Massachusetts Institute of Technology

Physics 8.03 Fall 2016

Exam 2

Instructions

Please write your solutions in the white booklets. We will not grade anything written on the exam copy. This exam is closed book. No electronic equipment is allowed. All phones, blackberry, blueberry, raspberry Pi, tablets, computers etc. must be switched off.

Formula Sheet Exam 2

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A\cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2\Gamma^2}}$$

and

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:
For $\Gamma = 0$ (undamped system):

$$x(t) = R\cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\,t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.

Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudies for the first ($c_1(\omega_d)$) and second ($c_2(\omega_d)$) objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$$

Eigenvalues (β) and eigenvectors ($\mathcal{A}$) of this 2×2 $\mathcal{S}$ matrix:

(1) $\beta = -1$, $\mathcal{A} = \begin{matrix} 1 \\ 1 \end{matrix}$

(2) $\beta = 1$, $\mathcal{A} = \begin{matrix} 1 \\ -1 \end{matrix}$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β, the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of jth(0th) object in the system.

Consider an one dimentional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node} : \sum_i I_i = 0 \ \text{Loop} : \sum_i \Delta V_i = 0$$

$$\text{Capacitors} : \Delta V = \frac{Q}{C} \quad \text{Inductors} : \Delta V = -L\frac{dI}{dt} \quad \text{Current} : I = \frac{dQ}{dt}$$

Trigonometric equalities:

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Integrals involving sin and cos:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$
$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$
$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$
$$\int x\sin(x)dx = \sin(x) - x\cos(x) + C$$
$$\int x\cos(x)dx = \cos(x) + x\sin(x) + C$$

Maxwell Equations in vacuum

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \; ; \; \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \; ; \; \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\epsilon_0\frac{\partial E_z}{\partial t} \; ; \; \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \; ; \; \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0\epsilon_0\frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \; ; \; \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Wave equation for EM fields in vacuum

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \text{ where } \; i = x, y, z$$

$$\frac{\partial^2 B_i}{\partial x^2} + \frac{\partial^2 B_i}{\partial y^2} + \frac{\partial^2 B_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_i}{\partial t^2} \text{ where } \; i = x, y, z$$

For EM plane waves in vacuum:

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\hat{k} \times \vec{E}(\vec{r}, t)$$

$$\vec{E}(\vec{r}, t) = c\vec{B}(\vec{r}, t) \times \hat{k}$$

Linear energy density in a string with tension T and mass density ρ_L

$$\frac{dK}{dx} = \frac{1}{2}\rho_L\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

EM energy per unit volume and Poynting vector:

$$U_E = \frac{1}{2}\epsilon_0\vec{E}^2 \qquad U_B = \frac{1}{2\mu_0}\vec{B}^2 \qquad \vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Transmission and reflection

$$R = \frac{z_1 - z_2}{z_2 + z_1} \qquad\qquad T = \frac{2z_1}{z_2 + z_1}$$

Phase velocity and impedance:

$$v = \sqrt{\frac{T}{\rho_L}} \qquad\qquad Z = \sqrt{T\rho_L} \;\; \text{(string)}$$

$$v = \sqrt{\frac{1}{LC}} \qquad\qquad Z = \sqrt{\frac{L}{C}} \;\; \text{(transmission line)}$$

Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Fourier transform

$$f(t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t}$$

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt = \delta(\omega - \omega')$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

Problem 1 (30 pts)

Solve the following short questions. (If you found that you spend a lot of time on one problem, you are probably not on the right track.)

A. (6 pts) A progressing plane electromagnetic wave is moving toward a perfect conductor, where charges in the conductor can move freely without energy dissipation. What is the boundary condition for the electric field of electromagnetic wave at the **surface** of the conductor (not inside the conductor)?

B. (6 pts) An AM radio station with radio frequency f recently received bad reviews on the audio quality. The manager of this AM station asked you for advice. You found that one possibility to improve the audio quality was to change the bandwidth Δf of the signal emitted by this station such that the station can send signals with better time resolution. Would you suggest to increase or decrease the bandwidth Δf? Why do you think so?

C. (6 pts) In a room of size $L \times L \times 16L$, what is the lowest angular frequency of normal mode oscillations of the air in the room? (You can assume that the speed of sound is v).

D. (6 pts) Consider a massive string with fixed ends, of length $2L$, tension T and linear mass density ρ_L. At $t = 0$, this string has an initial shape $\psi(x, 0)$ as shown in Figure 1. This string is then released carefully so that the initial velocity of the string $\dot{\psi}(x, 0)$ is 0. How long does it take for this string, a non-dispersive medium, to return to its initial shape after it is released at $t = 0$? Please give your explanations without actually doing Fourier decomposition.

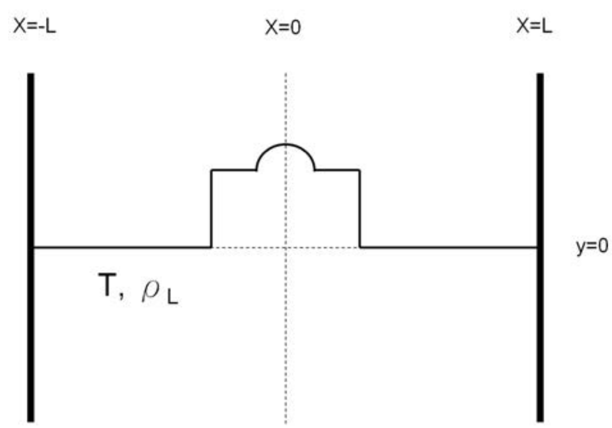

Figure 1: MIT string

E. (6 pts) A light beam travels through vacuum ($n_1 = 1$) before reaching two transparent plates with indices of refraction n_2 and n_3. It reaches first a transparent plate of index n_2 at an incident angle of $\alpha = 60°$. The beam traverses it, reaches another transparent plate with index n_3, traverses that, and enters a fourth material with index n_4, propagating in that medium at an angle $\beta = 45°$ to the normal axis. The configuration of this optical experiment is shown in Figure 2. What is the value for n_4?

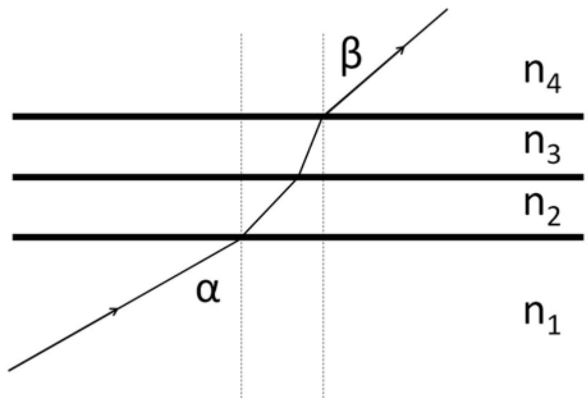

Figure 2: Light experiment

Problem 2 (17 pts)

The charge density fluctuations in a plasma are governed by a wave equation, which for one dimensional distortions reduces to

$$a^2 \frac{\partial^2 \rho(x,t)}{\partial x^2} - \Omega_p^2 \rho(x,t) = \frac{\partial^2 \rho(x,t)}{\partial t^2}$$

Here $\rho(x,t)$ is a small deviation of the charge density from its equilibrium value and Ω_p is a constant parameter referred to as the plasma frequency.

a. (3 pts) Write down a harmonic progressing wave solution which describes the deviation of the charge density $\rho(x,t)$, going toward the $-\hat{x}$ direction. We assume that the wavenumber of this progressing wave is k, the angular frequency is ω and the amplitude is A. At $t = 0$ and $x = 0$, the deviation of the charge density $\rho(0,0)$ is zero.

b. (7 pts) Find the dispersion relation $\omega(k)$. Draw $\omega(k)$ as a function of k. What are the angular frequencies of harmonic progressing waves which are allowed to exist in the plasma?

c. (4 pts) What is the group velocity of this medium? Is this a dispersive medium?

d. (3 pts) What is the limiting phase velocity at large wavenumber k of this medium?

Problem 3 (23 pts)

The electric field of a uniform plane wave in vacuum, traveling at the speed of light c, is given as

$$\vec{E}(\vec{r}, t) = E_0(3\hat{x} + a\hat{y})\cos(\omega t - 4x + 3y)$$

where a is a constant. (This is a problem in the three dimensional space.)

a. (4 pts) What is the direction of propagation of this plane wave? (Represent the direction as a Unit Vector).

b. (6 pts) Calculate the angular frequency ω and the wavelength of this EM wave by using given numbers

c. (4 pts) What is the value of a?

d. (6 pts) What is the magnetic field, $\vec{B}$, associated with this wave?

e. (3 pts) What is the directional energy flux density (the rate of energy transfer per unit area) of this electromagnetic wave?

Problem 4 (30 pts)

A string with tension T, mass per unit length ρ_L and length L is made into the shape shown in Figure 3. The string is only allowed to move up and down in the $\hat{y}$ direction. The left end of the string is fixed to a wall and the right end is attached to a massless ring which can move freely in the transverse direction without any friction. Gravity is ignored in this problem, and you may assume the small angle approximation for the whole string. The string is released from rest at $t = 0$.

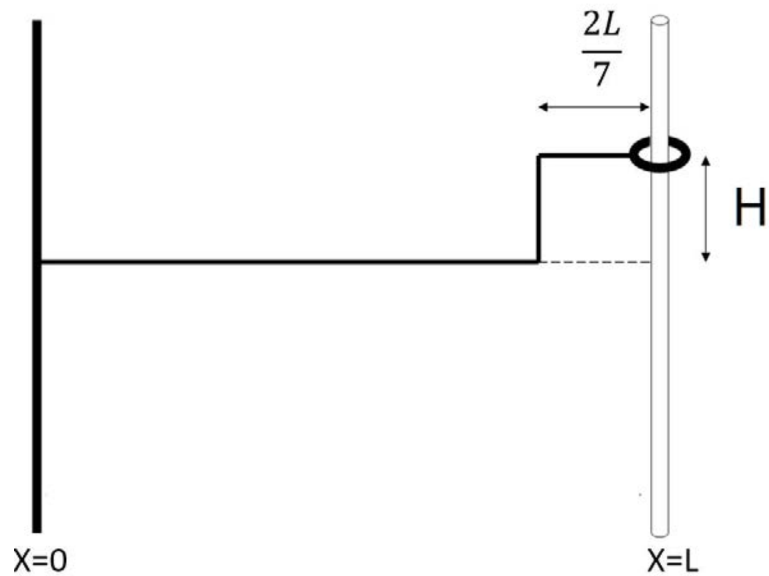

Figure 3: Initial shape of the string at time $t = 0$

a. (8 pts) Write down the boundary conditions at $x = 0$ and $x = L$. You can write down the condition in terms of $\psi(x, t)$ which is the displacement of the massive string in the $\hat{y}$ direction with respect to the equilibrium position.

b. (6 pts) Sketch the shape (ψ as a function of x) of the three lowest normal modes (regardless of whether they are excited or not) and give the corresponding angular frequencies.

c. (10 pts) For the initial shape shown in Figure 3, calculate the amplitude of the n^{th} normal mode in terms of n and H.

d. (6 pts) Which normal modes will not be excited? Express your results in terms of n.