Z. Abel, B. Chapman, E. Demaine revised Tuesday $23^{\text{rd}}$ April, 2024

# Problem Set 9

- **Due Date:** 11:59pm on **Monday $\mathbf{29^{th}}$ April, 2024**
- **Days Covered**: 17, 18, and 19 (including Lecture, Warm-Up, and Recitation)

## Problem 1. Double Counting [9 points]

Use a double-counting argument to prove that for all natural numbers $n \leq a \leq b \leq c$,

$$\sum_{i,j,k} \binom{a}{i}\binom{b}{j}\binom{c}{k} = \binom{a+b+c}{n},$$

where the sum ranges over all $i, j, k \in \mathbb{N}$ satisfying $i + j + k = n$.

## Problem 2. Pigeonhole Principle [10 points]

Let $S \subseteq \{1, 2, 3, \ldots, 2n\}$ where $|S| = n + 1$. Using the Pigeonhole Principle, show that there must be distinct elements $a, b \in S$ such that $a/b$ is a power of 2. Clearly indicate what are the pigeons, holes, and rules for assigning a pigeon to a hole.

*Hint:* Factor each number into the product of an odd number and a power of 2.

## Problem 3. Dice Dice Dice Dice Dice [21 points]

Ben Bitdiddle finds five standard six-sided dice in his fridge. He rolls them, because this is a math problem.

**(a)** [2 pts] What is the probability space for this problem? You should explain:

- The set of outcomes. (Please describe these carefully.)
- The number of outcomes.
- The probability of each outcome.

**(b)** [2 pts] Let $A_{29}$ be the event that the sum of the dice is at least 29. Write $A_{29}$ explicitly as a set of outcomes, and then compute $\Pr[A_{29}]$.

**(c)** [3 pts] Let $A_{18}$ be the event that the sum of the dice is 18 or larger. Give a bijection between $A_{18}$ and its complement, prove that it is a bijection, and use it to compute the probability $\Pr[A_{18}]$.

**(d)** [3 pts] Let $B_k$ be the event that there are *exactly* $k$ sixes among the five dice rolls, and let $C_k$ be the event that there are *at least* $k$ sixes among the five dice rolls (where $0 \leq k \leq 5$). Give expressions for $\Pr[B_k]$ and for $\Pr[C_k]$. Evaluating your expressions for $\Pr[C_2]$ and $\Pr[C_3]$ should give $\Pr[C_2] = \frac{1526}{6^5}$ and $\Pr[C_3] = \frac{276}{6^5} = \frac{46}{6^4}$. (If not, check your work!)

**(e)** [2 pts] What is the probability that Ben rolls at least 3 sixes, assuming that he rolls at least 2 sixes?

**(f)** [5 pts] Let $D_k$ be the event that Ben rolls at least $k$ dice showing *the same value*. Compute $\Pr[D_3 \mid D_2]$.

*Hint:* the answer is different from the previous part.

**(g)** [4 pts] Ben misplaced two of his dice (shh, they're in his other pocket, but don't tell him!), so he is now down to 3 dice. What is the probability that some pair of dice will add up to exactly 6?

*Hint:* You may wish to use Inclusion/Exclusion.

## Problem 4. Unreliable Helpers [10 points]

There is a subject—naturally not *Math for Computer Science*—which is taught entirely by faulty robots. Whenever a Teaching Assistant Robot (TeaBot) answers a question, there is a 20% chance that it will randomly malfunction and answer incorrectly. Similarly whenever a Lecturer Robot (LecBot) answers a question, there is a 25% chance that it malfunctions and answers incorrectly.

Unfortunately, 10% of the assigned problems contain errors.

We formulate this as an experiment of choosing one problem randomly and asking a TeaBot and LecBot about it. Define the following events:

$$\begin{aligned}
E &:= [\text{the problem has an error}], \\
T &:= [\text{the TeaBot says the problem has an error}], \\
L &:= [\text{the LecBot says the problem has an error}].
\end{aligned}$$

**(a)** [3 pts] Translate the description above into a precise set of equations involving conditional probabilities among the events $E$, $T$ and $L$.

**(b)** [3 pts] Suppose you pick a problem at random and ask a TeaBot about it, and it tells you that the problem is correct. To double-check, you ask a LecBot, who says that the problem has an error. Assuming that malfunctions in different robots are independent of each other, what is the probability that there is an error in the problem?

*Hint:* Draw a tree diagram. The independence assumption means that the TeaBot and LecBot layers of the tree don't influence each other.

**(c)** [4 pts] is $\Pr[T \mid L]$ equal to $\Pr[T]$? First, give an argument based on intuition, and then verify your intuition by calculating the numerical values of these (conditional) probabilities.