)$ GUT on lattice

- In the $SO(10)$ GUT in 3+1D, we have 16 massless right-hand Weyl fermion forming a 16-dim. spinner representation of $SO(10)$.

- Is such GUT anomalous or not?
- Can we put puch such a chiral GUT on a 3+1D lattice?
(The long standing **chiral fermion problem**)

- We have seen that 16 massless right-hand Weyl fermion with $U^{16}(1)$ symmetry cannot be put on 3+1D lattice. But can be put on 3+1D lattice if we reduce the symmetry to Z_2^{16}.

  Can we put n massless $d+1$D fermions with G symmetry on $d+1$D lattice? Wen arXiv:1305.1045

  Yes if (1) there is a mass term that give all fermions a mass (which may break the symmetry G down to G_Ψ), and (2) $\pi_n(G/G_\Psi) = 0$ for $n \leq d+2$.

  $\to$ We can put $SU(10)$ GUT on 3+1D lattice.

- The above condition is only sufficient. What is a necessary and sufficient condition?

Spectrum: relation between spaces of gapped states of non-interacting fermions in different dimensions

For two gapped states of non-interating fermions, existance of a gapped boundary $\leftrightarrow$ existance of a deformation path without closing gap.

- Let $\mathcal{M}_n$ be the space of gapped states of non-interacting fermions in n-dimensional space. Let $\mathcal{M}_n(\alpha), \alpha \in \pi_0(\mathcal{M}_n)$ be the α^{th} component. Let $\alpha = 0$ correspond to the trivial phase (the product states).

\- **The space of gapped boundaries** of a trivial state is **the space of the based loops in** $\mathcal{M}_n$ with base point in $\mathcal{M}_n(0)$ (which is the **loop space** $\Omega\mathcal{M}_n$. Check Wiki) Gaiotto Johnson-Freyd, arXiv:1712.07950

\- Physically, the space of gapped boundary of a trivial state is (or homotopically equivalent to) the space of gapped states in one lower dimension:

$$\Omega\mathcal{M}_n(0) \sim \mathcal{M}_{n-1}$$

\- For loop space, we have $\pi_k(\Omega\mathcal{M}) = \pi_{k+1}(\mathcal{M})$. Thus the space $\mathcal{M}_n$ of the space of gapped states of non-interacting fermions satisfies

$$\pi_k(\mathcal{M}_n) = \pi_l(\mathcal{M}_{n-k+l}) \quad \to \quad \pi_0(\mathcal{M}_n) = \pi_l(\mathcal{M}_{n+l}).$$

Classify gapped phases of 0+1D free fermions with no symmetry Z_2^f symmetry

- Fermion systems with no symmetry = Fermion system with Z_2^f symmetry. They correspond to fermionic superconductors.
- 0+1D free fermion system with Z_2^f symmetry is described by the following many-body Hamiltonian

$$\hat{H} = \sum_{ab} M_{ab}\hat{c}_a^\dagger \hat{c}_b + \sum_{ab}(\frac{1}{2}\Delta_{ab}\hat{c}_a\hat{c}_b + h.c.) = \frac{1}{4}\sum_{\alpha,\beta} A_{\alpha\beta}\,\mathrm{i}\,\hat{\eta}_\alpha\hat{\eta}_\beta + \#$$

$$\hat{c}_a = \frac{\hat{\eta}_{a,1} + \mathrm{i}\,\hat{\eta}_{a,2}}{2},\ \{\hat{c}_a^\dagger, \hat{c}_b\} = \delta_{ab},\ \{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = 2\delta_{\alpha\beta},\ \ A^\top = -A,\ A^* = A.$$

- To see the relateion between M and A, let $M = M^S + \mathrm{i}\,M^A$ and $\Delta = 0$.

$$\hat{H} = \sum_{ab} \frac{\mathrm{i}}{4}(\hat{\eta}_{a,1}M^S_{ab}\hat{\eta}_{b,2} - \hat{\eta}_{a,2}M^S_{ab}\hat{\eta}_{b,1}) + \frac{\mathrm{i}}{4}(\hat{\eta}_{a,1}M^A_{ab}\hat{\eta}_{b,1} + \hat{\eta}_{a,2}M^A_{ab}\hat{\eta}_{b,2}) + \#$$

Let us write $M = \mathrm{i}(M^A - \mathrm{i}\,M^S)$. We find that A is obtained by replacing 1 by σ^0 and i by $-\varepsilon$ in the bracket:

$$A = \sigma^0 \otimes M^A - (-\varepsilon) \otimes M^S = \sigma^0 \otimes M^A + \varepsilon \otimes M^S$$

- To see the relateion between Δ and A, let $M = 0$ and $\Delta = \Delta^R + \mathrm{i}\Delta^I$

$$\hat{H} = \sum_{ab} \frac{\mathrm{i}}{8}(\hat{\eta}_{a,1}\Delta^R_{ab}\hat{\eta}_{b,2} - \hat{\eta}_{a,2}\Delta^R_{ab}\hat{\eta}_{b,1}) + \frac{\mathrm{i}}{8}(\hat{\eta}_{a,1}\Delta^I_{ab}\hat{\eta}_{b,1} + \hat{\eta}_{a,2}\Delta^I_{ab}\hat{\eta}_{b,2}) + h.c.$$
$$= \sum_{ab} \frac{\mathrm{i}}{4}(\hat{\eta}_{a,1}\Delta^R_{ab}\hat{\eta}_{b,2} - \hat{\eta}_{a,2}\Delta^R_{ab}\hat{\eta}_{b,1}) + \frac{\mathrm{i}}{4}(\hat{\eta}_{a,1}\Delta^I_{ab}\hat{\eta}_{b,1} + \hat{\eta}_{a,2}\Delta^I_{ab}\hat{\eta}_{b,2}).$$

Let us write $\Delta = \mathrm{i}(\Delta^I - \mathrm{i}\Delta^R)$. We find that A is obtained by replacing 1 by σ^0 and i by $-\varepsilon$ in the bracket:

$$A = \sigma^0 \otimes \Delta^I - (-\varepsilon) \otimes \Delta^R = \sigma^0 \otimes \Delta^I + \varepsilon \otimes \Delta^R$$

- The superconductor is fully characterized by a $2n \times 2n$ anti-symmetric real matrix A. We will concentrate on A. Non-zero eigenvalues of $\mathrm{i}A$ appear in pairs $\pm\epsilon$. Up to homotopic equivalence, we may assume non-zero eigenvalues of $\mathrm{i}A$ to be ± 1.
- Gapped $\to$ A has no zero eigenvalue. Space of 0+1D gapped non-interacting fermion systems with Z_2^f symmetry $\mathcal{R}_0^0 \cong_{\text{homotopic}}$ space of anti-symmetric real matrix matrices with $\pm\mathrm{i}$ eigenvalues.

The classifying space $\mathcal{R}_0^0$

$$A = O_{O(2n)} \begin{pmatrix} \varepsilon & 0 & \cdots \\ 0 & \varepsilon & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} O_{O(2n)}^\top \qquad \text{where } \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= O_{O(2n)} O_{U(n)} \begin{pmatrix} \varepsilon & 0 & \cdots \\ 0 & \varepsilon & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} O_{U(n)}^\top O_{O(2n)}^\top \quad \to \quad \mathcal{R}_0^0 = \frac{O_{O(2n)}}{O_{U(n)}}_{n\to\infty}$$

- What is $\mathcal{R}_0^0 = \frac{O_{O(2n)}}{O_{U(n)}}$ for $n = 1$? From $\{U(1)\}_{1\times 1} = \{\cos\theta + \mathrm{i}\sin\theta\} \to$

$$\{O_{U(1)}\}_{2\times 2} = \left\{ \cos\theta - \varepsilon\sin\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}_{\text{replace i by } \varepsilon}.$$

$$\{O(2)\}_{2\times 2} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{\det=1}, \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}_{\det=-1} \right\}$$

Setting $\theta = 0$, we find $\mathcal{R}_0^0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ as a set of O's.

As a set of A's, we have $\mathcal{R}_0^0 = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} = \mathbb{Z}_2$

Many-body picture of the classifying space $\mathcal{R}_0^0$

- Fermion-number-parity: $\hat{N}_a = \hat{c}_a^\dagger \hat{c}_a = \frac{1+\mathrm{i}\,\hat{\eta}_{2a-1}\hat{\eta}_{2a}}{2}$
  $\to \hat{P}_f = \prod_a (1-2\hat{N}_a) = \prod_a(-\mathrm{i}\,\hat{\eta}_{2a-1}\hat{\eta}_{2a}) = (-\mathrm{i})^n \prod_{\alpha=1}^{2n} \hat{\eta}_\alpha$
- $\hat{P}_f$ is always a symmetry for fermion system

$$[\hat{P}_f, \hat{H}] = 0$$

  We denote this symmetry as Z_2^f, since $\hat{P}_f^2 = \mathrm{id}$.
- Assume A is "diagonal"

$$A = \begin{pmatrix} \pm\varepsilon & 0 & \cdots \\ 0 & \pm\varepsilon & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \to \quad \hat{H} = \pm \underbrace{\mathrm{i}\,\hat{\eta}_1\hat{\eta}_2}_{2\hat{c}_1^\dagger\hat{c}_1 - 1} \pm \underbrace{\mathrm{i}\,\hat{\eta}_3\hat{\eta}_4}_{2\hat{c}_2^\dagger\hat{c}_2 - 1} + \cdots$$

  $\mathcal{R}_0^0 = \mathbb{Z}_2$ corresponds to $\hat{P}_f = \pm 1$ ground states of $\hat{H}$.

The $U^f(1)$ symmetry for non-interacting fermion systems

- $\hat{H}$ commutes with the fermion-number operator

$$\hat{N} \equiv \sum_a (\hat{c}_a^\dagger \hat{c}_a - \frac{1}{2}) = \sum_a (\frac{\hat{c}_a^\dagger \hat{c}_a - \hat{c}_a \hat{c}_a^\dagger}{2}) = \frac{\mathrm{i}}{4} \sum_{\alpha\beta} Q_{\alpha\beta} \hat{\eta}_\alpha \hat{\eta}_\beta$$

where $Q = \varepsilon \otimes I$, $Q^2 = -1, Q^* = Q, Q^\top = -Q = Q^{-1}$, $\varepsilon \equiv \mathrm{i}\sigma^2$.

- The symmetry group $\{U^f(1)\} = \{\mathrm{e}^{\mathrm{i}\theta \hat{N}}\}$. $Z_2^f = \{\mathrm{id}, \mathrm{e}^{\mathrm{i}\pi \hat{N}}\} \subset U^f(1)$.
- $[\hat{H}, \hat{N}] = 0$ requires that

$$AQ = QA, \quad Q^2 = -1.$$

- Such a real anti-symmetric matrix A has the form $A = \sigma^0 \otimes M_a + \varepsilon \otimes M_s$, where M_s is real symmetric and M_a real antisymmetric. We can convert such a $2n \times 2n$ real antisymmetric matrix A into a $n \times n$ Hermitian matrix $M = M_s + \mathrm{i} M_a$, by replacing ε by i. This reduces the problem to the one that we discussed before (with fermion number conservation).

Z_2 symmetry: $Z_2 \times Z_2^f$ or Z_4^f symmetry

- A $Z_2 \times Z_2^f$ or Z_4^f transformation is generated by $\hat{P}_f$ and $\hat{C}$.
  (1) $\hat{C}^2 = \mathrm{id} \to Z_2 \times Z_2^f$. (2) $\hat{C}^2 = \hat{P}_f \to Z_4^f$.
  Note that $Z_2^f \subset Z_4^f$ or $Z_2 \times Z_2^f$.
- Matrix representation of $\hat{C}$:

$$\hat{C}\hat{\eta}_\alpha\hat{C}^\dagger = C_{\alpha\beta}\hat{\eta}_\beta, \quad \hat{C}^\dagger = \hat{C}^{-1}, \quad \hat{\eta}_\alpha^\dagger = \hat{\eta}_\alpha, \quad \{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = 2\delta_{\alpha\beta}, \quad .$$

- $(\hat{C}\hat{\eta}_\alpha\hat{C}^\dagger)^\dagger = \hat{C}\hat{\eta}_\alpha\hat{C}^\dagger = C^*_{\alpha\beta}\hat{\eta}_\beta \to C^* = C$.
- C must be an orthogonal matrix $C^\top = C^{-1}$ to keep $\{\hat{\eta}_\alpha, \hat{\eta}_\beta\} = 2\delta_{\alpha\beta}$ invariant.
- $C^2 = s_C$. (1) $s_C = + \to Z_2 \times Z_2^f$. (2) $s_C = - \to Z_4^f$.
- A $Z_2 \times Z_2^f$ or Z_4^f symmetry: $\hat{C}\hat{H}\hat{C}^{-1} = \hat{H}$ implies that A satisfies

$$CA = CA, \qquad C^2 = s_C.$$

$U^f(1)$ and Z_2 symmetries

- If we have both $U^f(1)$ and Z_2 symmetries, then $\hat{C}\hat{N} = \hat{N}\hat{C}$ and

$$CQ = s_{UC} QC, \quad s_{UC} = +.$$

- $U^f(1)$ and $Z_2 \times Z_2^f$ symmetry:

$$AQ = QA,\ AC = CA,\ \ Q^2 = -1,\ C^2 = 1,\ CQ = QC.$$

Symmetry group $G^f = U^f(1) \times Z_2$.

- $U^f(1)$ and Z_4^f symmetry:

$$AQ = QA,\ AC = CA,\ \ Q^2 = -1,\ C^2 = -1,\ CQ = QC.$$

Symmetry group $G^f = \frac{U^f(1) \times Z_4^f}{Z_2^f}$.

$U^f(1)$ and Z_2 charge conjugation symmetries

- If we have $U^f(1)$ and Z_2 charge conjugation symmetries, then $\hat{C}\hat{N} = -\hat{N}\hat{C}$ and

$$CQ = s_{UC} QC, \quad s_{UC} = -.$$

\- $U^f(1)$ and $Z_2 \times Z_2^f$ charge conjugation symmetry:

$$AQ = QA, \ AC = CA, \ \ Q^2 = -1, \ C^2 = 1, \ CQ = -QC.$$

Symmetry group $G^f = U^f(1) \rtimes Z_2$.
Classification: We have $Q = \varepsilon \otimes I_n$ and $C = \sigma^1 \otimes I_n$. For A to have $U^f(1) \rtimes Z_2$ symmetry, $A = \sigma^0 \otimes \tilde{A}$, and no condition on $\tilde{A}$. Same as no symmetry (or Z_2^f symmetry).

\- $U^f(1)$ and Z_4^f charge conjugation symmetry:

$$AQ = QA, \ AC = CA, \ \ Q^2 = -1, \ C^2 = -1, \ CQ = -QC.$$

Symmetry group $G^f = \frac{U^f(1) \rtimes Z_4^f}{Z_2^f}$.

Time-reversal symmetry

- The time-reversal transformation $\hat{T}$ is antiunitary: $\hat{T}\,\mathrm{i}\,\hat{T}^{-1} = -\mathrm{i}$. In terms of the Majorana fermions, we have (just like Z_2 symmetry $\hat{C}$)

$$\hat{T}\hat{\eta}_\alpha\hat{T}^{-1} = T_{\alpha\beta}\hat{\eta}_\beta, \qquad T^\top = T^{-1}.$$

- For fermion systems, we may have $\hat{T}^2 = (s_T)^{\hat{N}}$, $s_T = \pm$. ($s_T = -$ for electrons). This implies that $\hat{T}^2\hat{c}_i\hat{T}^{-2} = s_T\hat{c}_i$ and $T^2 = s_T$.
- Symmetry group: (1) $s_T = + \to Z_2^T$. (2) $s_T = - \to Z_4^T$.
- The time-reversal invariance $\hat{T}\hat{H}\hat{T}^{-1} = \hat{H}$ for $\hat{H} = \frac{\mathrm{i}}{2}\sum_{\alpha\beta} A_{\alpha\beta}\hat{\eta}_\alpha\hat{\eta}_\beta$ implies that

$$T^\top A T = -A \quad \text{or} \quad AT = -TA, \qquad T^2 = s_T.$$

$AT = -TA$ is different from the unitary Z_2 symmetry.

Relations between U, C, and T

- The time-reversal transformation $\hat{T}$ and the $U^f(1)$ transformation $\hat{N}$ may have a nontrivial relation: $\hat{T}\mathrm{e}^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = \mathrm{e}^{s_{UT}\mathrm{i}\theta\hat{N}}$, $s_{UT} = \pm$, or $\hat{T}\hat{N}\hat{T}^{-1} = -s_{UT}\hat{N}$. This gives us

$$TQ = s_{UT} QT.$$

- $s_{UT} = + \to U^f_{\text{spin}}(1)$ (conservation of S^z spin in XY magnets).
- $s_{UT} = - \to U^f_{\text{charge}}(1)$ (conservation of electric spin).

- The commutation relation between $\hat{T}$ and $\hat{C}$ has two choices: $\hat{T}\hat{C} = s_{TC}^{\hat{N}}\hat{C}\hat{T}$, $s_{TC} = \pm$, we have

$$CT = s_{TC} TC.$$

- The commutation relation between $\hat{N}$ and $\hat{C}$ has two choices: $\hat{N}\hat{C} = s_{UC}\hat{C}\hat{N}$, $s_{UC} = \pm$, we have

$$CQ = s_{UC} QC.$$

- $s_{UT} = - \to C$ is a charge conjugation.
  $s_{UT} = + \to C$ is not a charge conjugation.

Summary of symmetry groups with $U^f(1)$, C, and T

Symmetry groups		Relations total 52 groups
$G_{s_C}(C)$	(2)	$\hat{C}^2 = s_C^{\hat{N}}, \quad s_C = \pm.$
$G_{s_T}(T)$	(2)	$\hat{T}^2 = s_T^{\hat{N}}, \quad s_T = \pm.$
$G_{s_C}^{s_{UC}}(U,C)$	(4)	$\hat{C}^2 = s_C^{\hat{N}},\ \hat{C}\hat{N}\hat{C}^{-1} = s_{UC}\hat{N},\ s_C, s_{UC} = \pm.$
$G_{s_T}^{s_{UT}}(U,T)$	(4)	$\hat{T}\mathrm{e}^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = \mathrm{e}^{s_{UT}\mathrm{i}\theta\hat{N}},\ \hat{T}^2 = s_T^{\hat{N}},\ s_{UT}, s_T = \pm.$
$G_{s_T s_C}^{s_{TC}}(T,C)$	(8)	$\hat{T}^2 = s_T^{\hat{N}},\ \hat{C}^2 = s_C^{\hat{N}},\ \hat{C}\hat{T} = (s_{TC}^{\hat{N}})\hat{T}\hat{C},\ s_{TC}, s_T, s_C = \pm.$
$G_{s_T s_C}^{s_{UT} s_{TC} s_{UC}}(U,T,C)$	(32)	$\hat{C}\hat{N}\hat{C}^{-1} = s_{UC}\hat{N},\ \hat{T}\mathrm{e}^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = \mathrm{e}^{s_{UT}\mathrm{i}\theta\hat{N}},\ \hat{T}^2 = s_T^{\hat{N}},$ $\hat{C}^2 = s_C^{\hat{N}},\ \hat{C}\hat{T} = (s_{TC}^{\hat{N}})\hat{T}\hat{C},\ s_T, s_C, s_{UT}, s_{TC}, s_{UC} = \pm.$

- **Topological insulator** Electrons with $U^f(1)$-charge and T: symmetry group $G_-^-(U,T) = (U^f(1)_{\text{charge}} \rtimes Z_4^T)/Z_2^f$
- **Topo. S_z superconductor** Electrons with $U^f(1)$-spin and T: symmetry group $G_-^+(U,T) = (U^f(1)_{\text{spin}} \times Z_4^T)/Z_2^f$
- **Topological T superconductor** Electrons with T: symmetry group $G_-(T) = Z_4^T$
- **Topological $\tilde{T}$ superconductor** Electrons with $\tilde{T}$: symmetry group $G_+(T) = Z_2^T$ ($\tilde{T} = T \times \pi$-spin-rotation)

Including the Z_2^f FNP symmetry and fermionic symmetry

The fermion systems always has FNP Z_2^f symmetry. But for the symmetry groups in the above list, some conatin Z_2^f and are complete; some do not conatin Z_2^f and are incomplete.

Symmetry groups	Total fermion symmetry groups G^f
$G_{s_C}(C)$	$G_+(C) \times Z_2^f$, $G_-(C) \supset Z_2^f$.
$G_{s_T}(T)$	$G_+(T) \times Z_2^f$, $G_-(T) \supset Z_2^f$.
$G_{s_C}^{s_{UC}}(U, C)$	$G_{s_C}^{s_{UC}}(U^f, C) \supset Z_2^f$
$G_{s_T}^{s_{UT}}(U, T)$	$G_{s_T}^{s_{UT}}(U^f, T) \supset Z_2^f$
$G_{s_T s_C}^{s_{TC}}(T, C)$	$G_{++}^{+}(T, C) \times Z_2^f$, others $\supset Z_2^f$
$G_{s_T s_C}^{s_{UT} s_{TC} s_{UC}}(U, T, C)$	$G_{s_T s_C}^{s_{UT} s_{TC} s_{UC}}(U^f, T, C) \supset Z_2^f$

If the full symmetry group is $G^f = G_b \times Z_2^f$, then the Z_2^f is missing.

Symmetry of fermion systems is described by

$$1 \to Z_2^f \to G^f \to G_b \to 1$$

or by the full symmetry group G^f and its central Z_2^f subgroup:

$$(G^f, Z_2^f \overset{\text{cen}}{\subset} G^f)$$

Some 0d superconductors

- **Superconductors** with no symmetry ($G^f = Z_2^f$)
  Classifying space $\mathcal{R}_0^0$ = space of real anti-symmetric matrices A with eigenvalue $\pm\mathrm{i}$ (*ie* with $A^2 = -1$).
- T **superconductors** with symmetry $G_-(T) = Z_4^T = G^f$

$$TA = -AT, \quad T^2 = -1$$

  Classifying space $\mathcal{R}_0^1$ = space of real anti-symmetric matrices A, $A^2 = -1$, that anti commute with an orthogonal matrix that square to -1.
- $\tilde{T}$ **superconductors** with symmetry $G_+(T) = Z_2^T$ ($G^f = G_+(T) \times Z_2^f$)

$$TA = -AT, \quad T^2 = 1$$

  Classifying space $\mathcal{R}_1^0$ = space of real anti-symmetric matrices A, $A^2 = -1$, that anti commute with an orthogonal matrix that square to 1.

Some 0d topological superconductors

- S_z, T **superconductors** with $G_-^+(U,T) = (U^f(1) \times Z_4^T)/Z_2 = G^f$

$QA = AQ,\ Q = \varepsilon \otimes I,\ TA = -AT,\ TQ = TQ,\ T^2 = -1,\ T = \varepsilon \otimes T_M$

- A has the form $A = \sigma^0 \otimes M_a + \varepsilon \otimes M_s \to M = M_s + \mathrm{i}\, M_a = M^\dagger$.

$$T_M M = -M T_M, \quad T_M^2 = 1.$$

Classifying space $\mathcal{C}_1$ = space of hermitian matrix M, $M^2 = 1$, that anti-commute with an unitary matrix whose square is 1.

In comparison

- **Insulators** with symmetry $G^f = U^f(1)$.
  Classifying space $\mathcal{C}_0$ = space of hermitian matrix M, with $M^2 = 1$.
- The above $\mathcal{C}_0$ and $\mathcal{C}_1$ agrees with our previous definition of classifying space $\mathcal{C}_d$ using γ-matrices.

0d insulator with $U^f(1)$-charge and time-reversal symm.

- **Insulator** with symmetry $G_-^-(U,T) = (U^f(1) \rtimes Z_4^T)/Z_2 = G^f$

$$QA = AQ,\ Q^2 = -1,\ TA = -AT,\ TQ = -TQ,\ T^2 = -1.$$
$$\rho_i A = -A\rho_i,\ \rho_1 = T,\ \rho_2 = TQ,\ \rho_1\rho_2 = -\rho_2\rho_1,\ \rho_1^2 = \rho_2^2 = -1.$$

Classifying space $\mathcal{R}_0^2$ = space of real anti-symmetric matrices A, $A^2 = -1$, that anti commute with two anti-commuting orthogonal matrices that square to -1.

- **Insulator** with symmetry $G_-^+(U,T) = U^f(1) \rtimes Z_2^T = G^f$ (Here time reversal is $\tilde{T} = T_{\text{elec}} \times \pi$-spin-rotation)

$$QA = AQ,\ Q^2 = -1,\ TA = -AT,\ TQ = -TQ,\ T^2 = 1.$$
$$\rho_i A = -A\rho_i,\ \rho_1 = T,\ \rho_2 = TQ,\ \rho_1\rho_2 = -\rho_2\rho_1,\ \rho_1^2 = \rho_2^2 = 1.$$

Classifying space $\mathcal{R}_2^0$ = space of real anti-symmetric matrices A, $A^2 = -1$, that anti commute with two anti-commuting orthogonal matrices that square to 1.

The classifying spaces $\mathcal{R}_p^q$ and $\mathcal{R}_p$

- Classifying space $\mathcal{R}_p^q$ is formed by anti-symmetric real matrix A satisfying ($i, j = 1, \cdots, p + q$)

$$\rho_i A = -A\rho_i, \qquad A^2 = -1,$$
$$\rho_i^\top = \rho_i^{-1}, \quad \rho_i\rho_j = -\rho_i\rho_j, \quad \rho_i^2|_{i=1,\cdots,p} = 1, \quad \rho_i^2|_{i=p+1,\cdots,p+q} = -1.$$

- Classifying space $\mathcal{R}_p$ is formed by symmetric real matrix A satisfying

$$\rho_i A = -A\rho_i, \qquad A^2 = 1,$$
$$\rho_i^\top = \rho_i^{-1}, \quad \rho_i\rho_j = -\rho_i\rho_j, \quad \rho_i^2|_{i=1,\cdots,p} = 1.$$

Properties of the classifying spaces $\mathcal{R}_p^q$

- $\mathcal{R}_p^q = \mathcal{R}_{p+1}^{q+1}$

- From $\tilde{A} \in R_p^q$ that satisfies

$$\tilde{A}\tilde{\rho}_i = -\tilde{\rho}_i\tilde{A}, \quad \tilde{A}^2 = -1, \qquad \tilde{\rho}_j\tilde{\rho}_i + \tilde{\rho}_i\tilde{\rho}_j|_{i\neq j} = 0,$$
$$\tilde{\rho}_i^2|_{i=1,\dots,p} = 1, \quad \tilde{\rho}_i^2|_{i=p+1,\dots,p+q} = -1,$$

we can define

$$A = \tilde{A} \otimes \sigma^3, \quad \rho_i|_{i=1,\dots,p} = \tilde{\rho}_i \otimes \sigma^3, \quad \rho_{p+1} = I \otimes \sigma^1,$$
$$\rho_i|_{i=p+1+1,\dots,p+1+q} = \tilde{\rho}_{i-1} \otimes \sigma^3, \quad \rho_{p+1+q+1} = I \otimes \varepsilon.$$

We can check that $A \in \mathcal{R}_{p+1}^{q+1}$

$$A\rho_i = -\rho_i A, \quad A^2 = -1, \qquad \rho_j\rho_i + \rho_i\rho_j|_{i\neq j} = 0,$$
$$\rho_i^2|_{i=1,\dots,p+1} = 1, \quad \rho_i^2|_{i=p+1+1,\dots,p+1+q+1} = -1,$$

Properties of the classifying spaces $\mathcal{R}_p^q$

- For a $A \in \mathcal{R}_{p+1}^{q+1}$, we always choose a basis such that $\rho_{p+1} = I \otimes \sigma^1,\ \rho_{p+1+q+1} = I \otimes \varepsilon$. Then we have

$$A = \tilde{A} \otimes \sigma^3,\ \ \rho_i|_{i=1,\dots,p} = \tilde{\rho}_i \otimes \sigma^3,\ \ \rho_{p+1} = I \otimes \sigma^1,$$
$$\rho_i|_{i=p+1+1,\dots,p+1+q} = \tilde{\rho}_{i-1} \otimes \sigma^3,\ \ \rho_{p+1+q+1} = I \otimes \varepsilon.$$

We find $\tilde{A} \in \mathcal{R}_p^q$.

Properties of the classifying spaces $\mathcal{R}_p^q$ and $\mathcal{R}_p$

• $\mathcal{R}_0^q = \mathcal{R}_{q+2}$

- From $\tilde{A} \in R_0^q$ that satisfies

$$\tilde{A}\tilde{\rho}_i = -\tilde{\rho}_i\tilde{A}, \quad \tilde{A}^2 = -1, \qquad \tilde{\rho}_j\tilde{\rho}_i + \tilde{\rho}_i\tilde{\rho}_j|_{i\neq j} = 0,$$
$$\tilde{\rho}_i^2 = -1, \quad \tilde{\rho}_i^\top = \tilde{\rho}_i^{-1} \quad i,j = 1,\cdots,q$$

we can define

$$A = \tilde{A} \otimes \varepsilon, \;\; \rho_i|_{i=1,\dots,q} = \tilde{\rho}_i \otimes \varepsilon, \;\; \rho_{q+1} = I \otimes \sigma^1, \;\; \rho_{q+2} = I \otimes \sigma^3.$$

We can check that $A \in \mathcal{R}_{q+2}$

$$A\rho_i = -\rho_i A, \quad A^2 = 1, \qquad \rho_j\rho_i + \rho_i\rho_j|_{i\neq j} = 0,$$
$$\rho_i^2 = 1, \quad \rho_i^\top = \rho_i^{-1}, \quad i,j = 1,\cdots,q+2$$

- We can also show the reverse, by choosing a basis such that $\rho_{q+1} = I \otimes \sigma^1, \; \rho_{q+2} = I \otimes \sigma^3$.

Clifford algebra $Cl(0, 8n)$

16 dimensional real symmetric representation of Clifford algebra $Cl(0, 8)$:

$$\gamma_i \gamma_j + \gamma_j \gamma_i \underset{i \neq j}{=} 0, \qquad \gamma_i^2 \underset{i=0,...,8}{=} 1.$$

$$\gamma_1 = \varepsilon \otimes \sigma^3 \otimes \sigma^0 \otimes \varepsilon, \qquad \gamma_2 = \varepsilon \otimes \sigma^3 \otimes \varepsilon \otimes \sigma^1,$$
$$\gamma_3 = \varepsilon \otimes \sigma^3 \otimes \varepsilon \otimes \sigma^3, \qquad \gamma_4 = \varepsilon \otimes \sigma^1 \otimes \varepsilon \otimes \sigma^0,$$
$$\gamma_5 = \varepsilon \otimes \sigma^1 \otimes \sigma^1 \otimes \varepsilon, \qquad \gamma_6 = \varepsilon \otimes \sigma^1 \otimes \sigma^3 \otimes \varepsilon,$$
$$\gamma_7 = \varepsilon \otimes \varepsilon \otimes \sigma^0 \otimes \sigma^0, \qquad \gamma_8 = \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0,$$

where $\varepsilon = \mathrm{i}\sigma^2$. Also $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8 = \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0$ anticommute with γ_i: $\gamma\gamma_i = -\gamma_i\gamma$, and $\gamma^2 = 1$.

- $Cl(0, 16)$:

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i \underset{i \neq j}{=} 0, \qquad \Gamma_i^2 \underset{i=0,...,16}{=} 1.$$

where $\Gamma_i = \gamma_i \otimes 1, \Gamma_{i+8} = \gamma \otimes \gamma_i$ (32-dimensional representation).

Properties of the classifying spaces $\mathcal{R}_p^q$ and $\mathcal{R}_p$

- $\mathcal{R}_p^q = \mathcal{R}_{p+8}^q$
  From $\tilde{A} \in R_p^q$ that satisfies

$$\tilde{A}\tilde{\rho}_i = -\tilde{\rho}_i\tilde{A}, \quad \tilde{A}^2 = -1, \qquad \tilde{\rho}_j\tilde{\rho}_i + \tilde{\rho}_i\tilde{\rho}_j|_{i\neq j} = 0,$$
$$\tilde{\rho}_i^2|_{i=1,\dots,p} = 1, \quad \tilde{\rho}_i^2|_{i=p+1,\dots,p+q} = -1,$$

  we can define

$$A = \tilde{A} \otimes \gamma, \quad \rho_i|_{i=1,\dots,p} = \tilde{\rho}_i \otimes \gamma, \quad \rho_{p+i}|_{i=1,\cdots,8} = I \otimes \gamma_i,$$
$$\rho_i|_{i=p+8+1,\dots,p+8+q} = \tilde{\rho}_{i-8} \otimes \gamma,$$

  We can check that $A \in \mathcal{R}_{p+8}^q$

$$A\rho_i = -\rho_i A, \quad A^2 = -1, \qquad \rho_j\rho_i + \rho_i\rho_j|_{i\neq j} = 0,$$
$$\rho_i^2|_{i=1,\dots,p+8} = 1, \quad \rho_i^2|_{i=p+8+1,\dots,p+8+q} = -1,$$

- The above implies that $\mathcal{R}_p^q = \mathcal{R}_{p+8}^q = \mathcal{R}_p^{q+8}$.
  $\mathcal{R}_p^q = \mathcal{R}_{q-p+2}$ and $\mathcal{R}_p = \mathcal{R}_{p+8}$.

Go to higher dimensions (complex cases)

- d-dimensional complex cases: $\hat{H} = \int \mathrm{d}^d \boldsymbol{x}\ \hat{c}^\dagger(\gamma^i \mathrm{i}\partial_i + M)\hat{c}$.
  We consider symmetries that anti-commute with M and $(\gamma^i \mathrm{i}\partial_i)$:
  $$M^\dagger = M, \quad M^2 = 1, \quad M\rho_a = -\rho_a M, \quad \rho_a^\dagger = \rho_a^{-1}, \quad \rho_a\rho_b + \rho_b\rho_a = 2\delta_{ab};$$
  Since $(\gamma^i \mathrm{i}\partial_i)\rho_a = -\rho_a(\gamma^i \mathrm{i}\partial_i)$, we have
  $$\gamma_i\rho_a = -\rho_a\gamma_i, \quad \gamma_i^\dagger = \gamma_i, \quad \gamma_i^2 = \mathrm{id}, \quad \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}, \quad \gamma_i M = -M\gamma_i.$$
  Thus the classifying space is $\mathcal{C}_{p+d}$.
  If the symmetry commute with single-body Hamiltonian (matrix), we can consider the common eigenspace, and "ignore" the symmetry.
- We can show that $\mathcal{C}_p = \mathcal{C}_{p+2}$. Let $\tilde{M} \in \mathcal{C}_p$, satisfying
  $$M^\dagger = M, \quad M^2 = 1, \quad M\rho_a = -\rho_a M, \quad \rho_a\rho_b + \rho_b\rho_a = 2\delta_{ab}.$$
  Let $\tilde{M} = M \otimes \sigma^3$, $\tilde{\rho}_i = \rho_i \otimes \sigma^3$, $\tilde{\rho}_{p+1} = I \otimes \sigma^1$, $\tilde{\rho}_{p+2} = I \otimes \sigma^2$.
  Then $\tilde{M} \in \mathcal{C}_{p+2}$.
- IQH states in 2D (1980):
  $\pi_0(\mathcal{C}_2) = \mathbb{Z}$. vonKlitzing-Dorda-Pepper, PRL **45** 494, (80)

Go to higher dimensions (real cases)

- d-dimensional real cases: $\hat{H} = \mathrm{i} \int \mathrm{d}^d \boldsymbol{x}\ \eta^\top(\gamma^i \partial_i + M)\eta$, where $M = M^* = -M^\top, \quad M^2 = -1, \quad M\rho_a = -\rho_a M, \quad \rho_a \rho_b + \rho_b \rho_a = \pm 2\delta_{ab}$;
  Symmetry also requires $(\gamma^i \partial_i)\rho_a = -\rho_a(\gamma^i \partial_i) \rightarrow$
  $\gamma_i \rho_a = -\rho_a \gamma_i, \quad \gamma_i^\top = \gamma_i, \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}, \quad \gamma_i M = -M\gamma_i.$
  Classifying space $= \mathcal{R}^q_{p+d} = \mathcal{R}_{q-p-d+2}$.

Go to higher dimensions (real cases)

- d-dimensional real cases: $\hat{H} = \mathrm{i}\int \mathrm{d}^d\boldsymbol{x}\ \eta^\top(\gamma^i\partial_i + M)\eta$, where
  $M = M^* = -M^\top,\quad M^2 = -1,\quad M\rho_a = -\rho_a M,\quad \rho_a\rho_b + \rho_b\rho_a = \pm 2\delta_{ab};$
  Symmetry also requires $(\gamma^i\partial_i)\rho_a = -\rho_a(\gamma^i\partial_i) \to$
  $\gamma_i\rho_a = -\rho_a\gamma_i,\quad \gamma_i^\top = \gamma_i,\quad \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij},\quad \gamma_i M = -M\gamma_i.$
  Classifying space $= \mathcal{R}^q_{p+d} = \mathcal{R}_{q-p-d+2}$.

- Topo. $d + \mathrm{i}d/p + \mathrm{i}p$ SC in 2D (1999):
  $\mathcal{R}^0_{0+2} = \mathcal{R}_0 \to \pi_0(\mathcal{R}_0) = \mathbb{Z}$.
  Senthil-Marston-Fisher cond-mat/9902062
  Read-Green cond-mat/9906453

- Topological p-wave SC in 1D (2001):
  $\mathcal{R}^0_{0+1} = \mathcal{R}_1 \to \pi_0(\mathcal{R}_1) = \mathbb{Z}_2$.
  Kitaev cond-mat/0010440

- Topological insulator in 2D (2005):
  $\mathcal{R}^2_{0+2} = \mathcal{R}_2 \to \pi_0(\mathcal{R}_2) = \mathbb{Z}_2$.
  Kane-Mele cond-mat/0506581

- Topological insulator in 3D (2006):
  $\mathcal{R}^2_{0+3} = \mathcal{R}_1 \to \pi_0(\mathcal{R}_1) = \mathbb{Z}_2$.
  Moore-Balents cond-mat/0607314; Fu-Kane-Mele cond-mat/0607699

Gapped phases of non-interacting fermions

Real cases (blue entries for interacting classification):

Symm. group G^f	$U^f(1)\rtimes Z_2^T$	$\mathbb{Z}_2^T\times Z_2^f$	Z_2^f	Z_4^T $Z_4^T\times Z_2$	$\frac{U^f(1)\rtimes Z_4^T}{Z_2}$ $\frac{Z_4^f\rtimes Z_4^T}{Z_2}$	$\frac{U^f(1)\rtimes Z_4^T\times Z_4^f}{Z_2^2}$	$SU^f(2)$	$\frac{SU^f(2)\times Z_4^T}{Z_2}$
$\mathcal{R}_p\vert_{\text{for } d=0}$	$\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times\mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
class	AI	BDI	D	DIII	AII	CII	C	CI
$d=0$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$d=1$	0 $(\mathbb{Z}_2)$	$\mathbb{Z}$ $(\mathbb{Z}_8)$	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$d=2$	0	0	$\mathbb{Z}$ $(\mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$d=3$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$d=4$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$d=5$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$d=6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$d=7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Example	insulator w/ coplanar spin order $\tilde{T}$	supercond. w/ coplanar spin order $\tilde{T}$	supercond. (no symm.)	supercond. w/ time reversal T	insulator w/ time reversal T	insulator w/ time reversal and intersublattice hopping	spin singlet supercond.	spin singlet supercond. w/ time reversal T

Ryu-Schnyder-Furusaki-Ludwig arXiv:0912.2157, Kitaev cond-mat/0010440

Complex cases:

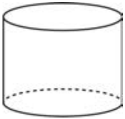
Wen arXiv:1111.6341

Symm. group	$\mathcal{C}_p\vert_{\text{for } d=0}$	class	$p\backslash d$	0	1	2	3	4	5	6	7	example
$U^f(1)$ Z_4^f	$\frac{U(l+m)}{U(l)\times U(m)}\times\mathbb{Z}$	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	(Chern) insulator supercond. with collinear spin order
$U^f(1)\times Z_2^T$ $Z_4^f\times Z_2^T$	$U(n)$	AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	supercond. w/ real pairing and S_z conserving spin-orbital coupling

Classifying spaces $\mathcal{R}_p$

p mod 8	0	1	2	3	4	5	6	7
$\mathcal{R}_p$	$\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times\mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
$\pi_0(R_p)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$\pi_1(R_p)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_2(R_p)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_3(R_p)$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_4(R_p)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$\pi_5(R_p)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\pi_6(R_p)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$\pi_7(R_p)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0

- Let $\mathcal{M}_d$ be the space of gapped $d+1$D fermion systems. Then $\mathcal{M}_d \sim \Omega\mathcal{M}_{d+1} \to \pi_{n-1}(\mathcal{M}_d) = \pi_n(\mathcal{M}_{d+1})$
  $\Omega\mathcal{M}$ is the loop space of $\mathcal{M}$: the space of the based loops in $\mathcal{M}$. For example: point $\sim \Omega S^2$, $Z \sim \Omega S^1$.



- Consider a 2D system H_g that form a cylinder. As we go around the cylinder, g goes around a loop in $\mathcal{M}_2$. We may also view the cylinder as a 1D system. Thus we obtain a map $\Omega\mathcal{M}_2 \to \mathcal{M}_1$.

- $\mathcal{M}_d \sim \mathcal{R}_{q-p+2-d} \to \mathcal{R}_p = \Omega\mathcal{R}_{p-1}$, $\pi_{n-1}(\mathcal{R}_p) = \pi_n(\mathcal{R}_{p-1})$

Why classification is useful apart from deep understanding?

- K-theory classification is constructive, which allow us to constructive all possible free-fermion gapped phases.

- An universal model for complex classes of topological phases of non-interacting fermions $H_{\text{one-body}} = \gamma^i \otimes I_n \mathrm{i}\partial_i + M,\ \{\gamma^i, \gamma^j\} = 2\delta_{ij}$

- An universal model for real classes of top. phases of non-interacting fermions $H_{\text{one-body}} = \mathrm{i}(\gamma_R^i \otimes I_n \partial_i + A_R),\ \{\gamma_R^i, \gamma_R^j\} = 2\delta_{ij}$

- **Example in 2D**: Fermion hopping on honeycomb lattice $\to$ two 2-component massless Dirac fermions (R,L pairs)

$$H_{\text{one-body}} = \mathrm{i}\sigma^1 \otimes \sigma^0 \partial_x + \mathrm{i}\sigma^3 \otimes \sigma^3 \partial_y, \quad \text{complex case}$$
$$= \mathrm{i}(\sigma^1 \otimes \sigma^0 \partial_x + \sigma^3 \otimes \sigma^3 \partial_y). \quad \text{complex case}$$

To obtain one-body Hamiltonian in Majorana basis, we replace 1 by σ^0 and i by $-\varepsilon$ in the above bracket, to obtain (see page 14 of this file)

$$H_{\text{one-body}} = \sigma^0 \otimes \sigma^1 \otimes \sigma^0 \partial_x + \sigma^0 \otimes \sigma^3 \otimes \sigma^3 \partial_y. \quad \text{real case}$$

Why classification is useful apart from deep understanding?

n-layers of honeycomb lattice $\to$ $2n$ 2-component massless Dirac fermions (n 4-component massless Dirac fermions)

$$H_{\text{one-body}} = \mathrm{i}\sigma^1 \otimes \sigma^0 \otimes I_n \partial_x + \mathrm{i}\sigma^3 \otimes \sigma^3 \otimes I_n \partial_y, \quad \text{complex case}$$
$$H^R_{\text{one-body}} = \mathrm{i}(\sigma^0 \otimes \varepsilon \otimes \sigma^0 \otimes I_n \partial_x + \sigma^0 \otimes \sigma^1 \otimes \varepsilon \otimes I_n \partial_y), \quad \text{real case}$$

- Adding a proper mass term according to the K-theory classification $\to$ a designed free-fermion gapped state.

$$H_{\text{one-body}} = \mathrm{i}\sigma^1 \otimes \sigma^0 \otimes I_n \partial_x + \mathrm{i}\sigma^3 \otimes \sigma^3 \otimes I_n \partial_y + M, \quad \text{complex case}$$
$$H^R_{\text{one-body}} = \mathrm{i}(\sigma^0 \otimes \sigma^1 \otimes \sigma^0 \otimes I_n \partial_x + \sigma^0 \otimes \sigma^3 \otimes \sigma^3 \otimes I_n \partial_y + A_R), \text{ real case}$$

Choose $n = 1$:

$$H^R_{\text{one-body}} = \mathrm{i}(\sigma^0 \otimes \sigma^1 \otimes \sigma^0 \partial_x + \sigma^0 \otimes \sigma^3 \otimes \sigma^3 \partial_y + A), \quad A = A^* = -A^\top.$$

- $U^f(1)$-symmetry $Q = \varepsilon \otimes \sigma^0 \otimes \sigma^0$, which satisfies

$$Q\sigma^0 \otimes \sigma^1 \otimes \sigma^0 = \sigma^0 \otimes \sigma^1 \otimes \sigma^0 Q, \quad Q\sigma^0 \otimes \sigma^3 \otimes \sigma^3 = \sigma^0 \otimes \sigma^3 \otimes \sigma^3 Q,$$
$$QA = AQ, \quad Q^2 = -1.$$

T-symmetry $T = \sigma^3 \otimes \varepsilon \otimes \sigma^0$:

$$T\sigma^0 \otimes \sigma^1 \otimes \sigma^0 = -\sigma^0 \otimes \sigma^1 \otimes \sigma^0 T, \quad T\sigma^0 \otimes \sigma^3 \otimes \sigma^3 = -\sigma^0 \otimes \sigma^3 \otimes \sigma^3 T,$$
$$TA = -AT, \quad T^\top = T^{-1}, \quad T^2 = -1, \quad TQ = -QT.$$

A continuum model for 2d top. insulator ($U^f(1) \rtimes Z_4^T / Z_2^f$)

- The conditions on A

$$A\sigma^0 \otimes \sigma^1 \otimes \sigma^0 = -\sigma^0 \otimes \sigma^1 \otimes \sigma^0 A, \quad A\sigma^0 \otimes \sigma^3 \otimes \sigma^3 = -\sigma^0 \otimes \sigma^3 \otimes \sigma^3 A,$$
$$A\sigma^3 \otimes \varepsilon \otimes \sigma^0 = -\sigma^3 \otimes \varepsilon \otimes \sigma^0 A, \quad A\varepsilon \otimes \sigma^0 \otimes \sigma^0 = \varepsilon \otimes \sigma^0 \otimes \sigma^0 A,$$

- From the last relation: $A = \#\sigma^0 \otimes \sigma^\mu \otimes \sigma^\nu + \#\varepsilon \otimes \sigma^\mu \otimes \sigma^\nu$.
- Adding the first relation: $A = \#\sigma^0 \otimes \sigma^{3,\varepsilon} \otimes \sigma^\nu + \#\varepsilon \otimes \sigma^{3,\varepsilon} \otimes \sigma^\nu$.
  where $\sigma^\varepsilon = \varepsilon$.
- Adding the second relation: $A = \#\sigma^0 \otimes \sigma^3 \otimes \sigma^{1,\varepsilon} + \#\sigma^0 \otimes \varepsilon \otimes \sigma^{0,3}$
  $+\#\varepsilon \otimes \sigma^3 \otimes \sigma^{1,\epsilon} + \#\varepsilon \otimes \varepsilon \otimes \sigma^{0,3}$.
- Adding the conidtion $A^\top = -A$:
  $A = \#\sigma^0 \otimes \sigma^3 \otimes \varepsilon + \#\sigma^0 \otimes \varepsilon \otimes \sigma^0 + \#\sigma^0 \otimes \varepsilon \otimes \sigma^3 + \#\varepsilon \otimes \sigma^3 \otimes \sigma^1$.
- Adding the third relation $\to$ A must has a form $A = m\sigma^0 \otimes \sigma^3 \otimes \varepsilon$
  $m > 0$ is one phase and $m < 0$ is another phase (maybe since $n = 1$).

- We know the two phases are different, but we do not know which is trivial and which is non-trivial. Within the field theory, we cannot know. Only after adding lattice reularization, we can know.

- A Dirac fermion realization of 2d topological insulator with symmetry $U^f(1) \rtimes Z_4^T/Z_2^f$, Majorana fermion basis:

$$H^R_{\text{one-body}} = \mathrm{i}(\sigma^0 \otimes \sigma^1 \otimes \sigma^0 \partial_x + \sigma^0 \otimes \sigma^3 \otimes \sigma^3 \partial_y + m\sigma^0 \otimes \sigma^3 \otimes \varepsilon)$$

$$Q = \varepsilon \otimes \sigma^0 \otimes \sigma^0, \quad T = \sigma^3 \otimes \varepsilon \otimes \sigma^0.$$

- Complex fermion basis ($\sigma^0 \to 1$ and $\varepsilon \to -\mathrm{i}$ for the first position):

$$H^R_{\text{one-body}} = \mathrm{i}(\sigma^1 \otimes \sigma^0 \partial_x + \sigma^3 \otimes \sigma^3 \partial_y + m\sigma^3 \otimes \varepsilon)$$

$$Q = -\mathrm{i}\sigma^0 \otimes \sigma^0, \quad T = ?.$$

The T action is explicit only in Majorana fermion basis.

Do we have an universal physical probe to detect all non-interacting fermionic topological phases?

- Boundary states are universal physical probe that can detect all topological phase, but not one-to-one.

Holographic principle of topological phases: Boundary completely determine the bulk, but bulk does not determine the boundary.

The bulk = the anomaly of the boundary effective theory