Z. Abel, B. Chapman, E. Demaine

Problem Set 3

- **Due Date:** 11:59pm on **Monday 26th February, 2024**
- **Days Covered**: 04 and 05 (including Lecture, Warm-Up, and Recitation)

Problem 1. Sorting by Reversals [15 points]

We have a permutation of the numbers $\{0, 1, \ldots, n-1\}$, and we'd like to manipulate it so it becomes sorted in increasing order. For this problem, we're only allowed to make *reversal* moves: from a list $(a_0, a_1, \ldots, a_{n-1})$, we can pick a pair of indices $i < j$ and reverse the entire subsequence from a_i to a_j:

$$(a_0, \ldots, a_{i-1}, \underbrace{a_i, a_{i+1}, \ldots, a_{j-1}, a_j}, a_{j+1}, \ldots, a_{n-1}) \to$$

$$(a_0, \ldots, a_{i-1}, \underbrace{a_j, a_{j-1}, \ldots, a_{i+1}, a_i}, a_{j+1}, \ldots, a_{n-1}).$$

If (i, j) is an *inverted* pair (i.e., $i < j$ but $a_i > a_j$), call this move a *good reversal*.

In this problem we'll show that if we start with any permutation $(x_0, \ldots, x_{n-1})$ of $\{0, \ldots, n-1\}$ and keep making *good* reversal moves, then no matter which ones we choose, we will eventually terminate with a sorted sequence.

(a) [3 pts] Describe this scenario as a state machine, where the states are the permutations of $\{0, 1, \ldots, n-1\}$. What are the transitions? the start state?

(b) [3 pts] What are the *final* states? How do you know? Conclude partial correctness: *if* an execution of the state machine terminates, then the list must be sorted.

(c) [6 pts] In lecture, we proved termination of swap-sort by proving that the number of inverted pairs strictly decreases as we make swaps. This is *not* true for this problem! For example, $(9, 1, 2, 3, 4, 5, 6, 7, 8, 0)$ can transition to $(0, 8, 7, 6, 5, 4, 3, 2, 1, 9)$, but the former has 17 inverted pairs (0 and 9 are inverted with everything else) while the latter has 28 inverted pairs!

Even so, please find a derived variable for this problem that (i) has only nonnegative integer values and (ii) is strictly decreasing. Be sure to prove that your derived variable satisfies both properties.

Hint: Think of $(a_0, \ldots, a_{n-1})$ as digits of a base n number.

(d) [3 pts] Use parts (b) and (c) to prove total correctness: every execution must eventually terminate with a fully sorted list.

Problem 2. A Fibonacci Sum [10 points]

Recall that the *Fibonacci sequence* is given by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, so the sequence starts with $0, 1, 1, 2, 3, 5, 8, 13, \ldots$.

(a) [8 pts] Use the perturbation method to find the closed form of the sum

$$S = \sum_{n=1}^{\infty} \frac{F_n}{c^n},$$

where $c = 10^5$. You may assume the sum converges.

Hint: Try comparing S with S/c.

(b) [2 pts] Verify your answer with WolframAlpha (`www.wolframalpha.com`) or some other high-precision calculator, recalling that $c = 10^5$. What are the first 50 digits of your result? What is the pattern and does it make sense?

Problem 3. Integral Method [15 points]

Let $f(x) := \dfrac{x}{(5+x)^3}$, and let $S_n := \displaystyle\sum_{x=n}^{\infty} f(x)$. In this problem, we will use the integral method to estimate the infinite sum S_0 to an accuracy of $\dfrac{1}{200}$. In other words, we will find bounds u and v such that $u \le S_0 \le v$ and $|v - u| \le \dfrac{1}{200}$.

Feel free to use a calculator (e.g., WolframAlpha) for algebraic/numerical computations, but be sure to justify all of your reasoning.

(a) [2 pts] Explain briefly why applying the integral method directly to S_0 will not work.

(b) [3 pts] Find the smallest index i such that the sum S_i can be approximated by applying the integral method directly.

(c) [4 pts] Use your answer from the previous part to find upper and lower bounds for S_0. What accuracy does your approximation achieve? (I.e., what is the difference between your upper and lower bounds?)

(d) [6 pts] Find the smallest index j such that the sum S_j can be approximated to an accuracy of $\dfrac{1}{200}$ by applying the integral method directly. Use your answer to find u and v (expressed as rational numbers) such that $u \le S_0 \le v$ and $|v - u| \le \dfrac{1}{200}$.

Hint: j should be a single digit integer.

Problem 4. Some Sums [10 points]

Find the closed form of each of the summations below.

(a) [5 pts]

$$\sum_{i=0}^{n} \frac{5^i - 7^i}{13^i}$$

(b) [5 pts]

$$\sum_{i=1}^{n} \sum_{j=1}^{k} (3^j - i)$$