6.1200J/18.062J *Mathematics for Computer Science* Tuesday 19th March, 2024
Massachusetts Institute of Technology, Spring 2024
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Problem Set 6

- **Due Date:** 11:59pm on **Monday 1st April, 2024**
- **Days Covered**: 11 and 12 (including Lecture, Warm-Up, and Recitation)

Problem 1. Failed Induction [10 points]

(a) [5 pts] In Recitation 11 Problem 3, it may seem like we've proved a stronger theorem:

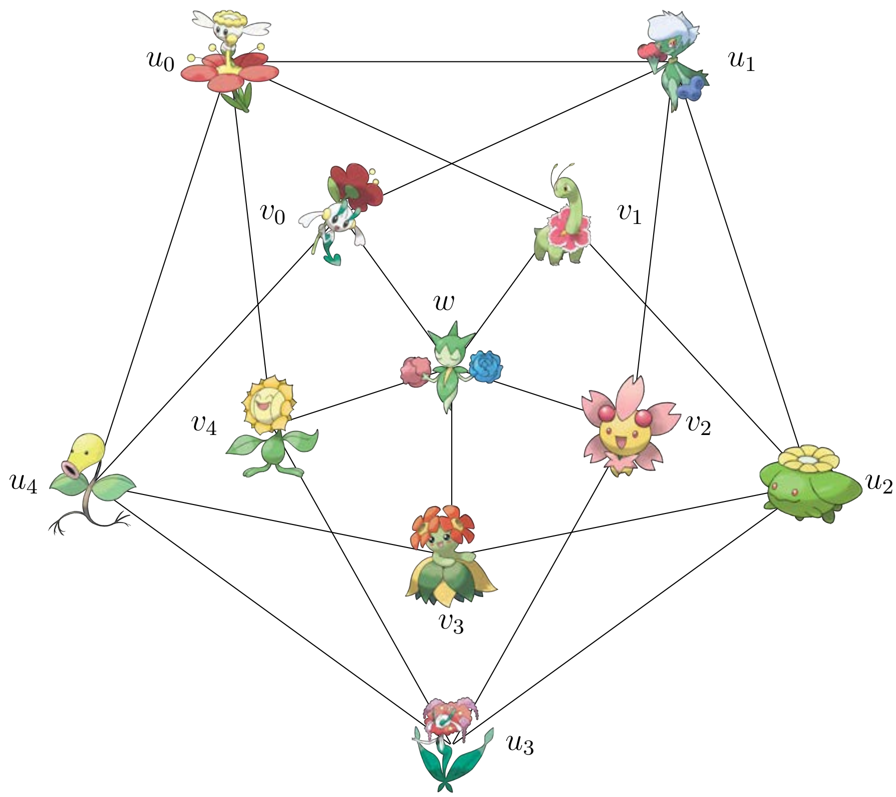
False Claim. *Every graph with n nodes and at most $3n - 6$ edges can be 6-colored.*

Show that this claim is actually false, by finding some integer n and some graph with n nodes and at most $3n - 6$ edges that cannot be 6-colored. Be sure to explain why your example cannot be 6-colored.

(b) [5 pts] Why doesn't the induction argument from Recitation 11 Problem 3(c) work for this stronger claim? Where does the proof break down?

Problem 2. High Chromatic Number Without Triangles [15 points]

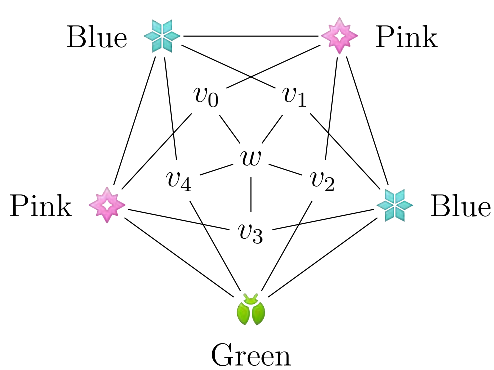
A basic example of a graph with chromatic number n is the complete graph on n vertices, that is $\chi(K_n) = n$. This implies that any graph with K_n as a subgraph must have chromatic number at least n. It's a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit an example countering this misconception, namely a graph with chromatic number four that contains no *triangle* (length three cycle) and hence no subgraph isomorphic to K_n for $n \geq 3$. Namely, let G be the 11-vertex graph below. The reader can verify that G is triangle-free.



Graph G with no triangles and $\chi(G) = 4$

(a) [6 pts] Prove that $\chi(G) \leq 4$.

(b) [3 pts] Assume that we had a coloring of G using only 3 colors. Prove that the coloring must match the partial coloring shown below, possibly after rotating the graph and/or relabeling the three colors.



(c) [6 pts] Prove that $\chi(G) > 3$.

Hint: Part b might be helpful.

Problem 3. Find A Pet [13 points]

Hasbro's market research has shown that giving their TV show protagonists animal sidekicks can increase toy sales. Meanwhile, in another universe, 4 ponies[1] were traveling across Equestria on their latest friendship quest and discovered an old animal sanctuary. Four of the local critters decided to join the group.

The following tables show everypony's and everycritter's preferences (most-favored to least-favored):

Pony	Critters
Hitch Trailblazer:	D, C, B, A
Izzy Moonbow:	A, C, B, D
Sunny Starscout:	C, B, D, A
Zipp Storm:	D, B, C, A

Critter	Ponies
Arlo the **A**rmadillo:	H, S, Z, I
Beatrice the **B**eaver:	H, I, S, Z
Cassandra the **C**apybara:	H, I, Z, S
Draco the **D**ragonfly:	H, Z, S, I

(a) [5 pts] Run the Matching Algorithm in the table below, with critters proposing to ponies. Please specify the proposals (new and recurring) and reservations/rejections that occur on each day. (We recommend drawing a box around the reserved partner on each day, as illustrated.) You may or may not need all the space given. (Note: the LaTeX source-code for this grid is available on Canvas.)

	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
H	A,B,C,[D]						
I	∅						
S	∅						
Z	∅						

(b) [5 pts] Run the Matching Algorithm, this time with ponies proposing to pets.

(c) [3 pts] Prove that any matching pairing H with C is unstable.

[1]Pipp Petals was absent from this episode because she already had a flying pomeranian, her voice actress was on vacation, and Zach wanted to make this problem a bit easier.

Problem 4. Stable Merriment [12 points]

In a stable matching between n Pokémon and n Trainers produced by the Stable Matching Algorithm (with Pokémon applying and Trainers evaluating), call a Pokémon or Trainer ***merry*** if they are matched with one of their top $\lceil n/2 \rceil$ choices. In this problem, we'll prove that at least one Pokémon or Trainer must be merry.

Let S be the total number of times a Trainer rejects a Pokémon.

(a) [6 pts] Prove that if there are no merry Trainers, then $S < n^2/2$.

(b) [6 pts] Conclude that at least one Pokémon or Trainer must be merry. *Hint:* What can be said about S if there are no merry Pokémon?