# 8.04, Quantum Physics I, Fall 2015

## TEST 1

**Tuesday October 20, 9:30-11:00 pm**

**You have 90 minutes.**

**Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).**

**There are five questions, totalling 100 points.**

**No books, notes, or calculators allowed.**

**Show your work CLEARLY!**

# Formula Sheet

- $\hbar c \simeq 197.3\ \text{MeV}\cdot\text{fm}\,, \quad m_e c^2 \simeq 0.511\ \text{MeV}\,, \quad m_p c^2 = 938\ \text{MeV}\,, \quad \frac{e^2}{\hbar c} \simeq \frac{1}{137}$
- Relativity: $p = \gamma\, mv\,, \quad E = \gamma mc^2\,, \quad E^2 = p^2c^2 + m^2c^4\,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \beta = \frac{v}{c}$
- Photons: $E = h\nu\,, \; p = \frac{h}{\lambda}\,,$ or $E = \hbar\omega\,, \; p = \hbar k$
- Wavelengths

$$\text{de Broglie: } \lambda = \frac{h}{p}\,, \qquad \text{Compton: } \lambda_C = \frac{h}{mc}\,.$$

- Momentum and position operators

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}\,, \quad [x,p] = i\hbar\,, \qquad \mathbf{p} = \frac{\hbar}{i}\nabla\,, \quad [x_i, p_j] = i\hbar\,\delta_{ij}$$

- Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x},t) = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\Big)\Psi(\mathbf{x},t)\,,$$

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0$$

$$\rho(\mathbf{x},t) = |\Psi(\mathbf{x},t)|^2\;; \qquad \mathbf{J}(\mathbf{x},t) = \frac{\hbar}{m}\text{Im}\,[\Psi^*\nabla\Psi]$$

- Fourier transforms:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int dk\,\Phi(k)e^{ikx}\,, \quad \Phi(k) = \frac{1}{\sqrt{2\pi}}\int dx\,\Psi(x)e^{-ikx}\,, \quad \int dx\,|\Psi(x)|^2 = \int dk\,|\Phi(k)|^2$$

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\,\Phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,, \; \Phi(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\,\Psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \; \int d^3x\,|\Psi(\mathbf{x})|^2 = \int d^3k\,|\Phi(\mathbf{k})|^2$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx = \delta(k)\,, \qquad \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x = \delta^{(3)}(\mathbf{k})$$

$$\int_{-\infty}^{+\infty} dx\,\exp\left(-ax^2+bx\right) = \sqrt{\frac{\pi}{a}}\exp\Big(\frac{b^2}{4a}\Big)\,, \quad \text{when } \text{Re}(a) > 0\,.$$

- Wavepackets

$$v_{group} = \frac{d\omega}{dk}\,, \quad \Delta k\,\Delta x \simeq 1\,, \quad \text{shape preserving}: \; t\,\Delta v \leq \Delta x$$

- Hermitian conjugation:

$$\int dx\,(K\Psi(x,t))^*\Psi(x,t) = \int dx\,\Psi^*(x,t)\,(K^\dagger\Psi(x,t))$$

If $K^\dagger = K$, then $K$ is Hermitian.

- Expectation values

$$\langle Q\rangle(t) \;=\; \int dx\, \Psi^*(x,t)(Q\Psi(x,t))$$

- Time evolution of expectation value. For $Q$ Hermitian

$$i\hbar\frac{d}{dt}\langle Q\rangle \;=\; \big\langle [Q,H]\big\rangle$$

- Commutator identity

$$[A,BC] = [A,B]C + B[A,C]$$

- Uncertainty $\Delta Q$ of a Hermitian operator $Q$

$$(\Delta Q)^2 \;=\; \langle Q^2\rangle - \langle Q\rangle^2 \;=\; \big\langle (Q-\langle Q\rangle)^2\big\rangle$$

- Uncertainty principle: $\Delta x\, \Delta p \geq \frac{\hbar}{2}$

1. **Estimate and units** [10 points]

   The size of a proton (more technically, the so-called charge radius) is about 0.9 fm. How does its Compton wavelength compare with the size? The possible answers are: The Compton wavelength is [ a lot bigger, a little bigger, a little smaller, and a lot smaller ] than the size. Which one is it?

2. **Group velocity** [10 points]

   For a relativistic particle $E^2 = p^2c^2 + m^2c^4$. Evaluate the group velocity $v_g = \frac{d\omega}{dk}$ recalling that $E = \hbar\omega$ and $p = \hbar k$. Leave your answer in terms of the velocity $v$ of the particle (without any approximation!).

3. **Free particle evolution** [15 points]

   Consider the state of a *free* particle of mass $m$ that at time equal zero is represented by the wavefunction

   $$\Psi(x,0) \;=\; \sin k_0 x\,, \quad k_0 \in \mathbb{R}\,.$$

   (a) Find the probability current $J(x,0)$.

   (b) If we measure the momentum of the particle (at time equal zero) what are the possible values that we may obtain?

   (c) Calculate $\Psi(x,t)$.

4. **Improving on bomb detection** [35 points]

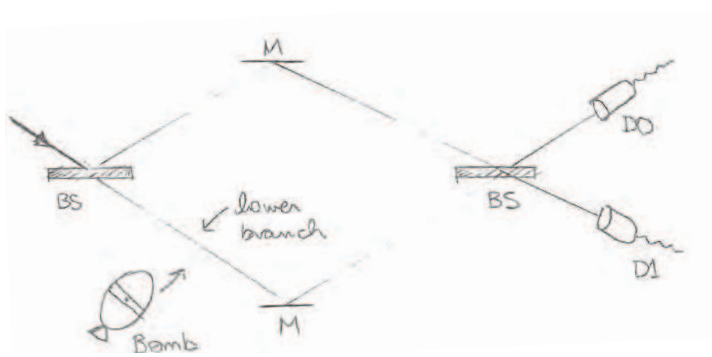
We modify the Mach-Zehnder interferometer to increase the percentage of Elitzur-Vaidman bombs that can be vouched to work without detonating them.

For this purpose we build a beam-splitter with reflectivity $R$ and transmissivity $T$. A photon incoming (from either port) has a *probability* $R$ to be reflected and a probability $T$ to be transmitted ($R+T=1$). Let $r$ and $t$ denote the *positive* square roots:

$$r \equiv \sqrt{R}, \quad t \equiv \sqrt{T}.$$

(a) Build the two-by-two matrix $U$ that represents the beam splitter. For this consider what happens when a photon hits the beam splitter from the top side (input here is $\begin{pmatrix}1\\0\end{pmatrix}$) and when it hits it from the bottom side (input here is $\begin{pmatrix}0\\1\end{pmatrix}$). To fix conventions $U$ will have all entries positive (and real) except from the bottom right-most element (the 2,2 element). Confirm that $U$ is unitary.


The interferometer with detectors D0 and D1 (shown below) uses two identical copies of the beam splitter. The incoming photon arrives from the top side.




(b) A defective bomb is inserted in the lower branch of the interferometer. What are the detection probabilities $P_0$ and $P_1$ at D0 and D1 respectively?

(c) A functioning bomb inserted in the lower branch of the interferometer. What is the detonation probability $P_{boom}$ and the detection probabilities $P_0$ and $P_1$? Express your answers in terms of $R$ and $T$.

(d) You test bombs until you are reasonably sure that either they malfunction or that they are operational. What fraction $f$ of the operational bombs can be certified to be good without detonating them? Give your answer in terms of $R$. What is the maximum possible value for $f$?

5. **Overlap between moving gaussians** [30 points]

We have learned (homework) that the overlap $\int dx \Psi_1^* \Psi_2$ between two different wave packets $\Psi_1$ and $\Psi_2$ is, on general grounds, time *independent*.

The following EXAMPLE seems to be in tension with this fact: Consider two gaussians that coincide at $t = 0$ but are moving in opposite directions with large momenta compared to their momenta uncertainties. Now we make two claims:

1. At $t = 0$ the overlap is big.

2. Once the centers of the packets are separated by a small multiple of the position uncertainty the overlap is small.

The purpose of this problem is to find a flaw in the claims of the EXAMPLE.

Consider a *free particle* and a normalized gaussian wavepacket $\hat{\Psi}$ at zero time:

$$\hat{\Psi}(x, 0) = \frac{1}{(2\pi)^{1/4}\sqrt{a}} \exp\Big(-\frac{x^2}{4a^2}\Big) .$$

From this we create two wavepackets $\Psi_1$ and $\Psi_2$:

$$\begin{aligned} \Psi_1(x, 0) \ &\equiv \ e^{iqx/\hbar} \ \hat{\Psi}(x, 0) , \\ \Psi_2(x, 0) \ &\equiv e^{-iqx/\hbar} \ \hat{\Psi}(x, 0) . \end{aligned}$$

Here $q$ is a real quantity with units of momentum.

(a) What is $\langle p \rangle$ for $\hat{\Psi}(x, 0)$ ? Will this expectation value change in time? Explain.

(b) What is $\langle p \rangle$ for $\Psi_1(x, 0)$ ? What is $\langle p \rangle$ for $\Psi_2(x, 0)$ ? Do these expectation values change in time?

(c) Compute the zero-time overlap $\gamma(0)$ of the two packets:

$$\gamma(0) \ = \ \int \Psi_1^*(x, 0)\Psi_2(x, 0) dx .$$

The following integral may be useful:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\Big(\frac{b^2}{4a}\Big) , \quad \text{when} \ \ \text{Re}(a) > 0 .$$

(d) Write an inequality that expresses the fact that the momentum of the wave-packets is *large* compared to the momentum uncertainty. What was wrong with the claims of the EXAMPLE ?