Problem Set 8

- **Due Date:** 11:59pm on **Tuesday 16th April, 2024**
- **Days Covered**: 14 and 15 (including Lecture, Warm-Up, and Recitation)

Problem 1. Strong Connectivity [7 points]

(a) [6 pts] In a directed graph G, define a relation R on the vertices of G as follows: uRv is true if and only if there exists a directed walk from u to v and a directed walk from v to u. Prove that R is an equivalence relation, by carefully explaining why R is reflexive, symmetric, and transitive.

(b) [1 pts] How do the equivalence classes of R relate to the strongly connected components of G?

Problem 2. Bijections via Inverses [9 points]

Let A and B be sets, and suppose we have total functions $f : A \to B$ and $g : B \to A$.

(a) [4 pts] Assuming $g(f(a)) = a$ for every $a \in A$, prove carefully that f is injective. In other words, if f has a ***left inverse***, then f is injective.

(b) [4 pts] Assuming $f(g(b)) = b$ for every $b \in B$ (but without assuming part (a) holds), prove carefully that f is surjective. In other words, if f has a ***right inverse***, then f is surjective.

(c) [1 pts] If both assumptions from parts (a) and (b) hold, conclude that f is a bijection.

NOTE: This provides a common pattern when trying to prove that a total function $f : A \to B$ is a bijection. Instead of directly showing that f is injective and surjective, it is enough (and often more convenient) to find a total function $g : B \to A$ that acts as a two-sided inverse for f. As above, this is enough to prove that f is a bijection, and g is indeed its inverse (and therefore also a bijection). We'll practice this technique in the next problem.

Problem 3. Increasing Correspondence [10 points]

Recall that the notation $\mathbb{N}^3$ means the set of ordered triples of natural numbers: $\mathbb{N}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{N}\}$. Define the subsets $A, B \subseteq \mathbb{N}^3$ as follows: A is the subset of *weakly increasing* ordered triples which add up to 1000. B is the subset of ordered triples (x, y, z) such that $x + 2y + 3z = 1000$. In set-builder notation:

$$A = \{(a, b, c) \in \mathbb{N}^3 \mid a \leq b \leq c \text{ AND } a + b + c = 1000\}$$
$$B = \{(x, y, z) \in \mathbb{N}^3 \mid x + 2y + 3z = 1000\}$$

For example $(1, 2, 997) \in A$ and $(2, 499, 0) \in B$.

In this problem, we will find a bijection between A and B.

(a) [3 pts] Define the function $f : A \to B$ by $f((a, b, c)) := (c - b, b - a, a)$. Prove that f is a well-defined total function from A to B; in other words, show that for every $(a, b, c) \in A$, the resulting triple $f((a, b, c))$ lies in B.

Hint: You must prove two things about $f((a, b, c))$.

(b) [6 pts] Find a total function $g : B \to A$ that is an inverse of f; in other words, $g(f((a, b, c))) = (a, b, c)$ for every $(a, b, c) \in A$, and $f(g((x, y, z))) = (x, y, z)$ for every $(x, y, z) \in B$. Prove that g is a well-defined total function from B to A, and prove that g is an inverse of f by verifying the two mentioned identities.

(c) [1 pts] Conclude that f is a bijection.

Problem 4. Counting Graphs [8 points]

(a) [4 pts] Let n be a positive number. How many DAGs $G = (V, E)$ have vertex set $V = \{1, 2, \ldots, n\}$ *and* have the list $(1, 2, \ldots, n)$ as a topological order? (Recall that DAGs are **directed**.)

(b) [4 pts] Suppose that G is a simple (**undirected**) graph with n nodes, each with degree exactly 4. Let's also assume that G has no cycle of length 3. How many length-3 paths are there? *Note*: we consider the path (a, b, c, d) to be **different** from the path (d, c, b, a).

Problem 5. Counting [16 points]

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) [4 pts] How many ways are there to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, u appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) [4 pts] How many ways are there to order the 26 letters of the alphabet so that there are *at least two* consonants immediately following each vowel?

(c) [4 pts] In how many different ways can $2n$ students be paired up?

(d) [4 pts] Two n-digit sequences of digits 0,1,. . . ,9 are said to be of the *same type* if the digits of one are a permutation of the digits of the other. For $n = 8$, for example, the sequences 03088929 and 00238899 are the same type. How many types of n-digit sequences are there?