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8.012 Physics I: Classical Mechanics  
Fall 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Physics

Physics 8.012

Fall 2007

**Final Exam**  
**Monday, December 17, 2007**

NAME: \_\_\_\_\_ **SOLUTIONS** \_\_\_\_\_

MIT ID number: \_\_\_\_\_

Instructions (PLEASE READ THESE CAREFULLY):

1. Do all **EIGHT (8)** problems. You have **3 hours**.
2. **Show all work.** Be sure to **circle your final answer.**
3. Read the questions carefully
4. **All work must be done in the white books provided.**
5. No books, notes, calculators or computers are permitted
6. A sheet of useful equations is provided on the last page.

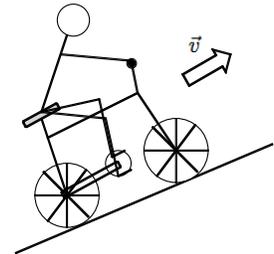
**Your Scores**

Problem	Maximum	Score	Grader
1	10		
2	10		
3	15		
4	10		
5	15		
6	15		
7	10		
8	15		
<b>Total</b>	<b>100</b>		

Problem 1: Quick Short Answer Problems [10 pts]

Answer all five problems. You do not need to show any work. **Be sure to record your answers in your solution book, NOT on this exam.**

(a) [2 pts] A bicycle rider pedals up a hill with constant velocity  $v$ . In which direction does friction act on the wheels?

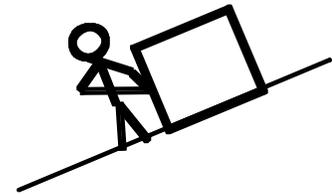


Uphill

Downhill

Friction does not act while the bike moves at constant velocity

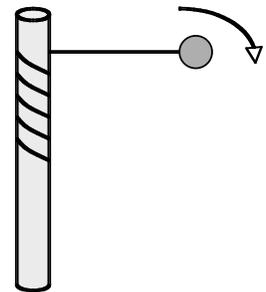
(b) [2 pts] An 8.012 student pushes a heavy object up a hill, and is prevented from slipping by friction between her shoes and the surface of the hill. While she is walking up, she picks up one of her feet. Will she be:



More likely to slip    Less likely to slip

Neither, as there is no change in the friction force acting

(c) [2 pts] A ball attached to rope is twirled around a stick as shown in the diagram at right. Ignore gravity and friction. Which of the following quantities is conserved in the motion of the ball? Be sure to write down all of the choices below that apply.



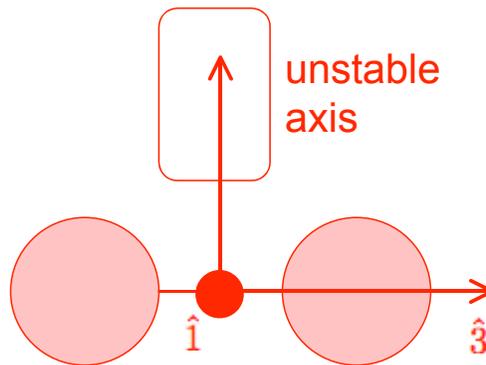
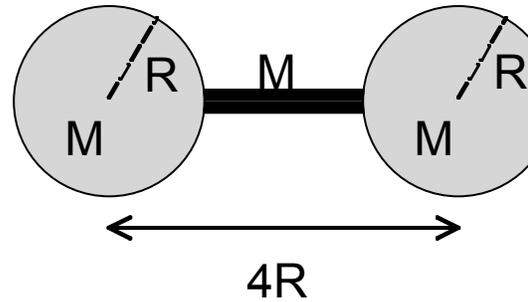
Energy

Momentum

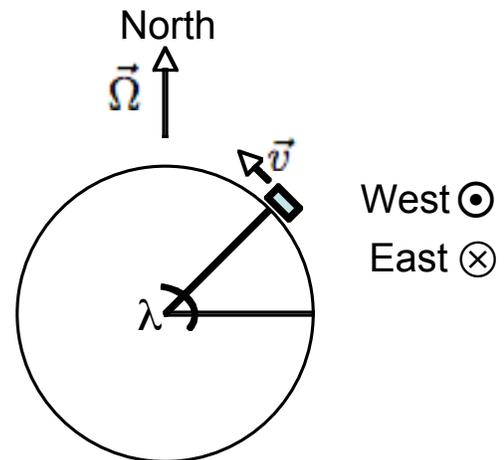
Angular  
Momentum

None of these are  
conserved

(d) [2 pts] (*Challenging*) Consider two uniform disks of mass  $M$ , radius  $R$  and negligible thickness, connected by a thin, uniform rod of mass  $M$ . The centers of the disks are separated by a distance  $4R$ . Reproduce the diagram at right in your solution book and draw the principle axes of this object centered at its center of mass [1 pt], indicating the axis about which torque-free rotations are unstable [1 pt]. Note: you do not need to calculate the moment of inertia tensor to solve this problem.



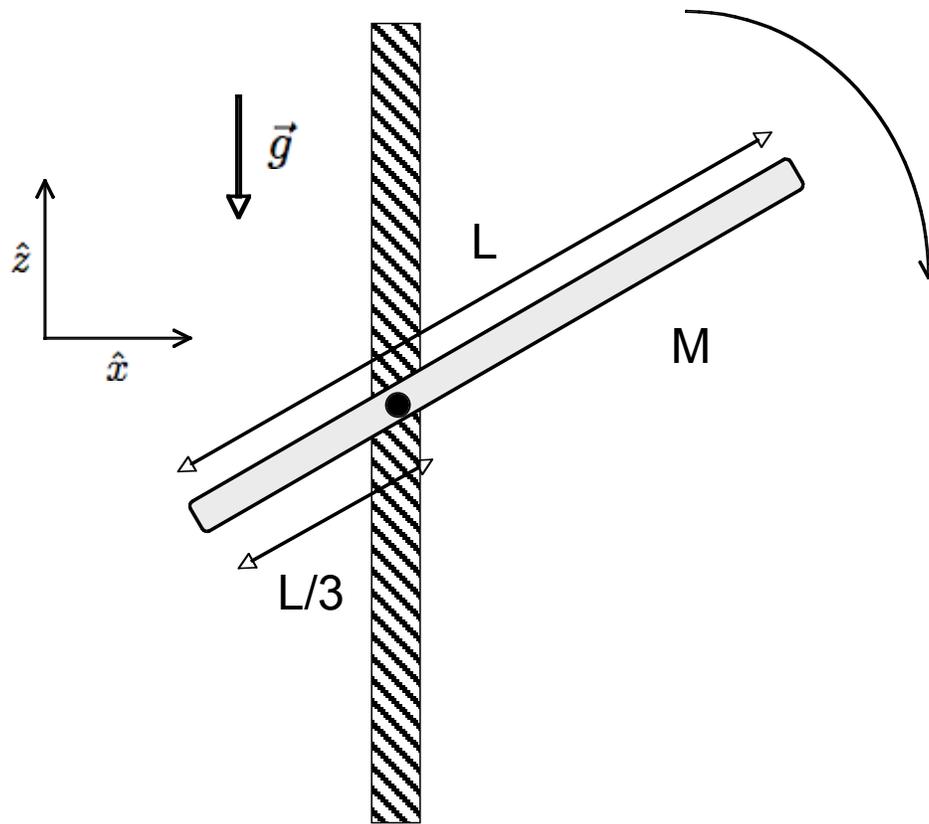
(e) [2 pts] A car at latitude  $\lambda$  on a rotating Earth drives straight North with constant velocity  $v$  as indicated in the diagram. In which direction does friction between the car's tires and the road act on the car to counteract the Coriolis force on the car?



East

West

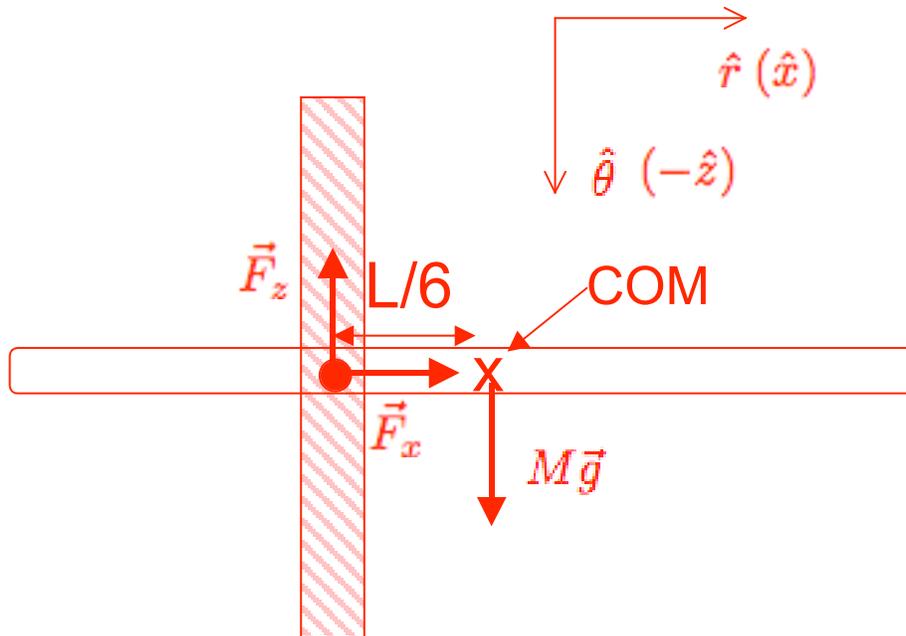
Coriolis force does not act on the car

Problem 2: Swing Bar Pendulum [10 pts]

A uniform bar of mass  $M$ , length  $L$ , and negligible width and thickness is pivoted about a fixed post at a point  $1/3$  along the length of the bar (see figure). The bar is initially released from rest when it is tipped just slightly off of vertical, causing it to swing downwards under the influence of constant gravitational acceleration  $g$  as shown above. Ignore friction.

(a) [5 pts] What is the total force the swinging bar exerts on the fixed post when it passes through horizontal? Express your answer as a vector with components in the coordinate system indicated above.

(b) [5 pts] What is the angular rotation rate of the bar as it swings past its lowest point (i.e., oriented vertically)?

**SOLUTION**

(a) The motion of the bar involves both translation of its center of mass (at distance  $L/6$  from the pivot point) and rotation about the center of mass. This motion is determined by both gravitational force and the constraint forces at the pivot. Assume that there are both horizontal ( $F_x$ ) and vertical ( $F_z$ ) components of the constraint force acting on the swinging bar (the forces the bar exerts on the fixed post will be equal and opposite these forces). The figure above shows the force diagram. Because the motion is rotational, use the polar coordinates  $r$  and  $\theta$  centered at the pivot point to describe the translational motion, where at the instant that the bar is horizontal the positive  $r$  and  $\theta$  directions correspond to  $x$  and  $-z$  in the prescribed rectangular frame.  $\theta$  is also the rotational coordinate of the bar about its center of mass, and points in the same direction as the translational motion (i.e. the bar rotates and swings in the same direction).

The equations of motion are (the coordinate  $r = L/6$  is constant):

$$\hat{r} : M(\ddot{r} - r\dot{\theta}^2) = -M\frac{L}{6}\dot{\theta}^2 = F_x$$

$$\hat{\theta} : M(r\ddot{\theta} + \dot{r}\dot{\theta}) = M\frac{L}{6}\ddot{\theta} = Mg - F_z$$

$$\text{Rotation} : I\ddot{\theta} = \frac{1}{12}ML^2\ddot{\theta} = \frac{L}{6}F_z$$

Combining the last two equations yields:

$$F_z = \frac{3}{4}Mg$$

To solve for  $F_x$ , we can use conservation of mechanical energy since only a conservative force (gravity) is doing work on the bar. The kinetic energy of the bar (translation of center of mass and rotation about center of mass) is drawn from its gravitational potential energy when it starts at vertical and its center of mass is  $L/6$  higher:

$$\begin{aligned} E &= Mg\frac{L}{6} = \frac{1}{2}Mv^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}M\left(\frac{L}{6}\dot{\theta}\right)^2 + \frac{1}{24}ML^2\dot{\theta}^2 = \frac{1}{18}ML^2\dot{\theta}^2 \\ \Rightarrow \dot{\theta}^2 &= 3\frac{g}{L} \end{aligned}$$

then using the  $r$  equation of motion:

$$F_x = -\frac{1}{2}Mg$$

The force that the swinging bar exerts on the fixed post is therefore:

$$F = -F_x\hat{x} - F_z\hat{z} = Mg\left(\frac{1}{2}\hat{x} - \frac{3}{4}\hat{z}\right)$$

Note that we could have also derived the rotation and energy equations as pure rotation about the pivot. In this case the rotational equation of motion, with torque coming from gravity acting at the center of mass, becomes:

$$I\ddot{\theta} = \left(\frac{1}{12}ML^2 + M\left(\frac{L}{6}\right)^2\right)\ddot{\theta} = \frac{1}{9}ML^2\ddot{\theta} = Mg\frac{L}{6}$$

$$\Rightarrow \ddot{\theta} = \frac{3}{2}\frac{g}{L}$$

which can be plugged into the  $\theta$  equation for the center of mass motion above to obtain  $F_z$ . The conservation of energy equation involves only rotational energy in this case because the pivot is not translating; hence:

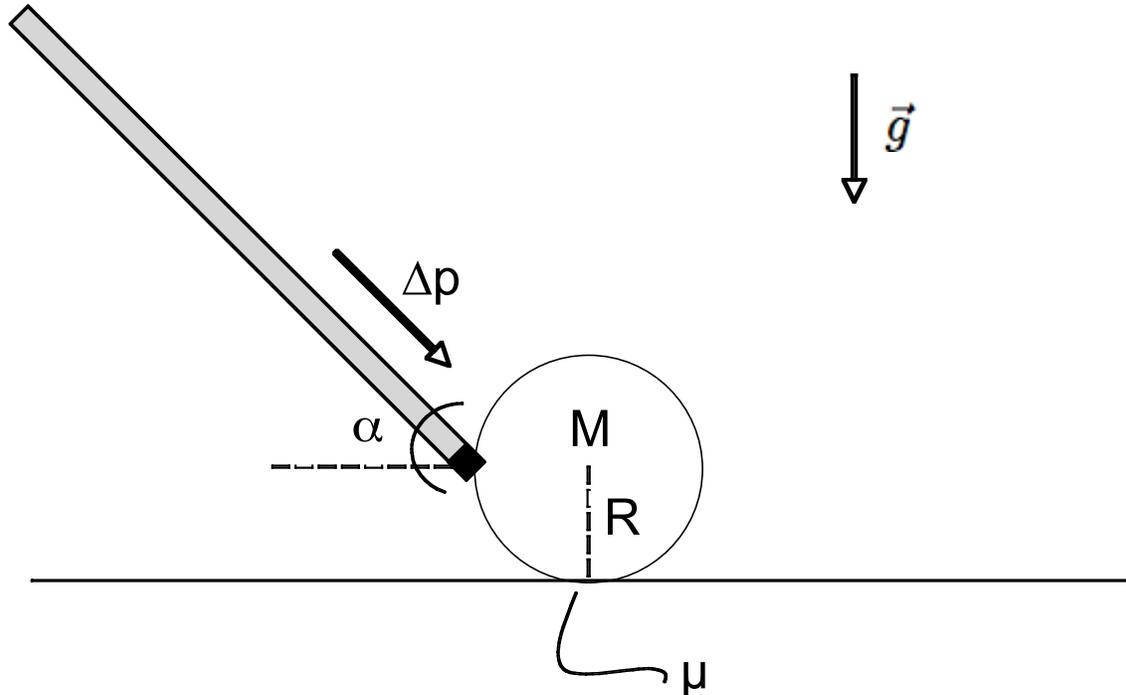
$$E = Mg\frac{L}{6} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{18}ML^2\dot{\theta}^2 \Rightarrow \dot{\theta}^2 = 3\frac{g}{L}$$

as derived above, and this can be plugged into the  $r$  equation of motion to obtain  $F_z$  as before.

(b) We can use conservation of energy again to compute the angular rotation rate at the bottom of the swing:

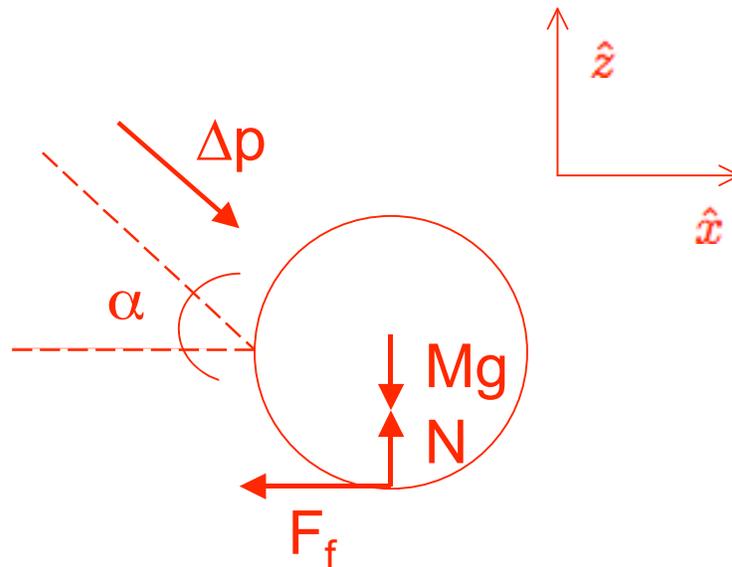
$$\Delta E = 2Mg\frac{L}{6} = \frac{1}{18}ML^2\dot{\theta}^2$$

$$\Rightarrow \dot{\theta} = \sqrt{6\frac{g}{L}}$$

Problem 3: Cue Ball Spin [15 pts]

A billiards player strikes a cue ball (a uniform sphere with mass  $M$  and radius  $R$ ) with a cue stick at the middle of the ball (i.e., at a height  $R$  above the table) and at an angle  $\alpha$  with respect to horizontal. The strike imparts an impulse  $\Delta p$  on the ball in the direction of the strike, causing it to move toward the right as well as “backspin” – spin in the direction opposite of rolling motion. The coefficient of kinetic friction between the ball and table surface is  $\mu$ . Assume that the ball does not rebound off of the table after the strike, and that constant gravitational acceleration  $g$  acts downwards.

- (a) [5 pts] What is the initial speed and angular rotation rate of the cue ball after it is struck?
- (b) [5 pts] For what angle  $\alpha$  will the ball eventually come to rest?
- (c) [5 pts] For the case of part (b), how far does the ball travel before it comes to rest?

**SOLUTION**

(a) There are two solutions for this part based on the ambiguity of whether  $\Delta p$  is the impulse imparted onto the ball or delivered by the stick. Both solutions were accepted. In the former (assumed) case, the horizontal translational momentum of the ball arises purely from the horizontal impulse, so that:

$$Mv_0 = \Delta p \cos \alpha \Rightarrow v_0 = \frac{\Delta p}{M} \cos \alpha$$

The vertical impulse imparts no vertical translation because of the normal force from the floor; however, it does impart an angular impulse on the ball:

$$I\omega_0 = \frac{2}{5}MR^2\omega_0 = \Delta L = \Delta p R \sin \alpha \Rightarrow \omega_0 = \frac{5}{2} \frac{\Delta p}{MR} \sin \alpha$$

Alternately, if we consider  $\Delta p$  to be purely the impulse imparted by the stick over a short time  $\Delta t$ , then one needs to also consider the other forces acting on the ball during that time, in particular friction between the ball and the floor. The friction force acting on the ball during the time  $\Delta t$  is:

$$F_f = \mu N$$

where during the time  $\Delta t$ :

$$N(t) = Mg + \frac{dp}{dt}(t) \sin \alpha$$

The normal force varies according to the details on how the strike force acts over the time  $\Delta t$ . Nevertheless, the change in momentum in the x direction over this period is:

$$\Delta p_x = Mv_0 = \Delta p \cos \alpha - \int_0^{\Delta t} \mu N(t) dt = \Delta p \cos \alpha - Mg\Delta t - \mu \Delta p \sin \alpha$$

If we assume that  $\Delta t$  is very small, then

$$\Rightarrow v_0 = \frac{\Delta p}{M} (\cos \alpha - \mu \sin \alpha)$$

The angular rotation rate remains the same as above.

(b) For this part assume the first solution for  $v_0$ . The ball comes to rest when it stops rotating and translating. Both motions are retarded by friction acting at the bottom surface of the ball.

$$M\dot{v} = -F_f = -\mu Mg$$

$$\Rightarrow v = v_0 - \mu g t_f = 0 \Rightarrow t_f = \frac{\Delta p}{\mu Mg} \cos \alpha$$

$$I\dot{\omega} = \frac{2}{5}MR^2\dot{\omega} = -RF_f = -R\mu Mg$$

$$\Rightarrow \omega = \omega_0 - \mu \frac{5}{2} \frac{g}{R} t_f = 0 \Rightarrow t_f = \frac{\Delta p}{\mu Mg} \sin \alpha$$

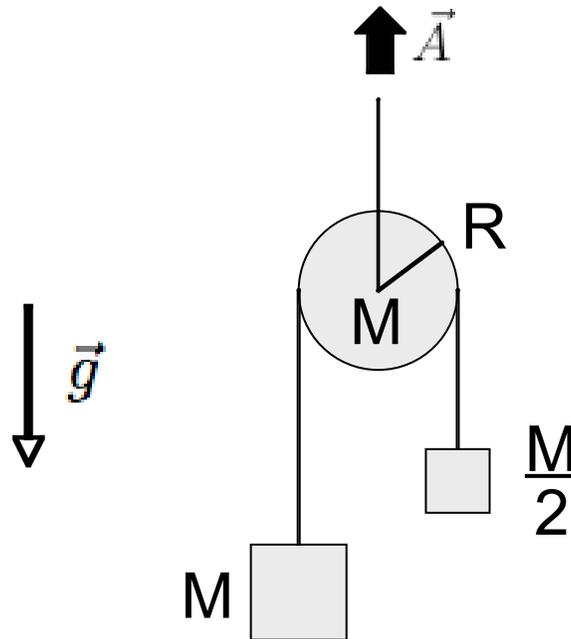
Since these times must be the same we find that:

$$\cos \alpha = \sin \alpha \Rightarrow \alpha = 45^\circ = \frac{\pi}{4}$$

(c) The distance traveled by the ball in this case is given simply by our usual expression for ballistic motion with constant acceleration:

$$\Delta x = v_0 t_f - \frac{1}{2} \mu M g t_f^2 = \frac{1}{2} \frac{\Delta p}{\mu M^2 g} \cos^2 \alpha = \boxed{\frac{1}{4} \frac{\Delta p}{\mu M^2 g}}$$

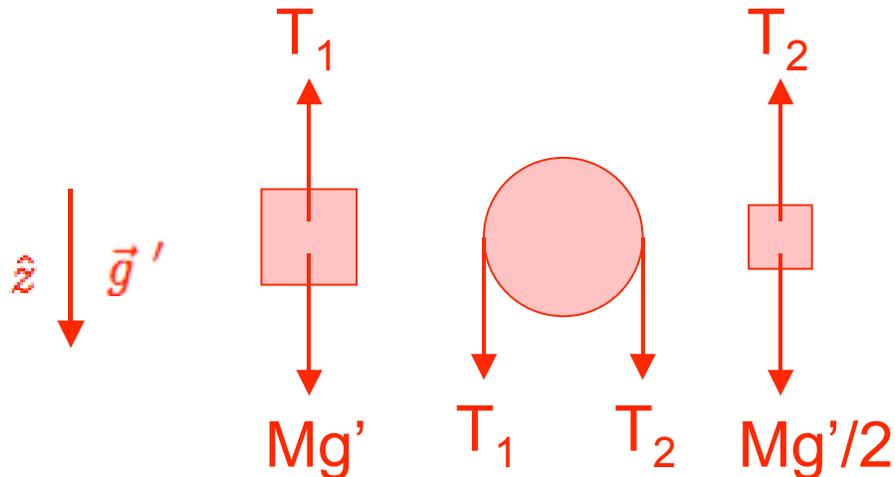
Problem 4: The Accelerated Atwood Machine [10 pts]



An Atwood machine consists of a massive pulley (a uniform circular disk of mass  $M$  and radius  $R$ ) connecting two blocks of masses  $M$  and  $M/2$ . Assume that the string connecting the two blocks has negligible mass and does not slip as it rolls with the pulley wheel. The Atwood machine is accelerated upward at an acceleration rate  $A$ . Constant gravitational acceleration  $g$  acts downward.

(a) [8 pts] Compute the net acceleration of the two blocks in an inertial frame of reference in terms of  $g$  and  $A$ . Do not assume that tension along the entire string is constant.

(b) [2 pts] For what value of  $A$  does the block of mass  $M$  remain stationary in an inertial frame?

**SOLUTION**

(a) This problem is straightforward if it is done in an accelerated reference frame, where the pulley and blocks move under an effective gravitational field  $g' = g + A$ . The force diagrams of the components of the system in this reference frame are shown above. The equations of motion for the pulley and blocks are (with positive  $z$  in the direction of effective gravity):

$$Ma'_1 = Mg' - T_1$$

$$\frac{1}{2}Ma'_2 = \frac{1}{2}Mg' - T_2$$

$$I\alpha = \frac{1}{2}MR^2\alpha = R(T_1 - T_2)$$

The massless string implies the constraint equations:

$$a'_1 = R\alpha = -a'_2 \equiv a$$

Combining these equations yields:

$$T_1 + T_2 = \frac{3}{2}Mg' - \frac{1}{2}Ma$$

$$T_1 - T_2 = \frac{1}{2}Ma$$

$$\Rightarrow T_1 = \frac{3}{4}Mg'$$

$$\Rightarrow a = g' - \frac{3}{4}g' = \frac{1}{4}(g + A)$$

To determine the accelerations in the inertial frame, add the system acceleration  $A$  to the accelerations of the blocks in the accelerated frame:

$$a_1 = a - A = \frac{1}{4}g - \frac{3}{4}A$$

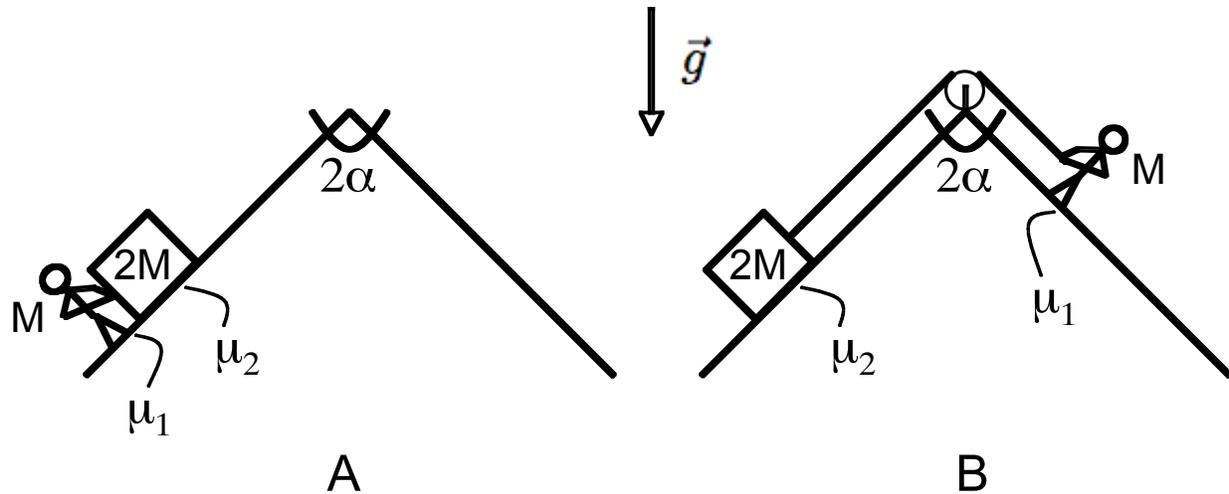
$$a_2 = -a - A = -\frac{1}{4}g - \frac{5}{4}A$$

These are downward accelerations, so the block of mass  $M/2$  actually accelerates upwards in the inertial frame.

(b) The first mass will remain stationary in the inertial frame if

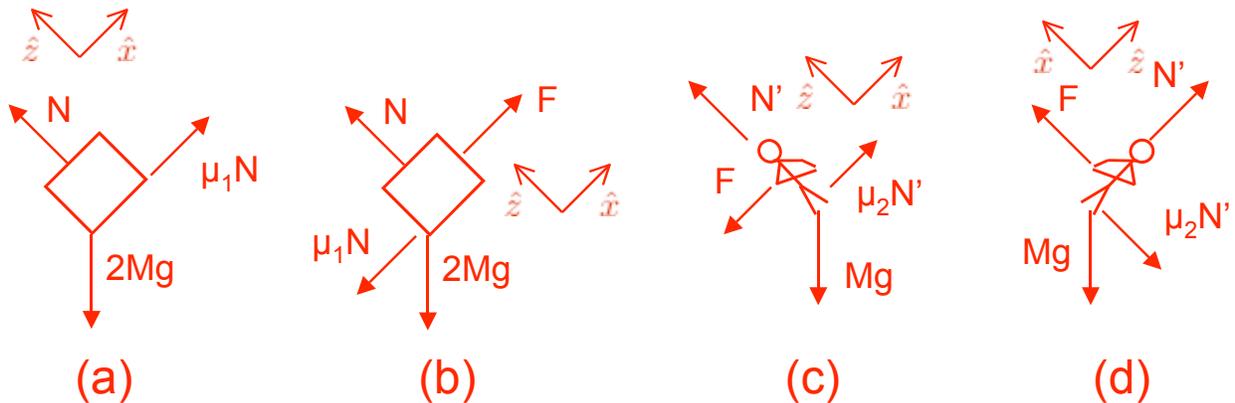
$$A = \frac{g}{3}$$

Problem 5: What is the Best Way to Move a Heavy Load up a Hill? [15 pts]



Two students, each of mass  $M$ , are attempting to push a block of mass  $2M$  up a symmetric triangular hill with opening angle  $2\alpha$ . Student A pushes the load straight up; student B pulls the load up by running a massless rope through a massless, frictionless pulley at the top of the hill, and pulling on the rope from the other side. The maximum coefficient of friction (assumed here to be equal to the coefficient of kinetic friction) is  $\mu_1$  between the students' shoes and the hill, and  $\mu_2$  between the block and the hill. Assume  $\mu_1 > 2\mu_2$ . Constant gravitational acceleration  $g$  acts downwards.

- (a) [5 pts] For what minimum angle  $\alpha_{\min}$  (i.e., maximum steepness) does neither student need to apply any force to hold the load in place?
- (b) [5 pts] Calculate and compare the forces each student must exert on the block to move it up the hill at constant velocity. Does either student have an advantage here?
- (c) [5 pts] Calculate and compare the minimum angles  $\alpha < \alpha_{\min}$  that each student is able to move the block up the hill at constant velocity without their shoes slipping on the hill surface. Does either student have an advantage here?

**SOLUTION**

(a) The force diagram of the boxes when it is not being pulled by the students is shown in diagram (a) above, for the critical case in which it almost slips. The diagram is the same for both boxes. The equations of motion in the  $z$  and  $x$  directions are:

$$2M\ddot{z} = N - 2Mg \sin \alpha = 0 \Rightarrow N = 2Mg \sin \alpha$$

$$2M\ddot{x} = \mu_2 N - 2Mg \cos \alpha = 2Mg(\mu_2 \sin \alpha - \cos \alpha)$$

$$\Rightarrow \boxed{\cot \alpha_{\min} = \mu_2}$$

Note that the way  $\alpha$  is defined here, the trigonometric functions are reversed to what is normally derived.

(b) The force diagram of the boxes when it is pulled with constant velocity by the students is shown in diagram (b) above, and is the same for both boxes. For constant velocity, the net force on the block must be zero, so the equations of motion in the  $z$  and  $x$  directions are:

$$2M\ddot{z} = N - 2Mg \sin \alpha = 0 \Rightarrow N = 2Mg \sin \alpha$$

$$2M\ddot{x} = F - \mu_2 N - 2Mg \cos \alpha = F - 2Mg(\mu_2 \sin \alpha + \cos \alpha) = 0$$

$$\Rightarrow F = 2Mg(\mu_2 \sin \alpha + \cos \alpha)$$

Neither student has an advantage in this case.

(c) The force diagrams on each student are shown in diagrams (c) and (d) above, for the critical case where they almost slip. The force acting on the students is equal and opposite to the force the students exert to pull the blocks because the string is massless. For the first student, the equations of motion in the z and x directions are:

$$M\ddot{z} = N' - Mg \sin \alpha = 0 \Rightarrow N' = Mg \sin \alpha$$

$$M\ddot{x} = \mu_1 N' - Mg \cos \alpha - F$$

$$= Mg(\mu_1 \sin \alpha - \cos \alpha) - 2Mg(\mu_2 \sin \alpha + \cos \alpha) = 0$$

$$\Rightarrow \cot \alpha = \frac{1}{3}(\mu_1 - 2\mu_2) \quad (\text{Student 1})$$

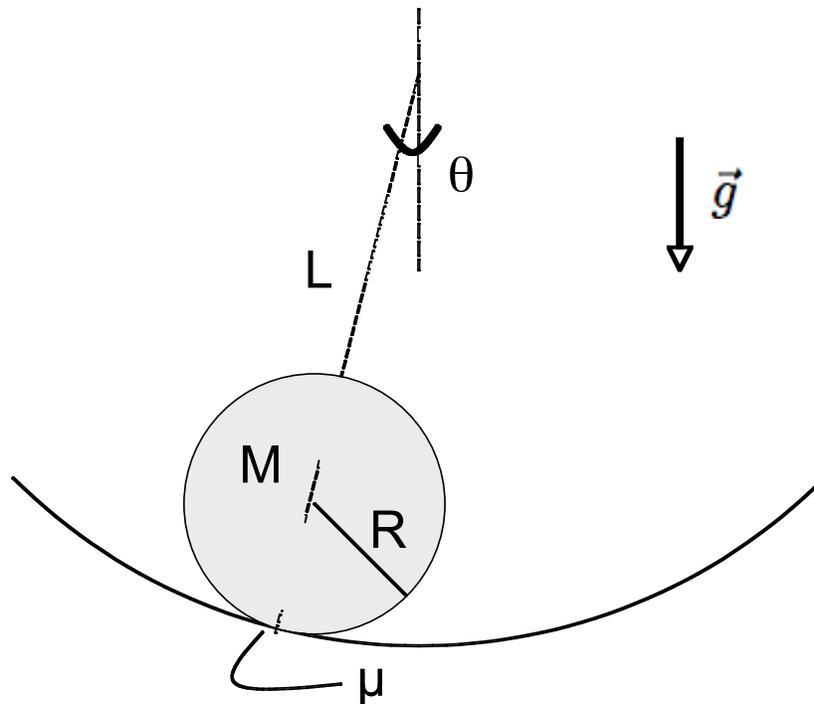
For the second student, the equations of motion in the z direction is the same and in the x direction:

$$M\ddot{x} = F - \mu_1 N' - Mg \cos \alpha$$

$$= 2Mg(\mu_2 \sin \alpha + \cos \alpha) - Mg(\mu_1 \sin \alpha + \cos \alpha) = 0$$

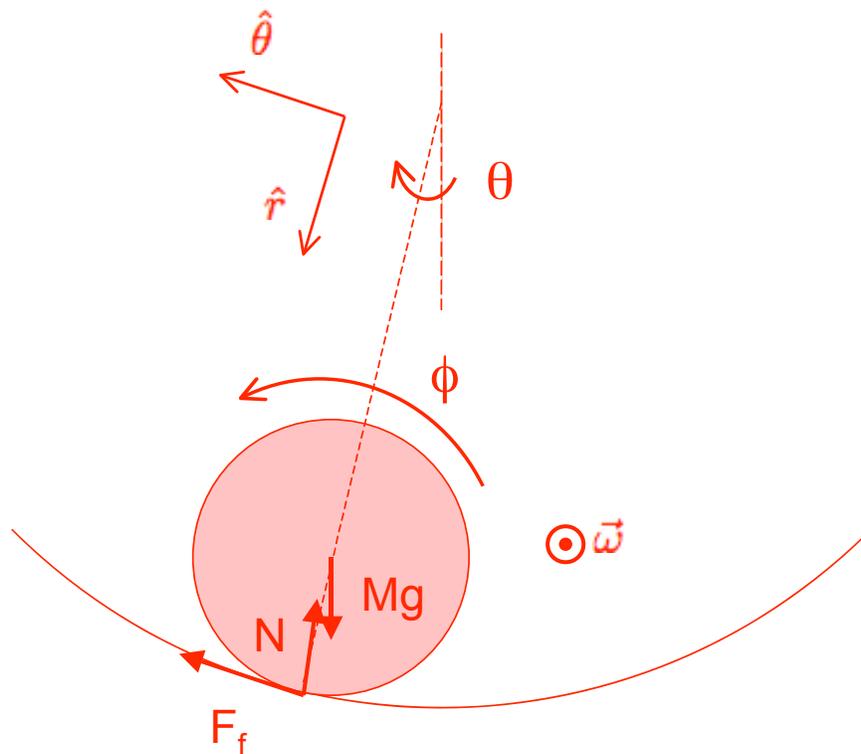
$$\Rightarrow \cot \alpha = (\mu_1 - 2\mu_2) \quad (\text{Student 2})$$

This angle is smaller than that for the first student, so Student 2 can pull up a block on a steeper incline than Student 1 can without slipping.

Problem 6: Ball Rolling in a Bowl [15 pts]

A solid uniform ball (a sphere) of mass  $M$  and radius  $R$  rolls in a bowl that has a radius of curvature  $L$ , where  $L > R$ . Assume that the ball rolls without slipping, and that constant gravitational acceleration  $g$  points downward.

- (a) [5 pts] Derive a single equation of motion in terms of the coordinate  $\theta$  (the position angle of the ball with respect to vertical) that takes into account both translational and rotational motion, for any point along the ball's trajectory. Be careful with your constraint equation!
- (b) [5 pts] Find the position angle of the ball along the bowl's surface as a function of time in the case that  $\theta$  is small. Assume that the ball is started from rest at a position angle  $\theta_0$ .
- (c) [5 pts] At what maximum initial position angle  $\theta_0$  can the ball be placed and released at rest and still satisfy the rolling without slipping condition throughout its motion? Note that  $\theta_0$  does not have to be small in this case.

**SOLUTION**

(a) The force diagram for the rolling ball is shown above. Note that friction points in the direction of rolling, as it acts to reduce the spin of the ball as it slows down going up the bowl's incline (if the ball is rolling down the side of the bowl, friction would also point in this direction to spin up the ball). Because we are dealing with circular motion, choose a polar coordinate system centered at the focal point of the bowl with positive  $\theta$  increasing as shown. Let  $\phi$  to be the coordinate describing the spin of the ball, with positive direction as shown (the spin vector points out of the page). The equations of motion for translation of the center of mass and rotation about the center of mass are:

$$\hat{r} : M(\ddot{r} - r\dot{\theta}^2) = -M(L - R)\dot{\theta}^2 = Mg \cos \theta - N$$

$$\hat{\theta} : M(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = M(L - R)\ddot{\theta} = F_f - Mg \sin \theta$$

$$\text{Rotation : } I\ddot{\phi} = \frac{2}{5}MR^2\ddot{\phi} = -RF_f$$

Note that the fixed radial coordinate  $r$  of the center of mass is  $(L-R)$ , not  $L$ . The constraint equation is:

$$(L - R)\ddot{\theta} = R\ddot{\phi}$$

We can use these equations to solve for the friction force  $F_f$ :

$$M(L - R)\ddot{\theta} = MR\ddot{\phi} = -\frac{5}{2}F_f = F_f - Mg \sin \theta$$

$$\Rightarrow F_f = \frac{2}{7}Mg \sin \theta$$

which gives a single equation for  $\theta$ :

$$\ddot{\theta} = -\frac{5}{7} \frac{g}{L - R} \sin \theta$$

This solution could also be found by appealing to energy conservation, since in the case of rolling without slipping there are no dissipative forces acting. Taking the zero point for gravitational potential energy to be at the bottom of the bowl, the total mechanical energy at any point along the bowl is (using the constraint equation):

$$\begin{aligned} E &= (L - R)Mg(1 - \cos \theta) + \frac{1}{2}M(L - R)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 \\ &= (L - R)Mg(1 - \cos \theta) + \frac{7}{10}M(L - R)^2\dot{\theta}^2 \end{aligned}$$

The change in energy with time is 0, hence:

$$\begin{aligned} \frac{dE}{dt} &= 0 = (L - R)Mg \sin \theta \dot{\theta} + \frac{7}{5}M(L - R)^2\dot{\theta}\ddot{\theta} \\ \Rightarrow \ddot{\theta} &= -\frac{5}{7} \frac{g}{L - R} \sin \theta \end{aligned}$$

(b) For small angles  $\theta$ , the equation of motion reduces to:

$$\ddot{\theta} = -\frac{5}{7} \frac{g}{L-R} \theta = -\omega^2 \theta$$

which is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5}{7} \frac{g}{L-R}}$$

The general solution of this equation is:

$$\theta(t) = A \cos \omega t + B \sin \omega t$$

with the initial conditions

$$\theta(0) = \theta_0, \dot{\theta}(0) = 0$$

which gives:

$$\theta(t) = \theta_0 \cos \omega t$$

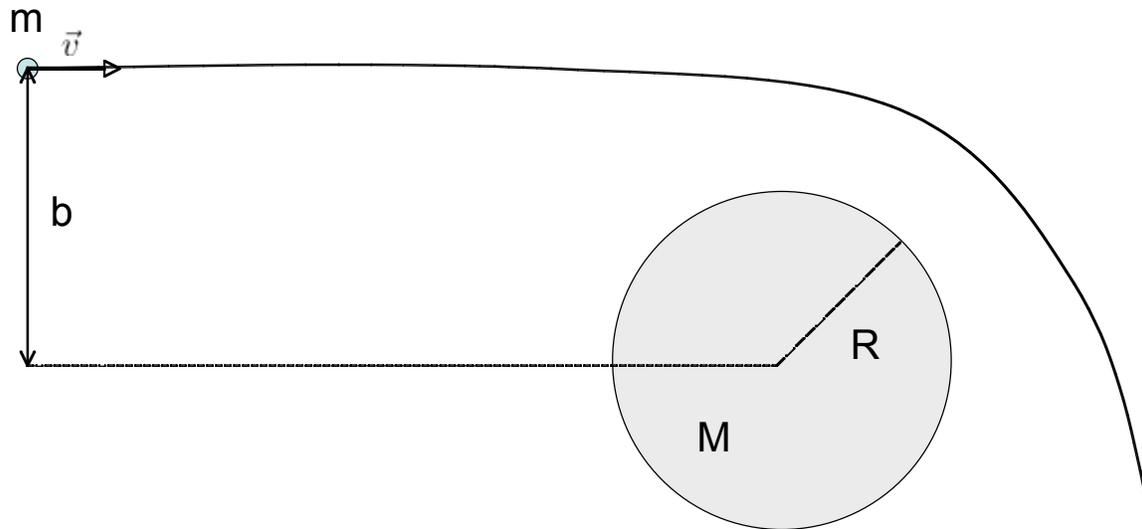
(c) There are a number of ways to derive the solution for this part, but consider that the condition for rolling without slipping implies that  $F_f \leq \mu N$  at every point along the motion. The maximum angular acceleration due to gravity, and hence the maximum slippage between the ball and bowl surfaces, occurs at the top of the roll at which point  $\dot{\theta} = 0$ . From the radial equation of motion:

$$N = Mg \cos \theta$$

and from the expression for  $F_f$  above we get:

$$F_f = \frac{2}{7} Mg \sin \theta \leq \mu N = \mu Mg \cos \theta$$

$$\Rightarrow \tan \theta \leq \frac{7}{2} \mu$$

Problem 7: Gravitational Focusing in an Asteroid Field [10 pts]

A small asteroid of mass  $m$  moves with speed  $v$  when it is far away from a large planet of mass  $M$ . The “impact parameter”,  $b$ , is the distance between the centers of mass of the planet and asteroid perpendicular to the initial velocity vector of the asteroid (see figure). As the asteroid passes by the planet, its path is deflected by the gravitational force acting between the two bodies. Assume that  $M \gg m$ , so that the planet does not move appreciably as a result of a single interaction with an asteroid.

(a) [5 pts] What is the minimum impact parameter  $b_{\min}$  that allows the asteroid to miss the planet? Hint: in this case the asteroid just grazes the planet surface.

(b) [5 pts] (*Challenging*) Now assume that the planet moves with velocity  $v$  through a space that is filled with many small asteroids of mass  $m$  that are distributed uniformly with average number density  $N$  (this parameter has dimension  $[L]^{-3}$ ) and are initially at rest. Any asteroid that collides with the planet sticks to it and adds to the planet’s total mass. Determine the rate at which the planet gains mass ( $dM/dt$ ) as it moves through the asteroid field as a function of  $v$ . You may express your answer in terms of  $b_{\min}$ . Note: this is an important problem in understanding the formation of planets like the Earth,

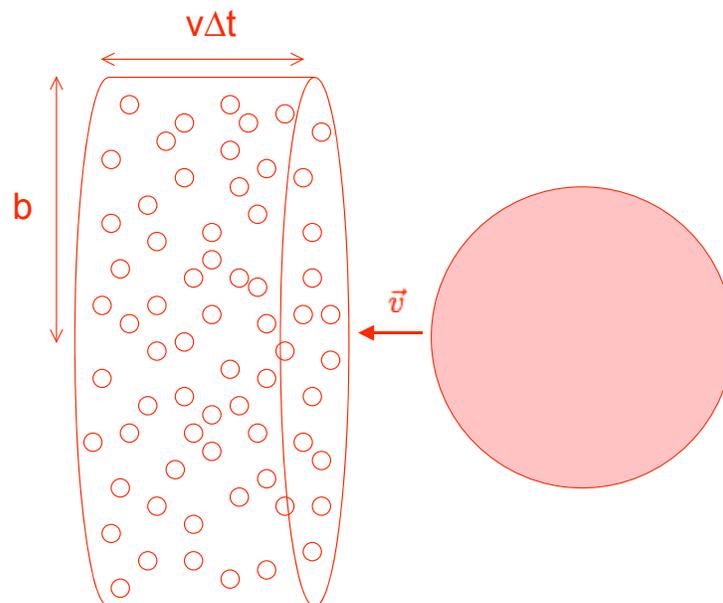
**SOLUTION**

(a) Because gravitational force is a central force law, there are no net torques acting on the asteroid; hence, its angular momentum is conserved. There are also no non-conservative forces acting, so total mechanical energy is also conserved. Using the hint in the question, and assuming the asteroid approaches from an effectively infinite (“far”) distance away, these conservation laws imply:

$$L = mbv = mRv_f \Rightarrow v_f = \frac{b}{R}v$$

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{\infty} = \frac{1}{2}mv_f^2 - G\frac{Mm}{R}$$

$$\Rightarrow b_{\min} = R\sqrt{1 + \frac{2GM}{Rv^2}}$$



(b) This situation is basically the same as part (a), except observed in the frame of reference of the asteroids. Hence the value of  $b_{\min}$  remains

unchanged. As the planet moves through the field of asteroids in a period  $\Delta t$ , it sweeps up everything in a volume (illustrated above):

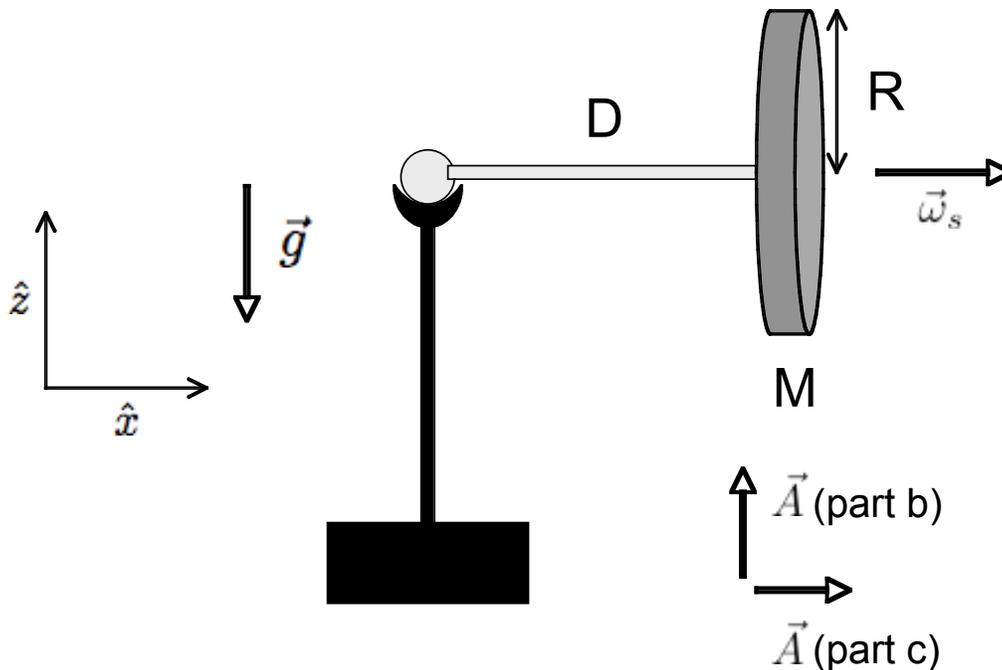
$$V = \pi b_{min}^2 v \Delta t$$

hence the total mass accreted onto the planet is:

$$\Delta M = mNV = mN\pi b_{min}^2 v \Delta t$$

$$\Rightarrow \frac{dM}{dt} = mN\pi b_{min}^2 v = mN\pi v R^2 \left(1 + \frac{2GM}{Rv^2}\right)$$

Problem 8: The Accelerated Gyroscope [15 pts]



A gyroscope wheel consists of a uniform disk of mass  $M$  and radius  $R$  that is spinning at a large angular rotation rate  $\omega_s$ . The gyroscope wheel is mounted onto a ball-and-socket pivot by a rod of length  $D$  that has negligible mass, allowing the gyroscope to precess over a wide range of directions. Constant gravitational acceleration  $g$  acts downward. For this problem, ignore both friction and nutational motion; i.e., assume the gyroscope only precesses uniformly. For all parts, express your solution as a vector (magnitude and direction) with components in the coordinate system shown above.

- (a) [5 pts] Calculate the total angular momentum vector of the uniformly precessing gyroscope in the orientation shown above; i.e., the total of the spin and precession angular momentum vectors.
- (b) [5 pts] The pivot mount is accelerated upward with magnitude  $A$ . Calculate the precession angular velocity vector in this case.
- (c) [5 pts] (*Challenging*) The pivot mount is accelerated toward the right with magnitude  $A$ . Calculate the precession angular velocity vector in this case.

**SOLUTION**

(a) Although it is not explicitly specified in the problem, it is optimal to calculate the angular momentum vector with respect to the pivot, since this point is at rest in the given reference frame and the directions  $x$  and  $z$  lie along principal axes of the gyroscope wheel in the orientation shown. This allows us to compute the components as:

$$\vec{L} = \vec{I} \cdot \vec{\omega} = I_{xx}\omega_s\hat{x} + I_{zz}\Omega\hat{z}$$

where  $\Omega$  is the constant precession angular velocity which points in the  $+z$  direction. The spin component has a moment of inertia:

$$I_{xx} = \frac{1}{2}MR^2$$

The precession component has a moment of inertia (using the parallel axis theorem):

$$I_{zz} = \frac{1}{4}MR^2 + MD^2$$

The spin angular velocity is given; the precession angular velocity can be calculated using our usual expression for torque acting to rotate a large spin angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F} = \vec{\Omega} \times \vec{L}$$

Here the total force  $F$  is just the gravitational force acting on the disk. For constant precession, only the component of angular momentum due to spin changes as the system rotates around, hence

$$\Rightarrow DMg = \Omega L_x = \frac{1}{2}MR^2\omega_s\Omega$$

$$\Rightarrow \Omega = \frac{2Dg}{R^2\omega_s}$$

Putting the components together:

$$\vec{L} = \frac{1}{2}MR^2\omega_s\hat{x} + \left(\frac{1}{4}R^2 + D^2\right)\frac{2DMg}{R^2\omega_s}\hat{z}$$

(b) If the gyroscope is accelerated upwards, this puts the gyroscope into an effective gravitational field of strength  $g+A$  pointing downwards. This changes the magnitude, but not direction, of the effective gravitational force acting on the disk. Hence, the precession rate simply changes to:

$$\Omega = \frac{2D(g+A)}{R^2\omega_s}$$

and the precession vector becomes:

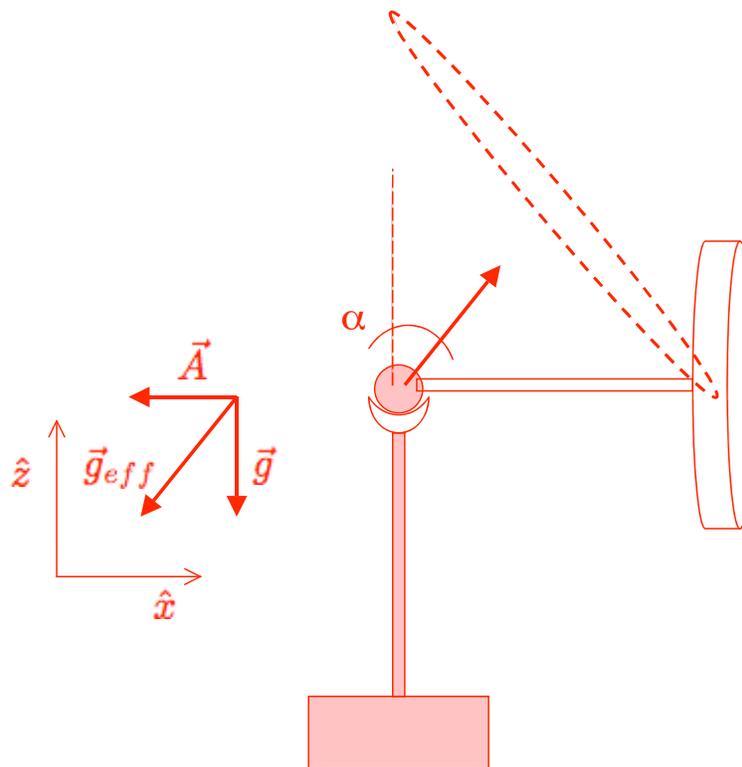
$$\vec{\Omega} = \frac{2D(g+A)}{R^2\omega_s}\hat{z}$$

(c) If the gyroscope is accelerated to the right, both the magnitude and direction of the effective gravity changes, and this problem effectively becomes identical to that of a the tilted gyroscope where the angle of tilt  $\alpha$  is given by:

$$\tan \alpha = \frac{A}{g}$$

The precession path is as shown below; if this is not readily obvious, consider the direction torque acts when the gyroscope wheel has rotated  $90^\circ$  and points into the page; in this case torque is moving the spin angular momentum vector upwards at an angle  $\alpha$ , consistent with the path shown. The torque acting on the gyroscope disk is:

$$\vec{\tau} = \vec{r} \times \vec{F} = DMg$$



The component of spin angular momentum that is precessing is:

$$|\vec{\Omega} \times \vec{L}_x| = \Omega L_x \cos \alpha = \Omega L_x \frac{g}{\sqrt{A^2 + g^2}}$$

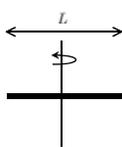
$$\Rightarrow \Omega = \frac{2D}{R^2 \omega_s} \sqrt{A^2 + g^2}$$

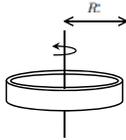
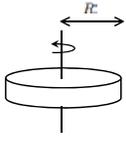
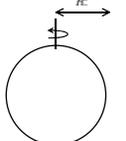
So the precession angular velocity vector can be written:

$$\vec{\Omega} = \Omega (\sin \alpha \hat{x} + \cos \alpha \hat{z}) = \frac{2D}{R^2 \omega_s} (A \hat{x} + g \hat{z})$$

Note that this reverts back to the expression derived in part (a) if  $A = 0$ . Also note that the precession vector is always parallel to effective gravity.

USEFUL EQUATIONS
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Velocity in polar coordinates	$\vec{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
Acceleration in polar coordinates	$\vec{a} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} + 2\dot{r}\dot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$
Center of mass of a rigid body	$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{1}{M} \int \rho \vec{r} dV$
Volume element in cylindrical coordinates	$dV = r dr d\theta dz$
Kinetic energy	$K = \frac{1}{2} M v^2 + \frac{1}{2} \vec{\omega} \cdot \bar{I} \cdot \vec{\omega}$
Work	$W = - \int \vec{F} \cdot d\vec{r} = - \int \vec{\tau} \cdot d\vec{\theta}$
Potential Energy (for conservative forces)	$U = - \int \vec{F}_c \cdot d\vec{r}$ where $\vec{\nabla} \times \vec{F}_c = 0$
Angular momentum	$\vec{L} = \vec{r} \times \vec{p} = \bar{I} \cdot \vec{\omega}$
Torque	$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$ Fixed axis rotation: $\tau_z = I_{zz} \dot{\omega}$
Moment of inertia for a uniform bar (about COM)	<div style="display: flex; align-items: center; justify-content: center;">  <math>I_{zz} = \frac{1}{12} M L^2</math> </div>

Moment of inertia for a uniform hoop (about COM)		$I_{zz} = MR^2$
Moment of inertia for a uniform disk (about COM)		$I_{zz} = \frac{1}{2}MR^2$
Moment of inertia for a uniform sphere (about COM)		$I_{zz} = \frac{2}{5}MR^2$
Scalar parallel axis theorem		$I = I_{COM} + MR^2$
Velocity from rotation		$\vec{v} = \vec{\omega} \times \vec{r}$
Moments of inertia tensor (permute x→y→z)		$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) = \int dV \rho(y^2 + z^2)$
Products of inertia tensor (permute x→y→z)		$I_{xy} = -\sum_i m_i x_i y_i = -\int dV \rho xy$
Fictitious force in an accelerating frame		$\vec{F}_f = -m\vec{A}$
Fictitious force in a rotating frame		$\vec{F}_f = -2m\vec{\Omega} \times \vec{v}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$
Time derivative between inertial and rotating frames		$\frac{d\vec{B}}{dt}_{(in)} = \frac{d\vec{B}}{dt}_{(rot)} + \vec{\Omega} \times \vec{B}$
Taylor Expansion of f(x)		$f(x) = f(a) + \frac{1}{1!} \frac{df}{dx} \Big _a (x - a) + \frac{1}{2!} \frac{d^2 f}{dx^2} \Big _a (x - a)^2 + \dots$