PROFESSOR: So here comes the point that this quite fabulous about Hermitian operators. Here is the thing that it really should impress you. It's the fact that any, all Hermitian operators have as many eigenfunctions and eigenvalues as you can possibly need, whatever that means. But they're rich. It's a lot of those states. What it really means is that the set of eigenfunctions for any Hermitian operator-- whatever Hermitian operator, it's not just for some especially nice ones-- for all of them you get eigenfunctions.

And these eigenfunctions, because it has vectors, they are enough to span the space of states. That is any state can be written as a superposition of those eigenvectors. There's enough. If you're thinking finite dimensional vector spaces, if you're looking at the Hermitian matrix, the eigenvectors will provide you a basis for the vector space. You can understand anything in terms of eigenvectors. It is such an important theorem. It's called the spectral theorem in mathematics.

And it's discussed in lots of detail in 805. Because there's a minor subtlety. We can get the whole idea about it here. But there are a couple of complications that mathematicians have to iron out. So basically let's state we really need, which is the following. Consider the collection of eigenfunctions and eigenvalues of the Hermitian operator q. And then I go and say, well, q psi 1 equal q 1 psi 1 q psi 2 equal q2 psi 2.

And I actually don't specify if it's a finite set or an infinite set. The infinite set, of course, is a tiny bit more complicated. But the result is true as well. And we can work with it. So that is the set up. And here comes the claim. Claim 3, the eigenfunctions can be organized to satisfy the following relation, integral dx psi i of x psi j of x is equal to delta ij. And this is called orthonormality.

Let's see what this all means. We have a collection of eigenfunctions. And here it says something quite nice. These functions are like orthonormal functions, which is to say each function has unit norm. You see, if you take i equal to j, suppose you take psi 1 psi 1, you get delta 1 1, which is 1. Remember the [INAUDIBLE] for delta is 1 from the [INAUDIBLE] are the same. And it's 0 otherwise. psi 1 the norm of psi 1 is 1 and [INAUDIBLE] squared [INAUDIBLE] psi 1, psi 2, psi3, all of them are well normalized.

So they satisfied this thing we wanted them to satisfy. Those are good states. psi 1, psi 2, psi

3, those are good states. They are all normalized. But even more, any two different ones are orthonormal. This is like the 3 basis vectors of r3. The x basic unit vector, the y unit vector, the z unit vector, each one has length 1, and they're all orthonormal.

And when are two functions orthonormal? You say, well, when vectors are orthonormal I know what I mean. But orthonormality for functions means doing this integral. This measures how different one function is from another one. Because if you have the same function, this integral and this positive, and this all adds up. But for different functions, this is a measure of the inner product between two functions. You see, you have the dot product between two vectors. The dot product of two functions is an integral like that. It's the only thing that makes sense

So I want to prove one part of this, which is a part that is doable with elementary methods. And the other part is a little more complicated. So let's do this. And consider the case if qi is different from qj, I claim i can prove this property. We can prove this orthonormality. So start with the integral dx of psi i star q psi j. Well, q out here at psi j is qj. So this is integral dx psi i star qj psi j. And therefore, it's equal to qj times integral psi i star psi j.

I simplified this by just enervating it. Because psi i and psi j are eigenstates of q. Now, the other thing I can do is use the property that q is Hermitant and move the q to act on this function. So this is equal to integral dx q i psi i star psi j. And now I can keep simplifying as well. And I have dx. And then I have the complex conjugate of qi psi i psi i, like this, psi j. And now, remember q is an eigenvalue for Hermitian operator. We already know it's real. So q goes out of the integral as a number. Because it's real, and it's not changed. Integral dx psi i star psi j.

The end result is that we've shown that this quantity is equal to this second quantity. And therefore moving this-- since the integral is the same in both quantities, this shows that q i minus qj, subtracting these two equations, or just moving one to one side, integral psi i star psi j dx is equal to 0. So look what you've proven by using Hermiticity, that the difference between the eigenvalues times the overlap between psi i and psi j must be 0.

But we started with the assumption that the eigenvalues are different. And if the eigenvalues are different, this is non-zero. And the only possibility is that this integral is 0. So this implies since we've assumed that qi is different than qj. We've proven that psi i star psi j dx is equal to 0. And that's part of this little theorem. That the eigenfunctions can be organized to have orthonormality and orthonormality between the different points.

My proof is good. But it's not perfect. Because it ignores one possible complication, which is that here we wrote the list of all the eigenfunctions. But sometimes something very interesting happens in quantum mechanics. It's called degeneracy. And degeneracy means that there may be several eigenfunctions that are different but have the same eigenvalue. We're going to find that soon-- we're going to find, for example, states of a particle that move in a circle that are different and have the same energy. For example, a particle moving in a circle with this velocity and a particle moving in a circle with the same magnitude of the velocity in the other direction are two states that are different but have the same energy eigenvalue.

So it's possible that this list not all are different. So suppose you have like three or four degenerate states, say three degenerate states. They all have the same eigenvalue. But they are different. Are they orthonormal or not? The answer is-- actually the clue is there. The eigenfunctions can be organized to satisfy. It would be wrong if you say the eigenfunctions satisfy. They can be organized to satisfy. It means that, yes, those ones that have different eigenvalues are automatically orthonormal. But those that have the same eigenvalues, you may have three of them maybe, they may not necessarily be orthonormal. But you can do linear transformations of them and form linear combinations such that they are orthonormal. So the interesting part of this theorem, which is the more difficult part mathematically, is to show that when you have degeneracies this still can be done. And there's still enough eigenvectors to span the space.