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Lecture 05: Sums

1 Sums

Useful for recurrences, counting, probability, runtimes of algorithms, performances of large systems, machine learning, and much more!

1.1 Annuity

Winning 1M in lottery really comes is yearly installments. Imagine 50k per year for 20 years (or 1 per year for 1M years). Is this worth the same? No! Money now is more valuable than money in the future!

Picture getting 50k now or in 10 years. If I get it now, I can invest it, and accrue 10 years of interest on it! It'll be worth way more in 10 years.

So, choice: 50k per year for 20 years, OR 1M now? (Definitely 1M!) What about 700k now instead? 500k?

This is exactly how a loan works: lump sum now, paid back in installments, accounting for interest. Known as an *annuity*.

Let's simplify our assumptions: fixed interest rate, p. So \$1 now is worth $\$(1+p)$ in 1 year, $\$(1+p)^2$ in 2 years, and so on. Conversely, \$1 in 1 year is equivalent to $1/(1+p)$ today. \$1 in 10 years is worth $(1/(1+p))^{10}$ today.

So, imagine an n year, $\$m$ annuity, with interest rate p. (E.g., picture $p = 0.0533$, today's interest rate from the Federal Reserve.)

So, m now, m in 1 year, m in 2 years, . . . , m in $(n-1)$ years. By our assumptions, this is equivalent to a lump sum today with value

$$\begin{aligned} V_1 &= m + \frac{m}{1+p} + \frac{m}{(1+p)^2} + \cdots + \frac{m}{(1+p)^{n-1}} \\ &= \sum_{k=0}^{n-1} m \cdot \left(\frac{1}{1+p}\right)^k \\ &= m \cdot \sum_{k=0}^{n-1} x^k \qquad \text{where } x = \frac{1}{1+p}. \end{aligned}$$

Want closed form! (What is closed form? Basically, a formula you could enter into an arithmetic calculator, with no summations, ellipses, recursions, etc.)

Geometric series: $\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1}$. We know closed form: $\frac{1-x^n}{1-x}$.

We already proved this by induction, in Recitation 02. In general, **if you know (or can guess!) the answer, it's usually straightforward to prove it with induction**. This is known as the **Guess and Check** method. The hard part is **finding/guessing** the answer, not proving it.

(In this case, the inductive step reduces to proving $(1-x^n)/(1-x)+x^n = (1-x^{n+1})/(1-x)$, which can be checked with algebra.)

But how would we discover this closed form, if we didn't already know it?

1.2 Perturbation Method

Gauss famously solved $1 + 2 + 3 + \cdots + n = n(n+1)/2$ with this method.

Perturbation method: compare the sum to itself or a modified version of itself, and hope things combine nicely. In this case, multiplying by x creates a sum that looks very similar:

$$\begin{aligned} S &= 1 + x + x^2 + x^3 + \cdots + x^{n-1} \\ xS &= \quad\;\; x + x^2 + x^3 + \cdots + x^{n-1} + x^n \end{aligned}$$

$S - xS = 1 - x^n$, because everything else cancels! So $(1-x)S = (1-x^n)$, so $S = (1-x^n)/(1-x)$.

Back to annuity:

$$V_1 = m \cdot \frac{1-x^n}{1-x} = m \cdot \frac{1 - \left(\frac{1}{1+p}\right)^n}{1 - \frac{1}{1+p}}.$$

If $m = 50k$, $n = 20$, $p = .0533$, then $V_1 \approx \$638,340$.

What about our \$1, 1M year annuity? In fact, let's just say it goes on forever. What is $\sum_{k=0}^{\infty} x^k$? Assuming $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} x^k = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}.$$

$V_2 = \frac{1}{1-x} = 1 - \frac{1}{p+1} = (p+1)/p$. Same interest rate gives $V_2 \approx \$19.76$.

Perhaps counterintuitive that money paid every year *forever* would still have finite total value, but that's the interest rate at work.

These *perpetual bonds* are rare but do exist! Earliest surviving one was issued by a Dutch water company in 1624. Five such bonds are known to survive today. One was acquired by Yale for \$24k and now earns them €11.35 annually.

1.3 Educated Guessing, aka the Ansatz Method

How to evaluate $S := \sum_{k=1}^{n} k^2$?

Method 1: already know the answer! $\frac{n(n+1)(2n+1)}{6}$. Can prove by induction! Inductive step boils down to proving that $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$, which can be verified with algebra.

But what if we didn't know the answer? Make a partial guess, using variables for unknown numbers, and solve for their values. In this case, let's guess that it's a degree-3 polynomial:

$$\sum_{k=1}^{n} k^2 = an^3 + bn^2 + cn + d,$$

for some constants a, b, c, d. This would imply

$$\begin{aligned} 0a + 0b + 0c + d &= 0 \\ a + b + c + d &= 1^2 \\ 8a + 4b + 2c + d &= 1^2 + 2^2 \\ 27a + 9b + 3c + d &= 1^2 + 2^2 + 3^2 \end{aligned}$$

which is enough info to solve: $a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$.

So can we conclude that $\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$? Well, not yet; all we know is that this is true for $n \in \{0, 1, 2, 3\}$. How could we test whether it is true for all n? Induction! If the induction works, we'll have a proof that our formula is right! If not, the induction will fail, so we'll know our guess was wrong.

Already checked $n \in \{0, 1, 2, 3\}$. Inductive step can be verified with algebra.

Note: The summation $\sum_{n=1}^{0} f(n)$ is called the *empty summation* and has value 0.

1.4 Double Sums

Sometimes we have to evaluate sums of sums, otherwise known as *double summations*. E.g.

$$\sum_{i=1}^{n} \sum_{j=1}^{i} j$$

To evaluate such a sum, first find a closed form for the inner sum, and then use it to find a closed form for the outer sum.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i} j &= \sum_{i=1}^{n} \frac{i(i+1)}{2} \\
&= \sum_{i=1}^{n} \left(\frac{1}{2}i^2 + \frac{1}{2}i\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} i^2\right) + \frac{1}{2}\left(\sum_{i=1}^{n} i\right) \\
&= \frac{n(n+1)(2n+1) + 3n(n+1)}{12} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

Suppose instead we had $\sum_{i=1}^{n}\sum_{j=i}^{n} j$. One tool that is often useful for evaluating double sums is to *exchange the order of summation*. Altogether, we are summing over all pairs i, j such that $1 \leq i \leq n$ and $i \leq j \leq n$, i.e. $1 \leq i \leq j \leq n$. The key observation is that we can also express this as $1 \leq j \leq n$ and $1 \leq i \leq j$.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n} j &= \sum_{j=1}^{n}\sum_{i=1}^{j} j \\
&= \sum_{j=1}^{n} j^2 \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

What about $\sum_{j=1}^{n} j2^j$? One (perhaps counterintuitive) way to handle this is to *introduce a double sum* by expressing $j = \sum_{i=1}^{j} 1$, and then exchange the order of summation.

$$\begin{aligned}
\sum_{j=1}^{n} j2^j &= \sum_{j=1}^{n}\sum_{i=1}^{j} 2^j \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n} 2^j \\
&= \sum_{i=1}^{n}\left(2^{n+1}-2^i\right) \\
&= \left(\sum_{i=1}^{n} 2^{n+1}\right) - \left(\sum_{i=1}^{n} 2^i\right) \\
&= n2^{n+1} - 2^{n+1} + 2 \\
&= (n-1)2^{n+1} + 2
\end{aligned}$$

1.5 Approximating Sums: The Integral Method

What about $\sum_{k=1}^{n}\sqrt{k}$? No known closed form! Best we can do is **approximate**.

Suppose $f(x)$ is a *weakly increasing* function of x, and $S = \sum_{k=1}^{n} f(k)$. (For the preceding example, $f(x) = \sqrt{x}$.) Recall Riemann Sums from integral calculus: how can we approximate $I := \int_1^n f(x)\,\mathrm{d}x$?

If we underapproximate $f(x)$ with $f(\lfloor x \rfloor)$, then we can lower bound I:

$$\begin{aligned}
I &= \int_1^n f(x)\,\mathrm{d}x \\
&\geq \int_1^n f(\lfloor x \rfloor)\,\mathrm{d}x \\
&= \sum_{k=1}^{n-1}\int_k^{k+1} f(\lfloor x \rfloor)\,\mathrm{d}x \\
&= \sum_{k=1}^{n-1} f(k)
\end{aligned}$$

Equivalently, this upper bounds S in terms of I:

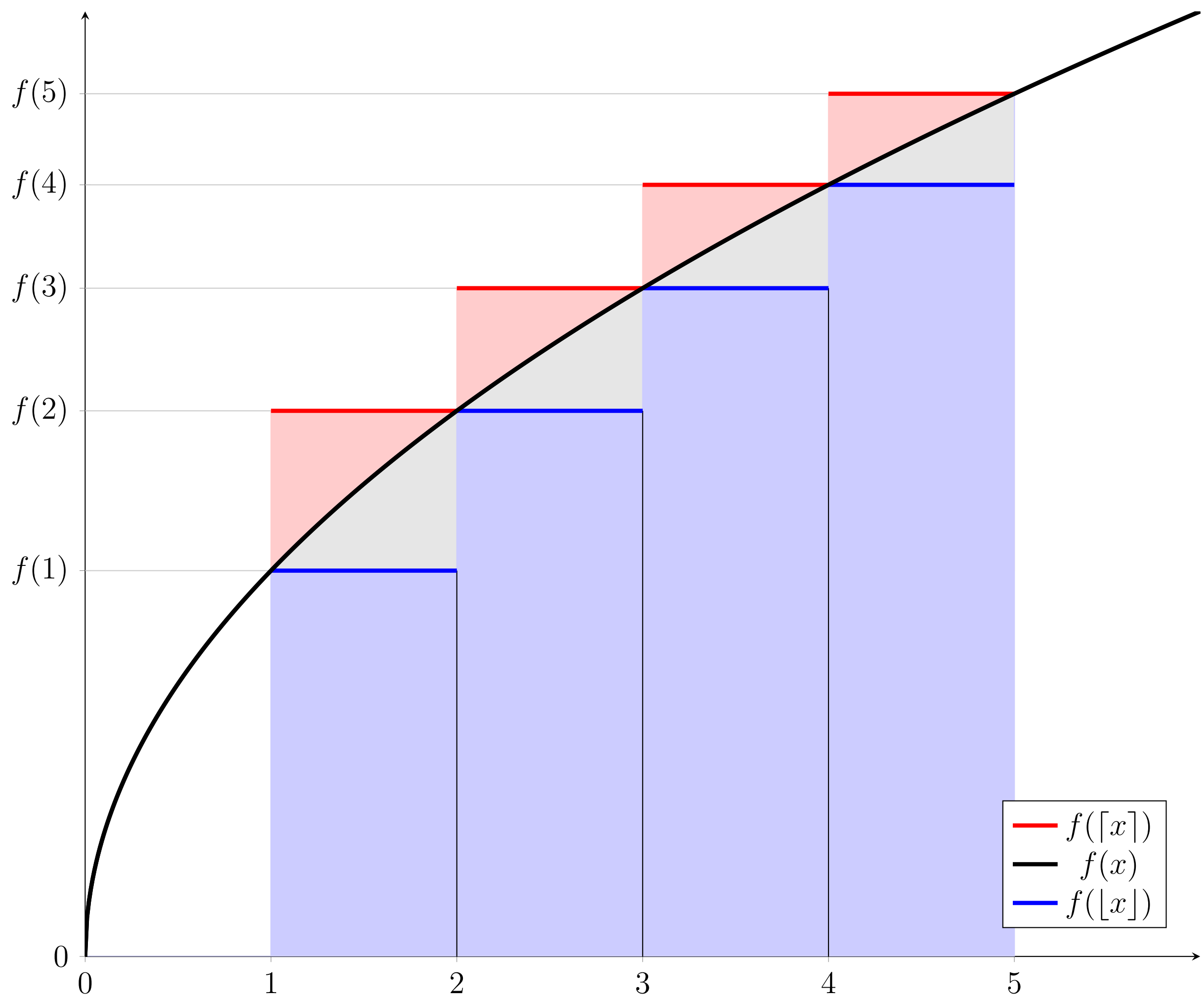
$$\begin{aligned}
S &= \sum_{k=1}^{n} f(k) \\
&= f(n) + \sum_{k=1}^{n-1} f(k) \\
&\leq f(n) + I
\end{aligned}$$

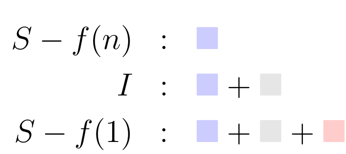
Similarly, we could instead over-approximate $f(x)$ with $f(\lceil x \rceil)$ to get

$$\begin{aligned} I &\leq \int_1^n f(\lceil x \rceil)\,\mathrm{d}x \\ &= \sum_{k=2}^{n} f(k) \end{aligned}$$

Equivalently, this lower bounds S in terms of I:

$$\begin{aligned} S &= f(1) + \sum_{k=2}^{n} f(k) \\ &\geq f(1) + I \end{aligned}$$





We now have both an upper and lower bound on $\sum_{k=1}^{n} f(k)$:

Theorem 1 (Integral Bound - Increasing)**.** *If $f : [1, n] \to \mathbb{R}$ is weakly increasing, then*

$$f(1) + \int_1^n f(x)\,\mathrm{d}x \quad \le \quad \sum_{k=1}^{n} f(k) \quad \le \quad f(n) + \int_1^n f(x)\,\mathrm{d}x.$$

Proof. Above □

For $f(x) = \sqrt{x}$, we have $\int_1^n \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x\sqrt{x}\right]_1^n = \frac{2}{3}n\sqrt{n} - \frac{2}{3}$, so

$$1 + \frac{2}{3}n\sqrt{n} - \frac{2}{3} \quad \le \quad S \quad \le \quad \sqrt{n} + \frac{2}{3}n\sqrt{n} - \frac{2}{3}.$$

So S is a touch larger than $\frac{2}{3}n\sqrt{n}$.

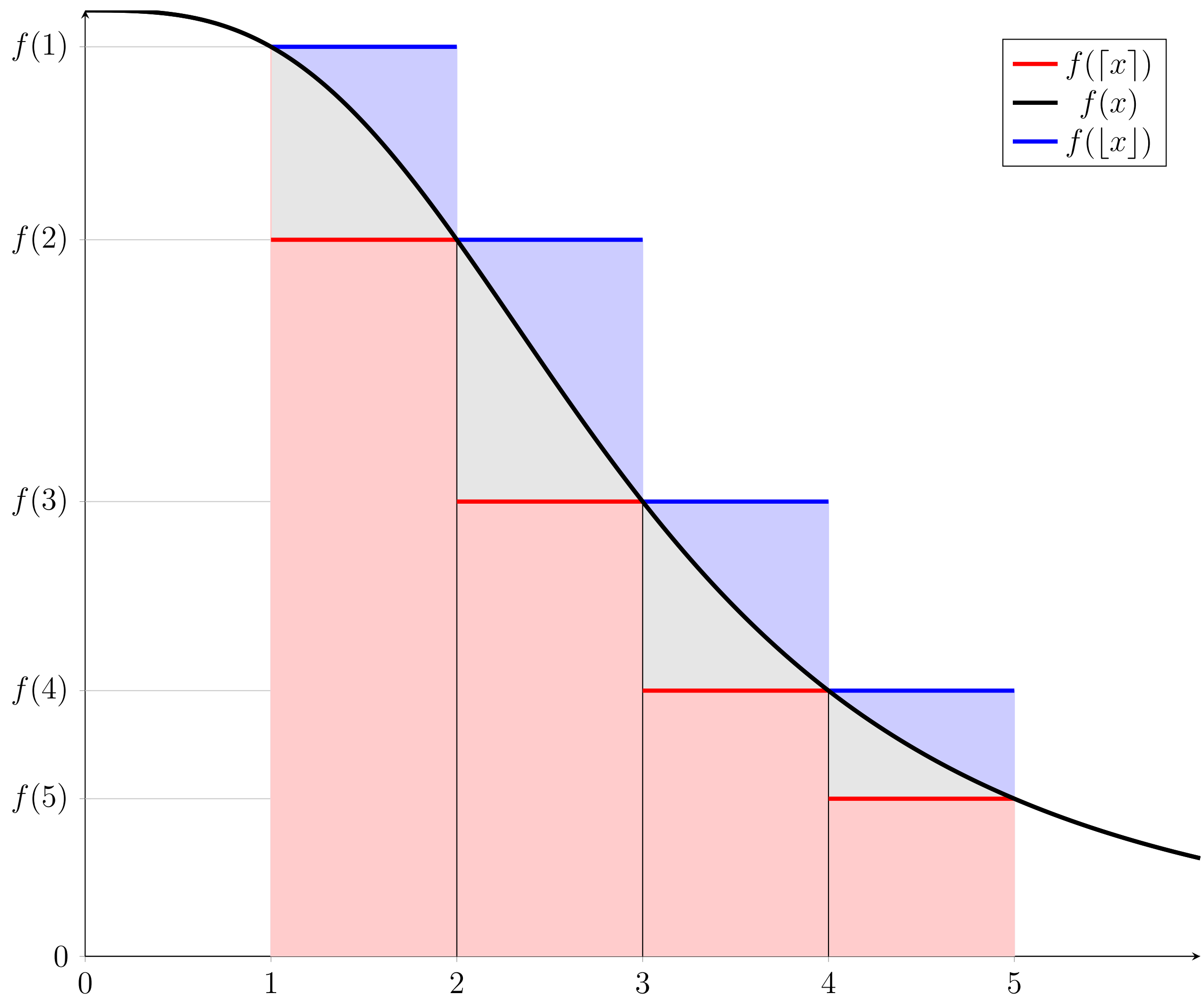
Same method proves:

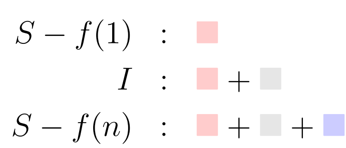
Theorem 2 (Integral Bound - Decreasing)**.** *If $f : [1, n] \to \mathbb{R}$ is weakly decreasing, then*

$$f(n) + \int_1^n f(x)\,\mathrm{d}x \quad \le \quad \sum_{k=1}^{n} f(k) \quad \le \quad f(1) + \int_1^n f(x)\,\mathrm{d}x.$$

Proof. Apply Theorem 1 to $g(x) := f(n + 1 - x)$. □

Note: there are other proofs. We could instead apply Theorem 1 to $h(x) := -f(x)$, or simply observe that in the previous proof, $f(\lfloor x \rfloor)$ becomes an *upper* bound on $f(x)$, while $f(\lceil x \rceil)$ becomes a *lower* bound.





Theorem 3 (Integral Bound - Improper)**.** *If $f : [1, \infty) \to \mathbb{R}$ is weakly decreasing, then the sum $\sum_{k=1}^{\infty} f(k)$ converges iff the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges. If they converge, then*

$$\int_1^{\infty} f(x)\,\mathrm{d}x \quad \leq \quad \sum_{k=1}^{\infty} f(k) \quad \leq \quad f(1) + \int_1^{\infty} f(x)\,\mathrm{d}x.$$

Proof. Take limits of Theorem 2 as $n \to \infty$. □

In this class, if we ask you to use the Integral Method (or Integral Bound), you should simply cite one of the above theorems. You do *not* need to rederive them!

Example: Suppose $f(x) = x^{-2}$. Define:

$$\begin{aligned} S &:= \sum_{k=1}^{\infty} k^{-2} \\ I &:= \int_1^{\infty} x^{-2}\,\mathrm{d}x \\ &= \lim_{n\to\infty} \left[-x^{-1}\right]_1^n \\ &= 1 \end{aligned}$$

Theorem 3 gives bounds $S \in [I, I + f(1)] = [1, 2]$, which is not a very precise approximation. How can we improve the approximation?

Idea: approximate tail of S, e.g.

$$\begin{aligned} S' &:= \sum_{k=4}^{\infty} x^{-2} \\ I' &:= \int_4^{\infty} x^{-2}\,\mathrm{d}x \\ &= \lim_{n\to\infty} \left[-x^{-1}\right]_4^n \\ &= \frac{1}{4} \end{aligned}$$

Now $S' \in [I', I' + f(4)] = \left[\frac{1}{4}, \frac{5}{16}\right]$, so $S = f(1) + f(2) + f(3) + S' \in \left[\frac{232}{144}, \frac{241}{144}\right]$. This is much more precise than our previous approximation!

OPTIONAL MATERIAL, NOT EXAMINED: As it turns out, $S = \frac{\pi^2}{6}$.

Proof. Let m be an even integer, and let $n = 2m + 1$. We first observe that

$$\frac{\cos(nx) + i\sin(nx)}{\sin^n x} = \frac{e^{inx}}{\sin^n x} = \left(\frac{e^{ix}}{\sin x}\right)^n = (\cot x + i)^n = \sum_{j=0}^{n} \binom{n}{j} i^{n-j} \cot^j x.$$

Taking only the imaginary parts (second term on LHS, even indices on the RHS) gives

$$\frac{\sin(nx)}{\sin^n x} = \sum_{k=0}^{m} \binom{n}{2k} \left(-\cot^2 x\right)^k = nP(\cot^2 x), \tag{1}$$

where

$$P(x) := \frac{1}{n} \sum_{k=0}^{m} \binom{n}{2k} (-x)^k.$$

P is a monic degree-m polynomial, and Equation 1 gives all of its roots. If x is an integer multiple of $\frac{\pi}{n}$, then the LHS of Equation 1 is 0. There are m distinct integer multiples of

$\frac{\pi}{n}$ in the interval $\left(0, \frac{\pi}{2}\right)$ (on which $\cot^2 x$ is strictly monotone), namely $\left\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{m\pi}{n}\right\}$. Therefore, $\left\{\cot^2 \frac{\pi}{n}, \cot^2 \frac{2\pi}{n}, \ldots, \cot^2 \frac{m\pi}{n}\right\}$ are *all* of the roots of P. Since P is monic and has degree m, the sum of its roots is the negation of the coefficient of x^{m-1}. This gives

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{n} = \frac{1}{n}\binom{n}{3} = \frac{(n-1)(n-2)}{6}.$$

After multiplying by $\frac{\pi^2}{n^2}$ and rewriting in terms of m, we have

$$\sum_{k=1}^{m} \left(\frac{\pi}{2m+1} \cot \frac{k\pi}{2m+1}\right)^2 = \frac{\pi^2}{6} \cdot \frac{2m(2m-1)}{(2m+1)^2}.$$

Using the identity $\csc^2 x = 1 + \cot^2 x$, we also have

$$\sum_{k=1}^{m} \left(\frac{\pi}{2m+1} \csc \frac{k\pi}{2m+1}\right)^2 = \frac{\pi^2}{6} \cdot \frac{2m(2m-1)}{(2m+1)^2} + \frac{\pi^2 m}{(2m+1)^2}.$$

We combine the previous two equations using the inequalities $\sin x < x < \tan x$ on the interval $\left(0, \frac{\pi}{2}\right)$:

$$\frac{\pi^2}{6} \cdot \frac{2m(2m-1)}{(2m+1)^2} < \sum_{k=1}^{m} k^{-2} < \frac{\pi^2}{6} \cdot \frac{2m(2m-1)}{(2m+1)^2} + \frac{\pi^2 m}{(2m+1)^2}.$$

Taking limits as $m \to \infty$ gives $\frac{\pi^2}{6} \leq S \leq \frac{\pi^2}{6}$, i.e. $S = \frac{\pi^2}{6}$. $\square$