GEORGE BARBASTATHIS: OK. So I don't know if everybody's awake, but I will start. So today, the topic is actually Hamiltonian optics, but we had some leftover from last time, leakage. So I'd like to go through that very briefly.

And then I will handover to Se Baek and Pepe, who will give a demonstration of aberrations in sort of experimental demonstration, and also how optical design software works. And then we'll get into Hamiltonian optics the second hour, and we'll see how far we go.

OK. So last time we talked about how many aberrations? We talked about spherical, coma, that's about it. So we talked, actually, about two aberrations. And we said that there is five of them-- the five so-called Seidel aberrations or side order aberrations. And today, I'll cover the other three.

So the first one in line is astigmatism, which is actually-- for me, at least, it is the most difficult to visualize. The best schematic I have seen actually is in your textbook. It's the middle picture over here. But before we get to that and we see what happens is, let's define to make sure that we know the terminology.

So astigmatism happens-- look at the top left diagram. Astigmatism happens with an object point that is off center. And since it is off center, imagine that there's a fundamental asymmetry. If you take the chief ray that goes through the center of the lens-- and again, here the assumption is that the aperture is at the lens itself. So the chief ray goes to the center of the lens.

So now you can take two ray-- you can think of two marginal rays. One is the marginal ray that goes through the meridional plane. So the meridional plane is the one shown like this, and the other is the marginal ray that goes through the sagittal plane, perpendicular to the meridional plane.

So astigmatism occurs because the two-- now take cross-sections of the rays in the two planes-- the sagittal and the meridional plane. That two ray bundles, they do not necessarily focus at the same place. In the sagittal plane, actually, if you tried to visualize it in the sagittal plane, there is no difference than normal incidence.

So you basically get-- so whereas in the meridional plane, because you are off-axis, you have this property that we discussed before, that you'll get different magnification at different points in the field. So as a result, these two might actually yield different focal distances. So this is depicted here, actually, where you're seeing-- it's actually a pretty good diagram. You can see the sagittal plane and the meridional plane.

And you can see that the sagittal plane actually focuses out here. It is F sub S for sagittal. Whereas the meridional plane focuses here. It is called F sub T for tangential. So if an optical system is subject to this problem, we just call astigmatic.

The typical way that people characterize astigmatism is with a spoke wheel. And most of you must have taken eye exams. They give you a similar test, actually, to test if your eyes are astigmatic and you need correction. They show you something like this, a spoke wheel. And they ask you, basically, if the spokes are simultaneously in focus with the circle. The outline circle.

So if the system is astigmatic, then as you move a piece of paper along the optical axis, you will see one of them come to a focus first, and the sagittal come to the focus later. OK.

So that's astigmatism. I hear some voices, but I think they're from the next classroom, not from Boston. Did anybody say anything there? I guess not. OK.

A related aberration, which has to do with the fact that-- well, let me say first. So a related aberration is called caricature of field. This one is actually easy to understand. It says the following. It says that if you derive the imaging condition for a point on-axis, then imagine you go off-axis, it says that the image may not necessarily focus on the same plane as the image on-axis.

In fact, what this aberration is about is that the image actually comes to focus on a curved surface. Center of the optical axis. Now, why do we care about the plane? Why do we want this to be a plane here? Well, in old fashion cameras, one used to put film at the image plane of the camera. So we have to be sure that the entire image is focused simultaneously on the film.

Nowadays, we don't use film. We use a digital chip. But in both cases, they're flat. So curvature of field is a problem then because it says that actually the off-axis image points will be out of focus because they will focus in front of the final plane. So these are the kind of things that if you could, for example, make a curve chip surface, you could solve it immediately. All we have to do is match the surface of the chip to the surface of the curvature of the field.

But in fact, actually, people are trying to do something like that. There's a lot of work on flexible electronics. And that's one of the motivations. But it is not yet widely available.

Nevertheless, there is one condition where the surfaces that the field comes to focus actually becomes flat. This was derived by a mathematician. I don't know when. I think it was in the early 1900s. His name is Petzval. So this condition, it has a more complicated expression. But for two lenses, it works out to something like this.

If the sum of these quantities where n1 is the index of refraction of the lens. And f1 is the focal length. So it's a little bit unusual now. We multiply the focal length by an index of refraction. We've never done that before. But this is what this guy derived.

If they two sum up to zero, these quantities for a pair of lenses, then the field of curvature vanishes. And what is even more sort of remarkable about this relation is that it does not involve the space in between the lenses. No matter where they are, if they satisfy this condition, the field of curvature will be eliminated.

And so for example, if they are composed of the same index of refraction, then the index of refraction also drops out of this equation. And then you can treat them as a composite, as a thick lens, basically, which this is a relationship that we already derived for a composite element.

And you can make sure that one of them is the negative of the other so that you can make this quantity vanish. That is, f1 is equal to minus f2. And then you can find the focal length from this equation over here. So it's kind of cool because it says that you can actually eliminate-- if it is really important to you, you can eliminate this problem by using the Petzval condition. And again, there's a more complicated condition that applies to multilens systems that have more than two lenses. But I didn't bother to draw it here.

And what is even more remarkable is that the Petzval surface and the field curvature is also related to astigmatism. So what you see in this plot here, also from the textbook, is the focal surfaces-- that is, the surfaces in which different points from a flat-- if you take a flat object and you take different points along this object, these three surfaces are correspondingly the tangential focus for the astigmatic system, the sagittal focus, and finally the Petzval surface. Yeah.

Let me go for [INAUDIBLE]. So what we discussed here is actually the Petzval condition. I haven't said yet what is the Petzval surface. So the Petzval surface is the following. It turns out that as you change the stopping and optical system-- that is, as you move the stop in the axial direction, you can achieve the condition where the tangential and the sagittal astigmatic focal surfaces, they collapse into one surface. This surface is called the Petzval surface, not to be confused with the Petzval condition, even though, of course, they are related.

So what you see here is basically, as you move the stop in an optical system-- this is borrowed from another textbook, not from-- I should have recorded, actually. This is from Jenkins and White, *Fundamentals of Optics.* So as you move the stop, you can find the situation where the two surfaces coincide, and that is the Petzval surface of the system.

And of course, this one does not satisfy the Petzval condition because the Petzval surface is curved. If I could somehow satisfy both the Petzval condition and collapse that to surfaces-- tangential and sagittal-- onto the Petzval, then I would have a flat system with a flat field, which would also be free of astigmatism.

So obviously we cannot get into too much detail about this without becoming quantitative. And this is quite difficult to deal with algebra, so I'm just being descriptive. And you will see some of this in action when Se Baek does the demo of the software in a little bit.

And the light distortion that I will discuss has to do with something slightly different. So the field of curvature occurs and astigmatism occurs because if you take object points off-axis, they fail to focus at the same plane. They focus at different planes and on different surfaces, the sagittal and tangential.

The distortion occurs because even if you manage to focus at one plane, the magnification is different. The magnification-- and this you can see very easily, actually, just looking at this diagram over here. If you take the on-axis point, the imaging and condition of the magnification are given by these distances. For example, the lateral magnification is the ratio of this distance to this distance.

But what about this point now? For this point, the lateral magnification will be the ratio of this distance to this distance. And for those of you in Boston, I'm pointing out to the tilted chief ray. So I have to measure the distance from the lens on the chief ray. And you can see very easily that this length is different than the length of the chief ray for the on-axis point.

So because of that, the magnification on-axis is different than the magnification off-axis. Therefore, if I attempt to image a checkerboard, like this, if the magnification is different, I will see this sort of-- I don't want to use the word "distorted." You will see this sort of altered-- let's say altered-- image. And this is the definition of distortion.

So in optics we have to be a little careful. Distortion, colloquially, means everything. All of the stuff that I described before. In everyday English, you can call them distortions. Well, in optics, in professional optics, when we mention distortion, we automatically mean this one. So we have to be careful not to use the word "distortion" in the colloquial sense because it is reserved to mean exactly that.

And I guess it has two forms. It can be of this type, where the magnification is higher off-axis. That is called pincushion. Or lower, that is called barrel. And they believe, in this case, the top one will give rise to barrel because in this case, because the stop is here, the aperture stop is here, the chief ray on the object side becomes longer off-axis.

Now, if you recall, the magnification is the lateral magnification equals S on the image side over S on the object side with a minus sign. But the minus is irrelevant here.

So since I made-- so in this case, S on the image side is approximately the same as on-axis. But S on the object side became longer by putting the stop here. So therefore, I get a lower magnification off-axis. Therefore, I get this type of distortion.

If I put the stop on the other side, then it is the opposite case. Now I make this length longer. Therefore, I get pincushion distortion. And of course, the way to eliminate distortion altogether is to make the system like this. This is actually a telecentric system. And to make sure that the stop is exactly at the center of the system. And also-- yes.

And so if you do that, then basically a little bit of thought will convince you that in this case the variation between this length is actually very small. So therefore, there is some distortion, but it is high order. The third order distortion is completely eliminated here. So you basically get a distortion-free image.

OK. So before I turn over to Se Baek and Pepe, last time I mentioned caustics. So I would like to show what caustics are. So this is actually from the last page of today's notes, which I haven't done yet. And the details here are not very important. By the end of the lecture today, you will know what has happened here.

This is a special type of lens called a gradient index lens. And I simulated it with off-axis incidence. So therefore, in this case, we have coma. So if you look at the ray, these are ray traces of this lens. For now, please accepted the fact that the rays are curved. But what I wanted to show you here, what is the caustic that I mentioned in the last lecture.

You can see that in the bottom side of this diagram that ray paths, they kind of collapse onto each other. So you see that the rays are relatively uniformly distributed up to this point approximately. Over here, they begin to accumulate near one side. So what you would expect in this case, if you put a piece of paper here, is you would expect very strong light intensity near the bottom of the beam. So that is what we call a caustic.

And I think in the examples that the guys will show you in the demo, you will also see some caustics form in real life. But this is what I was trying to describe last time. OK. So Pepe and Se Baek, I don't know if you guys are ready but--

PROFESSOR: Yeah.

GEORGE BARBASTATHIS: OK.

PROFESSOR: Can you help me with that.

GEORGE BARBASTATHIS: Take it away.

PROFESSOR: So this is a similar demo to the one that we showed. Now we're going to focus on a different thing of the aberrations that we've been learning. So so far we've seen two type of aberrations in the system that we can identify very easily.

But first, let me describe the components. So can you go back here. So for this one here we start with a laser, as usual. And now we know that we have a collimating system so used to condition the light and make a nice plane wave at the output here. And then this plane wave illuminates these lens, which is a cylindrical lens, shown here. And then we are just looking at the rays going on. And this is for a green laser. It's a wavelength of 532 nanometers.

So now let's think what's happening, actually. So we can reduce exposure. OK. So we can see at this point how the rays start converging. But it's a bit hard to see here, but they do not focus in a tight point. They actually focus at different points. And as we'll see in the software, this is basically the experimental version of a spherical aberration that we can see. And in this case, I'm illuminating the lens with an on-axis plane wave. So therefore, we can see its condition.

Now let's try to look at the case of coma as seen from the top. So what I'm doing is that I'm rotating the lens. And then we can see how the rays start forming an asymmetric pattern. And let me try to get this closer together.

So now for this off-axis plane wave, we see that that's some of the rays go straight through and some of the rays form these caustics here that we just mentioned-- George mentioned in the slides. If we were to put a page perpendicular to this one that I have here, which is hard because we need actually a camera in order to show you that with very, very tiny pixels because the actual point produces or the focus produced by this lens would be very small. But if we do that, we would see the comma shape that we described. And we're actually going to see that in the simulation software that we're going to show later.

GEORGE BARBASTATHIS: Pepe?

PROFESSOR: Yep?

GEORGE BARBASTATHIS: Is it possible to put the other business card-- can you put the back of the business card there?

PROFESSOR: You're saying perpendicular?

GEORGE BARBASTATHIS: Yeah.

PROFESSOR: You don't see anything, George. This is a cylindrical lens. This is cylindrical.

GEORGE BARBASTATHIS: Yeah.

PROFESSOR: Yeah. And you see what I'm saying? These are cylindrical lens. And even with a CCD, it's very, very small. You would need something, in order to measure it accurately, something that is called a Foucault knife edge test, for example, where you actually can measure what is called the point spread function. And we're going to learn that in the next part of the course.

All right. So let me show you another type of aberration. Now imaging our transparency. In this case, I just have a white light source that is illuminating this object, which is just a transparency with some GRIN. In this case, it's going to be just the GRIN that George was showing that is regular GRIN. And I have a single lens here. It's a spherical lens, biconvex, with a short focal length. So therefore, the curvature is pretty high, and we expect to have a lot of aberrations.

But the one that we're going to see here-- we can decrease exposure-- we're going to see the evidence of distortion, the aberration of distortion. And as you can see here-- let's just focus in. You see how the lines-- similar to the slide that George just showed, the lines are actually curved by this lens. Although these are supposed to be straight lines, they are curved. Let me just magnify a little bit this. There you go. Increase.

Now if we were to replace these with another lens-- so for example, this one that has a longer focal length-- the radius of curvature is larger. So it's basically not as curved as this one. And we're trying to do the same. Now what I'm going to do is I'm going to use this lens to image my transparency.

Now we can see how the lines look much more straight. That distortion aberration got reduced. So what type of aberration did you guys see? It was pincushions or barrel aberration, the one that you saw before? I'm going to put it again while you guys think.

Which one is this? I can hear. Just press the button, please.

AUDIENCE: Pincushion?

PROFESSOR: Yeah. Yeah. We see the evidence of the magnification of the lenses. It's maybe hard for you being able to see. But here, the lines basically get curved to outside.

So in the break, if you guys want to come on and just play with the demo and change the focal length and see where the images get formed, and also this one here, you're welcome to do so. Now we're going to switch to our numerical software to show how we actually deal with these in practice to optimize an imaging system.

SE BAEK OH: OK. So people, before we move on to next topic, which is a GRIN medium and Hamiltonian optics, we want to show how we actually build or design the actual imaging optical system in practice. Of course, so far we learn a lot of stuff. The Snell's law, and reflection and refraction, and the lens maker's law, and paraxial optics, and blah, blah, blah.

But in real life, we usually use a computer software. So just to give you an example, since most of you are mechanical engineer, including myself. So let's think about this. If you want to build some toy or some mechanical model for your bike or car or toy, what do you do is you start with sketching by hand. So make the conceptual model. And then you go to computer. Use the CAD program.

So your build the 2D or 3D model. And if you have multiple element, you can even assembly, in SolidWorks. And you can export the 3D model. And if your system is complicated enough, you can also use finite element model to analyze the stress or strain or the temperature variations, blah, blah, blah. Same idea.

You start with the conceptual model by drawing a few rays. And then you apply lens law and the 1 over object distance plus 1 over image distance should be 1 over f. And sometime you only use the [INAUDIBLE]. But after that you go to computer and use probably one of these commercialized software, which is Code V. So it's not V. It's a Roman number. So Code V or ZEMAX and Oslo.

So the Code V and ZEMAX, they are industrial standards. And they are pretty much the same. So if you can handle one of these, then it's very easy to do-- easy to use the other one. So essentially what it does is it draws some rays for you but very fast, and it's automated fashion.

Because if you have CAD program, you can use actually do this ray tracing. So you just draw a ray. And whenever you meet the interface, you complete the incident angle to the surface normal and apply the Snell's law and compute refraction angle and do the same thing. But it's very complicated, especially you have the 3D model.

So this program does that for you. And it actually has very different method to do the ray tracing. So first of all it's just-- actually, we learn the ray transfer matrices. So it does ray tracing based on the paraxial region. But probably one of the reasons that we use the software. It does the real ray tracing. Just as I described, it just trace every ray based on Snell's law.

And you can use it-- and actually, it also has many built-in features to analyze performance of optical system, like aberration. So we learn about the five geometric aberrations. You can analyze it. And also, point spread function or spot diagram, which means if you have ideal imaging system and lets say we have point object, the images should be also infinitely small point, assuming that they are conjugated.

But in real life, it never be like this because you always have the finite size of focused image. So point spread function or spot diagram tells you how big is your focus at the image plane. And it also analyze MTF or OTF, which is frequency analysis. We cover later of the course. And probably-- actually, another reason that we have to use this software is to optimize the optical system. I prepared some examples, so we will see.

So this is the-- so we are using the ZEMAX. Just the one of the industry's standard. And if you open the ZEMAX in this, we have this table, which basically are lens data editor. So filling in this table means you specify your optical system. I'll describe how to do it.

But once you have this-- once you fill in this table, then you can analyze very different stuff. So for example, it shows the actual layout of a system, like this one. This 2D layout. And it even made the 3D model for you, like this one. So it can check the actual shape of the system.

So this is called [INAUDIBLE]. So actually, the horizontal axis is actually your pupil plane, and vertical axis is deviation of the ray at the image plane. So if you have-- so if your system is aberration free, then these lines should be just horizontal here because all the rays meet at the point. But this is practical system so you have the deviation, which means you have aberration.

And these guys are spot diagram. So if you have point object in your object space, then you'll probably get this kind of focus at your image plane. And this is same thing but for different wavelengths because the glass is dispersive, which means the refractive index is also a function of the wavelength. So if you change the wavelength, then the shape of these spots also change. So it analyzed this dispersive property.

OK. So let me explain how to fit into those tables. So let's say we have this two lens system. So what you do is you just decompose your system to multiple surface. So for example, here I have the object. So this is by default surface zero. And the first surface should be the front surface of the first lens here. And surface two is the back surface of the first lens. And the surface three is the front surface of the second lens, and so on and so forth.

So in the table we have the different-- the multiple roles. So each role represent the one surface. And the radius means the radius curvature of the surface. So for example, here, since this object is flat, the radius curvature is infinity. And thickness is the distance from the surface to the next surface. So in this case, d0 is the distance from here to the first lens.

And in some case, if you deal with the infinite conjugate, like telescope, then this d0 could be infinite because your object is at infinity. And at the surface one, I have the lens which has the radius curvature R1. So I have R1. And thickness d1 represent the thickness of the actual length from here to here. And after the surface one, I have the glass. So I just can specify the type of glass, which is BK7 here. This is the most common optical glass.

And you just keep repeating this one. So once you have this and you have the-- you just fill in the table in the ZEMAX. And there are also other parameters you have to specify. So as I said, the wavelength. And you also need to choose the field parameter, which basically this means object information. So if your object is in finite distance, then field means object height or size of your object, or if it is infinite, you can set the incident angle to the system, just like the telescope.

So first example I prepare is actually the example the lecture three. So we had the two lens. Both of them have the focal length of 10 here and here. And if you went through the derivation, we had the several different ways to derive. But the image distance-- actually, it's not image distance. The imaging is located at 30 millimeter behind the second length, and the lateral magnification was negative 4.

And also, if you did the wavelength of matrices, then effective focal length was 20 over 3, which is about 7. And the front focal length and back focal length was 10 over 3. So about 3, right? So in ZEMAX, I already filled in the table. So this is the how the system looks like. I have the object here, and I have the image here.

And if you analyze the system, then it gives you all the specifications here. So here, the effective focal length is 6-point-- I don't know you can see, but the effective focal length is 6.92. So about 7. So we have a good agreement here.

And the back focal length is about 3.05. So about 3. And the paraxial magnification is negative 4.8. So we have a slightly bigger magnification in real system, and so on and so forth. And also, I put the aperture stuff here. So you can compute the diameter-- the size and position of the entrance pupil and exit pupil as well.

GEORGE BARBASTATHIS: Se Baek?

SE BAEK OH: Yes?

GEORGE BARBASTATHIS: I think the reason for the discrepancy is that the lenses have finite thickness in your model, right?

SE BAEK OH: Yes. So in real model, as you can see in the layout, this is not in length. And I just put the brief numbers-- the radius of the lens. So I just put the 10.5. And the thickness of the lens in 1 millimeter in this case. So it's not the real length. So it gives you slightly different numbers.

And the next example is actually the Hubble Space Telescope. So Professor Barbastathis already mentioned the last lecture about this telescope. So actually, NASA realized after they launched this telescope and find out the image is not good as it's supposed to be. And they form a committee to investigate what's the problem. And they eventually figured out, and they fix it.

But let's just start with the structure here. So this is a typical classic type telescope. We have two mirrors. So one is primary mirror, which is a concave hyperboloid mirror, and we have the secondary mirror, which is convex hyperboloidal mirror. And the size of the mirror, the diameter is about 2.4 meters. The primary mirror. And the second one is about, I guess, 60 centimeters or so. And the distance between these two mirror was 4.94 meters. So it's a huge telescope.

And after extensive investigation, the problem was, actually, the shape of the primary mirror is not same as the intended shape. So the overall shape-- I mean, the curvature of the mirror was slightly different than the design. And it turns out that even though-- actually, the diameter of this primary lens is 2.4 meter. But at the edge, the difference between the design and the actual shape was only 4 micrometer. So it's very small, the physical difference.

So just give you an idea. So the black hair, like the Asians, like myself, the diameter of the thick hair is about 100 micrometer. So 4 micrometer is very, very small distance compared to the diameter. But in general, it introduced a huge speckle aberration, which means ideally you have-- all these incoming rays should meet at one point, but it has a different extended focal point. So the effect was too catastrophic.

So the image of the galaxy it looked like this. But after they fix, it looked like this. So does anyone know how NASA fixed this problem? The telescope was already in orbit, so you cannot take it back to the ground. And to fabricate another big mirror also need huge resource. So how can you fix?

Yeah. So actually what they did is-- this is huge telescope. And they had a focal point. And they have different cameras, because you deal with different spectrum, like visible or the IR or UV. And what they did, actually-- and all these cameras show the same focal plane. So what they did, they actually sacrificed another camera and put another mirror system here to introduce this speckle aberration but in opposite fashion. So they kind of canceled the speckle aberration.

So I prepared an example in ZEMAX with the real parameter. So this is the intended design of the Hubble Space Telescope. And as you can see, the surface number two and surface number three, they are primary and secondary mirror. And this is hyperboloidal mirror. So you have-- we have here. So conic coefficient.

And this is the simulation result if you have-- simulation result of image if you have the letter F. But due to the manufacturing process, this conic coefficient was not this one. So actually it was negative 1.02324. So overall shape it doesn't change that much. But the image analysis looks very blurred. So that's why we had the very blurred image of galaxy. And that's why you need to deal with all these aberration with this computer software because you can analyze easily and compensate easily.

OK. So last example is optimization. So for example, let's think about this one. I want to design some optical system which has three lenses here. So these two positive lens is made by crown glass, and this negative lens is actually flint glass. And if you want to design this one, then you have many parameters you can tune.

So the radius curvature of each length-- so R1/R2, R3/R4, and R5/R6. And also the thickness of the lens, and also the distance between the lenses. So we have another six parameters here. So you have total 12 parameters you can choose.

And it's hard to find the best parameter because you need some specification to satisfy. So what you do is you can just run the optimization routine here. So first process-- first procedure is select variable. As I just described, you can tune the radius of curvature of the lens and the thickness and the distance between lenses.

So you want to choose the variable you can tune. And this column, you define the merit function, or sometimes it is called object function or cross function. Basically, you just select what you want to minimize. For example, you want to minimize some speckle aberration for given object point, or maybe you can minimize a spot diagram, like size or focus for different field points.

And also, you can add some constraint, like your total-- physical system should be like this. It cannot longer than certain distance. Or the thickness of the lens can be constrained due to the manufacturing process. So you can add some constraints.

And then what you do is compute the value of merit function with the different combination of the parameter. And this is the automated-- this is iterative process so ZEMAX can do it for you.

So this is the layout of the system. I start with just three each value-- I just choose the random value of this-- I mean, there's 12 values here. So we have the three lenses here. And parameter with the [INAUDIBLE], they are variables. So we have the 12 variables here.

And this is the result of the aberration at the initial state. And the spot diagram through the focus. So as the bottom left figure, I have four different incident angles. So 1, 2, 3, 4. The different colors here. So these four rows in spot diagram means they're corresponding to this different incident angle. And these five column means I had five different defocus. So center one is focused, and they are slightly defocused in front or back.

I also have the spot diagram with different wavelength. So this one is about 500 nanometer. And it's about 480 nanometer. I guess this one is 550 or so. 50 or so.

And this is merit function. I already defined the merit function. Actually, I used the default merit function in this case. So there are many, many different parameter you can tune, but the essence is you want to minimize some aberration at the image plane.

So I'm going to run this optimization with the auto update. So you're going to see the actual shape of this analysis for every iteration. So we start with this situation, and I run 50 cycles here. So every iteration is update the system. And it actually improve the performance of system.

So yeah. So after 50 iterations, this focus is much nicer. Focus is much nicer. And it doesn't change that much. But actually, the scales are different. So actually it improved a lot.

So after this is optimized-- so this is what usually optical engineers do in industry when you want to design some optical system. But is this the best design? I mean, can you say this is this the best design? Probably not because it's just one of the minimum. It can be just global minimum. So it's kind of sensitive to initial value or constraint or whatever.

And also, there are many, many issues-- I mean practical issues-- we have to consider. For example, in this case, let's see this third lens. The last lens. The radius of curvature of this lens is 60.8 on this one. And the second radius is negative 68.3. So they are very similar, which means it looks like convex lens. But they are very deep. They are slightly different, but they are not same.

So if you're on an assembly system, then you need to figure out which surfaces from surfaces shown is the back surface. But it's very similar, so it's hard to figure out. So sometimes you need to consider many, many practical issues because in this case, it's not good design because they are so similar. You better have the plane of convex to figure out which surface is front and which surface back surface.

So ZEMAX does that for you. I mean, the optimization. But you need to interpret the result, and you need to take into account many practical issues. So basically, which means it means nothing. So that's why we learn many, many different things. I mean, the geometrical optics, you need to interpret it. And you need the experience and insight. Yeah. That's what I prepared today.

PROFESSOR: The last example that we're going to show is on a gradient refractive index lens. So these are a very interesting type of lenses. I have this microphone here. Very interesting type of lenses because the index of refraction as opposed to normal lenses that you're familiar with and we've been showing over and over, which is just a piece of glass with a fixed index of refraction, in this case, as we've been mentioning, the index of refraction changes as a function, in this case, of radius.

So the lens itself, it looks like this rod here. So this is a 3D model. Just like a cylinder of glass. But its properties, or its optical properties, especially a refractive index, as you can see, changes. And in this case, commercially available green lenses, the way they're manufactured, they're constrained to be with a parabolic index of refraction variation.

So at the center-- so this is basically distance from the-- radial distance is at the center of the lens. Optical axis. And this as we propagate to one-- as we move to one of the extremes of the lens. And we see that we have the highest index of refraction at the center. And then the case parabolically to the sides.

And if you can notice here the variation from top to bottom is maybe hard to read. But it's very, very small. So it goes from something like 1.6 to 1.59 or something like that. So it's very, very small, the index variation. However, that is enough to guide the light and cause the rays to bend.

So one thing that we're going to see here is that Snell's law is basically applied in a little bit different way. Before, we're interested in curved surfaces. And a ray hitting a given curved surface will bend accordingly to Snell's law and the index of refraction. In this case, you see that the faces of the lens to both sides are flat. So if the index of these was basically uniform, the plane wave would just go straight through like a piece of glass, like in a window.

But the fact that this has a parabolic index of refraction, then it makes these rays to smoothly be guided and bent towards a focus. In this case, on-axis or off-axis. This is a 10 degree off-axis plane wave incidence. So what is the nice thing about these things? It's that instead of only using, you can think of the refractive power of these two surfaces. This type of lens uses the entire volume of the lens to guide the rays.

So therefore, if I put a screen here and see the intersections of all these rays in that screen-- this is the spot diagram that we've been seeing also in the other ones-- we see that for the on-axis point, we have a tight focus, which typically is smaller. In this case, the focal length, the effective focal length of this system is very small.

So if you try to do this with a normal refractive lens with curved surfaces, you would need to have a really, really curved lens. And as we know from the demos and from the discussions, those type of lenses will be really aberrated. However, these, even though it has a shorter focal length, it still focuses into a very tight focus because it uses, as I said, the entire volume.

Now here, this is the example of coma. So if I actually show-- I can even focus a little bit into how the rays are coming. And we see the similar caustics that you saw in the demo and in the slides before. But if we put a screen here, this is the actual shape of the rays intersecting that plane. So these two are the circles of confusion.

And just to give you a preview, this is basically for the on-axis point. So this is what geometrical optics predict. The bottom one. So the geometrical optics predict just how rays intersect a given plane. It doesn't take into account diffraction that we're going to learn in the next part of the course. However, diffraction is very important in optical system because it regulates resolution. And many of you have been asking about numerical aperture and how that depends on resolution and what happens changing the aperture stop and so on and so forth.

Well, this is also very related to this point spread function that you can see here. And as you can see, it's a bright spot, and then there's concentric rings which is the way you can see the diffraction effect of that point. So this is geometrical version. This is the diffraction version or the more realistic version of how light would diffract. And in order to compute this, we'll basically need the math and the tools that we're going to learn in the second part of the course. But this is just giving you the type of science and analysis that we could do in this software.

So any questions before-- because we're going to break, right, George?

GEORGE BARBASTATHIS: Yeah.

PROFESSOR: Any questions before the break?

GEORGE BARBASTATHIS: I think it's time to continue. Before I move on, are there any questions about-- I don't know. About anything. About the homeworks, about the lectures, about imaging, about focusing?

A bit of a milestone, I guess. I don't believe in quizzes. Actually, I hate giving exams. But I have to produce grades. The register asks for grades. So the accepted, I guess, established way is with exams. But anyway. And I also hate reviews because of the same reason because a review suggests somehow that the quiz is important.

Of course, the quiz is not important except for the register. The only significance, perhaps, exams have is that they really force you to study. I mean, if you didn't have an exam, then you would say, well, maybe I'll study later. Maybe I will study in the summer. When I go out water skiing, I can study optics while I water ski.

But anyway. So I try to-- in my classes, I try to de-emphasize exams as much as possible. And the assumption is that if throughout the course of the class you have learned the material, then you will do well in the exam. Whereas to do the opposite, to try to satisfy the needs of the exam is very limited because then, what you get out of it? You're wasting your time.

So that's why I try to not mention exams. As far as I can help it, I try not to mention them. But I know-- I've been a student also. At the time I would also be very worried about exams. Anyway. So this is the time. Is there any sort of-- whatever. Review questions or-- OK. I'm going to count to from 3 down. 3, 2, how they do in auctions, 1.

You can still interrupt. Also, we have the class forum. I mean the online chat room or whatever it's called. The forum the call it. So of course, you can post questions there. I will be diligently checking.

Also, I sent an announcement by email. You may have not seen it yet. But we've posted some solved problems. Actually, the problems have been posted since Monday. So the problems are solved. So you don't really have to-- so this is the homework for next week. You don't have to actually turn in the homework on the quiz day. That would have been cruel. But we actually gave you some solved problems. So you can go over them.

And of course the recommendation is to really pretend that you don't have the solution. Try to solve it yourself. And then check it-- check your solution against ours. And of course, you don't have to turn in your solution. And I'll stop talking about the exam now, unless someone has a question now or later.

So Pepe already gave an introduction to the topic of gradient index or GRIN optics. And he mentioned that the motivation is basically that if you can focus the light by using a non-uniform material, a material whose index of refraction is variable, perhaps you can get away by also using a material whose surface is flat.

So you don't have to use this awkward spherical shape that we know from everyday optics. Maybe you can have something that looks like a slab or a plate. And nevertheless, this flat-looking plate might have the ability to focus light.

So the way you can do-- first of all, can you do this kind of thing? Can you make an element with variable index or the fraction. So there's two-- well, there's many ways. But in industry, in manufacturing practices, there's two ways that are being used. And I will describe them.

One is called ion exchange. And what they do is they take a glass rod, which is doped with a certain kind of ions-- typically, they use sodium in the glass rod. Then they have the glass meld at the very high temperature, of course, with a different kind of ion. Typically lithium. And what happens is they slowly dip the glass rod into the glass meld.

And over a period of several hours, what's happens is the lithium ions, they kick out some of the sodium ions from the glass rod. They basically penetrate the rod, and they kick out some of the sodium ions. But of course, the index of refraction depends on what kind of ions you have inside of the glass.

And also, because the ions, they propagate at a finite speed, what will happen here is in the exterior of the glass rod, you get a lot of ion exchange. So the index changes a lot. But near the center, if you get your timing right, near the center you don't get any exchange at all. So you basically have-- in the center you only have sodium ions. Near the exterior to give both sodium and lithium. So you can get a situation where the index of refraction is different in the center than it is at the edge.

And then what they do, of course, is after they finish this process, they chop this glass rod into slabs. And then each one of those slabs is to become a GRIN lens. And as I said before, they do it typically so that the index is higher in the center and lower at the edges. So this kind of thing I will do in detail later. But if you look at it in cross-section, it focuses like so if you illuminate it with parallel bundle of rays.

You can think of the index of refraction as an attraction of sorts for rays. So the ray that enters at the lower index, it gets attracted by the high index. It bends. And as a result, by the time the rays come out, they actually come out as a focusing bundle. So you can use this element to focus light. So just like a lens, but it has a flat surface.

And because the ionic change is actually diffusion-driven-- I don't want to go into the mathematics of diffusion. But it turns out that if you get the specific form of the index of refraction that you get in this case, it's quadratic or parabolic. So it is maximum in the center. And then it drops as the square-- as the radius square, where r is the radius in the slab times some coefficient alpha.

And I will show in a second that this coefficient alpha and the thickness of the lens, the product of the two actually specifies the focal length. So this looks like a funny formula. But if you think about it, alpha must have dimensions that are inverse distance square so that the product alpha times r squared is non-dimensional. And d is distance. So therefore, the units here are distance. So indeed, the quantity over there is the distance. It is the focal length.

So let's see now why this is the case. So I guess we've done it several times. So you must be tired of it by now. I just lost my-- oh, maybe I'll use the-- oh, thanks. I also must not forget this. OK. So in typical fashion, we will use Fermat's principle in order to calculate what is happening here.

So Fermat says that if you-- here is my element. And I have two rays that both started at infinity. So the rays share their origin, the starting point. And then my desire here, my intended outcome, is that these two rays meet because I want the element to focus the light.

So Fermat says that the optical path length here on-axis and the optical path length out here off-axis, they must be equal. Because the rays started at the same point, they end up at the same point, the optical path lengths must be equal.

So what is the optical path length? So this derivation here is very highly handwaving and very, very inaccurate. But bear with me. In the next half hour, I will tell you how this calculation can be done exactly. But for now, this is kind of an interesting way to think about GRIN, so we'll do it anyway. It's also in the book.

So you don't really have to copy my derivation here. You already have it in two places. One is in the book, and the other is the notes. So maybe you can just look at it and don't bother to write it down.

So let's compare the two rays. Let's compare the on-axis ray, which basically goes straight. This is the optical axis. And the off-axis ray, which as Pepe already mentioned, the off-axis ray will actually do something like this. It will bend. And then, of course, after it comes out in free space, it will go straight. We know that for sure. But inside the gradient index medium, the ray will bend. It will follow a continuous kind of trajectory.

This is also counterintuitive because in our experience light rays go straight. But such is life. Physics sometimes have unexpected outcomes. If you put a ray in a gradient index medium, indeed it will bend onto a curved trajectory.

So let's put some notation here. So we'll call this r. This is the radial direction. We'll call this d. This is the thickness of the GRIN. We'll call this f. This is the focal length. And for the sake of this derivation, let me actually redraw it. I will redraw it. Forgive me.

I just want to emphasize this caricature. So we'll actually neglect this distance. This space over there, I will pretend it's not there. And this is consistent with a paraxial approximation. Here is r, here is d again, and here is f. And that's the optical axis.

OK. So well, let's do, first of all, the optical path length for the on-axis ray. The optical path length equals the index times the distance integrated over the path of the ray. So this would equal the index at radius equal 0. That is the index over here at the center of the lens times the thickness of the lens plus the focal length.

So of course, outside the lens, the index we assume to be 1. You can have an imaginary lens, but let's not do that now. So this is where it comes from.

AUDIENCE: Excuse me.

GEORGE BARBASTATHIS: Yeah?

AUDIENCE: According to Fermat's principle, it says that the ray travels in such a way that it tries to minimize the path. So why does the ray goes from lower index to higher index? Means if it goes into higher index material, actually it is increasing its optical path length?

GEORGE BARBASTATHIS: Ah. But it will bend. That's what Snell says. It will bend so that the total path length actually decreases. Did I answer your question or did you have something else?

AUDIENCE: I think-- well, I don't know, but maybe the answer to your question is that although-- I think what he says is that it will actually travel through a longer optical path in the medium because it goes into the bigger index.

GEORGE BARBASTATHIS: Ah. OK. I see.

AUDIENCE: But I think that's counteracted by the extra that would be outside, isn't it, in the air, I think.

GEORGE BARBASTATHIS: Yeah. And you will see that in the derivation. Because it will have to go through this hypotenuse over here, it will actually-- yeah. So the overall path is actually minimized. And this is the only way it can be minimized.

OK. So lo and behold, then, let's do what you just said. So what about the ray that arrives at the higher elevation? So here I'm going to cheat big time. I'm going to pretend that all of this-- the ray bends, of course. But I'm going to pretend that throughout all this travel here, the ray sees the same index of refraction. This is a gross approximation, but I will do it.

So therefore, this is-- so n0, another name for it is-- it is also equal to n max. The optical part for n1 will be equal n max. 1 minus alpha r square over 2, which is the index of refraction at the elevated position. And then, as you pointed out, it will go through this increased distance, which is the hypotenuse of the triangle.

So it is really r squared plus f squared. So again, I pretended that this r and this r are the same. I want to emphasize that this is an approximation. I'm cheating. OK.

The next is actually easier from now on. All I'm going to do-- and the reason I'm going to do it is to remind you. I'm going to do the paraxial approximation, which basically says that r is much less than f. So that means that square root can be written as-- I pull the f outside. 1 plus r square over f square approximately equal f1 plus r square over 2f square. And this is actually equal to f plus r square over 2f.

So now let's write down Fermat's principle. Fermat says that OPL 0 must be equal to OPL 1. So OPL 0 was n max times d plus f. And OPL 1 was-- again, if you look at it from this equation, OPL 1 was this expression times d, of course. I forgot to put the d here. Times this distance. So it is equal to n max 1 minus alpha r square over 2 times d plus f plus-- what was it-- r square over 2f.

Now a number of things will cancel. This will eat this one. This will eat this one. And all that is left is that n max times alpha r square over 2d with a minus sign plus r square over 2f equals 0. And now we can see that the quadratic dependence for the index of refraction that we draw is convenient because the same quadratic came out of the paraxial approximation of the square root.

So I can just cancel the two quadratics. And the twos also cancel. And I get that f equals the inverse of all of that, 1 over n max times alpha times d. This is how we find the focal length of the GRIN.

So this appears a little bit mysterious what happened here. How did I pull this out of the hat? Well, what it really means-- the fact that this has a solution means that this element actually succeeds in focusing. What I mean is the following. Suppose that what came out of this equation was not the nice quadratic that canceled, but suppose I got something like this.

Suppose I got minus n max alpha r cube, for example, over 2 times d plus r square over 2f equals 0. Well, if this is what I had gotten, I'm out of luck. This is not a proper lens. So what I'm trying to say is suppose that I made the GRIN with a cubic dependence of the index of refraction, another quadratic.

Then this is the question I would have arrived instead. Now I cannot cancel r anymore. So it means that each ray goes to a different focal point. So therefore, this element does not focus. So this would not be a good lens.

Actually, the cubic dependence has some other interesting properties, but these are perhaps for a more advanced class. Nevertheless, the quadratic dependence is very convenient because as you can see, it drops out of here. It means that I have a unique focal length at which all the rays focus, at least in the paraxial approximation.

So that is the story of the quadratic GRIN, how it comes about, and how we derive its focal length. And I should also say that even though my derivation here was cheating, basically the way I cheated is I assumed that d, the thickness of the lens, is also very small. So it is very similar to the thin lens approximation. Nevertheless, this equation is actually true, even for thicker lenses. So that's kind of interesting.

OK. The second way people make GRIN optics is actually a combination of melding and grinding. And the way they do that results in a different profile that now the index of refraction varies along the longitudinal-- the axial dimension. So what does this mean? What they do first is they start with a stack of glasses of different indices over refraction.

So here and in the rest of this lecture, the sort of darkness indicates index of refraction. So dark gray means high index. Light gray means low index. So they stack them. Then they meld them, which basically means that they mold and meld.

So after they meld them, basically the index of refraction, the variation becomes kind of continuous because as they meld, the different indices will kind of diffuse into each. So you will get a quasi-continuous variational index. And the final thing they do is they actually polish. This stack that they've got with a continuous index variation, they polish it.

So now this thing actually has two properties. First of all, it is a lens in a traditional sense because it has a spherical surface, so it focuses by virtue of refraction. But in addition, it has an axial index profile. So here the motivation is actually a little bit different than what Pepe and I described earlier.

So clearly I did not end up with a slab-looking element. I still have a curvature. But the motivation is different. It's a little bit similar to-- in the previous lecture I think we mentioned something called the Smith correction or the Smith telescope. So it's a very similar motivation.

If you take a uniform index sphere, we saw this a couple of times that in general, this produces an aberrated image, even if it is oriented properly. This is supposed to be a plane or convex lens oriented in the correct way. That is the plane wave goes with the spherical surface. But still, this does not completely cancel the aberration. There is still some residue.

So what do you do then, if you modify the index of refraction in the interior of the surface, what happens is this ray-- let's take the paraxial ray over here. It hits at a relatively high index, whereas this ray, it hits at a relatively lower index.

What this means, if you want to look at it from the geometrical optics point of view-- there's also a wave optics point of view that is interesting, but from the geometrical optics point of view, it means that the Snell's law of refraction here will be milder for this ray than for this ray.

Therefore, this ray, who's used to focus kind of violently and produce a spherical aberration here, now it will be pulled outwards. And therefore, it will be forced to go through the proper focal point over here as the paraxial lens. So this is a clever way to fix spherical aberration.

And I don't know who invented this first. I believe-- and maybe, Colin, you again correct with this. I believe in was Duncan Moore at the Rochester who came up with this particular-- at least I've seen a paper of his which is as old as I am, 1971, that discusses this optimization.

So in order to solve the basic principles of GRINs, I would like to spend a little bit of time-- I don't think we'll finish today. We'll probably have to leak into next Wednesday after the quiz. But I'll get us started on, how do we properly compute things like index? I mean, like focal length for a gradient index structure.

So we'd like to ask the question more generally. If I have a medium whose index of refraction varies as a function of position. So here, this cloud is meant to indicate a variable index. So you have high index, low index, in different positions.

So this is like a generalized GRIN if you wish. We saw two types. You saw quadratic and you saw axial. But it could be really-- in principle, I should be able to fabricate something like this. Not that it is readily doable, but let's consider it for the sake of value.

So you've seen this diagram before. In fact, I copied this from an earlier slide. One of the previous-- I think it was in the first lecture or second lecture. So we don't know really what path a ray will follow through these variable index medium, but we know one thing. We know that if you integrate this quantity, if you integrate the index of refraction times the length, the elemental length along the path of the ray, we know that Fermat requires that these quantities will be minimized.

So how do we deal with that? I mean, what can we do about this? Now, some of you who have taken advanced mechanics classes, this sounds a little bit familiar. What is the name that comes to mind when you see this kind of expression?

So let me repeat again. I have a situation where something-- in this case, a light ray, but it could be something else, a particle or some mechanical analogy-- can follow a number of different trajectories. It can go this way or it can go this way or this way and so on.

To give you another sort of trivial example, if I drop my marker here, it goes down straight. So you may have wondered, well, why does it go down straight? Why does it not do something like this or like this or whatever? Why does it not go up, for that matter?

Well, it goes straight because-- well, because gravity pulls it down right. Well, we're very happy with that. We know from experience that things fall straight. But we also know that gravity does not always produce straight trajectories. For example, the Earth. Thankfully, the Earth follows a circular, or actually elliptical, trajectory around the sun. And that's very fortunate because if the Earth had a way to escape from the sun, then we would all be-- we would not even exist.

So we know that gravity can produce straight trajectories. It can produce elliptical trajectories, hyperbolical trajectories, and so on and so forth. So there's more to it than the straight falling of the apple in Newton's case or of the marker in my case.

So anyway, so the reason I'm giving all this preamble is because this may be familiar for you from the field of mechanics. And the buzzword that comes to mind when you have this kind of integral with something inside and you demand that the integral itself is maximized is Lagrangian. It is the Lagrangian principle that you may have seen.

I don't think those of you took 2003 saw it. Did you or did you not, actually? I don't know. Is anybody in the class who took 2003? You should because 2004 is a prerequisite. And as a principle, if you are here, it means you have taken both 2003 and 2004.

We have lost our undergraduates, or they lost their voice, or they're too sleepy. Anyway. Or there's another possibility, that we don't remember if we were taught Lagrangians in 2003. Anyway. But I will not do it this way.

So it turns out I can solve this problem using a Lagrangian formulation. But I will not do it this way. Did someone want to speak? I saw the camera moving. Was someone about to speak?

AUDIENCE: I was just going to say, yes, we did have Lagrangian.

GEORGE BARBASTATHIS: Ah, you did. OK. Thank you. Thank you. Well, even if you did, I guess I will not do it with the Lagrangians because, well, the math is too complicated and would kind of cause a detour here.

Nevertheless, I will use a mechanical analogy. So for a moment, we'll take a break from optics. And I guess to justify why 2004 is a prerequisite for this class, I will show this. And I will spend, I guess, the next 15 or so minutes to establish a connection between this mechanical system and light rays.

For now, let's forget about light rays. I've given you-- it's actually a particle. So it looks like a car but it's really small. So I don't have to worry about moments of inertia and all of that stuff. It is also frictionless, so I don't have any dissipation at all in the system. This particle is attached to a rigid-- to an immobile, rigid wall with a spring.

And basically what I can do is I can inject some energy into the particles by pulling it. So if I do that, then we know that this system has two forms of energy. When I actually pull it, what I do is I store energy in the spring. The spring is not happy having its energy stored. It will want to give it away.

What does it mean to give away the energy? It means to convert it to kinetic. It means the spring will start pulling the particle back. As the particle is pulled back, it will accelerate. It will reach a maximum velocity-- we know that from classical mechanics-- approximately where the spring has zero displacement. And at this point, all of the energy has been converted to kinetic.

And of course it will not stop there because it is moving, and there's nothing to stop it. It will keep moving. But if it moves, now it is beginning to push, to squeeze this spring. So now the particle is giving back its energy into the spring until all of the energy has been given. Now the spring is squeezed. Doesn't like to be squeezed. It starts giving energy back to the particle.

And of course, on and on they go. This will execute and oscillate in motion forever because I neglected all dissipation. And of course, what is happening in this case is that at any given moment, the particle will have a mix-- the system will have a mix of kinetic energy and energy stored into the spring, which is referred to as potential energy.

So of course, the kinetic energy is given by 1/2 times the mass times the velocity squared. If you remember the definition of momentum, momentum equals mass times velocity. So I can also write it like this. Momentum squared over 2 times the mass. And the potential energy-- actually, the potential energy can have very different forms depending on what is exactly the energy storage element.

In the case of a spring-- actually, in the case of a hook spring, a linear spring, it is simply equal to the square of the displacement of the spring with respect to its rest position. So anyway, no matter what these things are, the sum must be constant because, again, energy cannot be gained or lost by its system. So the sum of the potential and kinetic energy is always conserved.

So that is one thing, and it's very good. Now, let me play a game. So first of all, I will write the energy. I will basically repeat what is on the slide. I will repeat it on the whiteboard here. So let me write the energy.

I will write it in a funny way. I will write it as a function of the momentum p and the position q. And now, in the previous slide, these were actually scalar quantities because this thing moves only along a straight line. But in general, of course, I can have positions and momenta that are vectors. So I switched to vector notation. I hope that doesn't confuse you horribly.

So let me write it down. So in the vector case, this will equal the momentum vector divided by 2m plus 1/2 m-- wait a minute. 1/2 k, the spring constant, times the position vector square.

Now what I'm going to do, for fun, is I'm going to take derivatives of this quantity with respect to the position and the momentum. Of course, since these became vectors, now these are not exactly derivatives, they are gradients. So basically what I will do is I will compute the gradient of H with respect to p.

This is a matter of notation. I will denote it as dH dp. So what is dH dp? Well, what I have to do is I have to compute-- clearly this term does not play because it only depends on q. But this term contains the momentum. So if I do it, it will actually-- you can do it. It is actually p times its-- p over m. That's it. Nothing else.

And then I'm going to do the same with respect to q. And I'm going to call it dH dq. Again, it is a gradient with respect to q. And for the same reasons as the other one, it will turn out to be k times 2.

OK. Then again for fun-- I'm not justifying what I'm doing. It's just coming sort of some-- coming out of some sort of epiphany. I will write down two differential equations. One of them, I will write this one. It is already on your screen there. dq dt. What is that? That's the time derivative of the displacement. Therefore, it is the velocity.

So we'll take it to be equal to the gradient of the quantity H of the energy, that we called it H for some strange reason, with respect to the opposite variable. This was the derivative of q. I will take its gradient with respect to p.

And let's see what I get. This way I get p over m. That's correct. Actually, this equation is not particularly insightful. This is the definition of momentum, isn't it? If you recall-- this means recall, like in cartoons. The little cloud. The momentum is defined as m times dq dt. That's the definition of momentum.

So by doing this trick, by taking the gradient of the energy with respect to the momentum and setting it equal to the velocity, I got a true result. I got a result that happens to be true.

Now let's see if I can pull the opposite trick. If I can take the derivative with respect to time of the momentum, that's like an acceleration quantity because momentum is velocity. So its derivative is acceleration. And let's see if I can get away by putting it equal to the derivative of this quantity H with respect to the other variable.

In this case, I cannot quite get away with it. I have to put a minus sign. And what do I get if I put a minus sign? kq. What's this now? kq? It is k times displacement. So what are the units of Hooke's constant time displacement? Force. What is on the left-hand side?

OK. I said acceleration. Actually, I should have corrected myself, and I should have said, proportional to acceleration. This is the derivative of the momentum. So therefore, it is force. So what is this then, the equation that I got? It is the force balance on the-- it is Newton's law, actually, on the particle.

It says that the force pulling the particle, it is the restoring force due to the spring. And what are the two equations combined? Actually, the two equations combined, they are what we call the equations of motion. The equations of motion, by definition, are first order differential equations that link the dynamical variables that is the position, the displacement, and the velocity.

So this is what I did. I got a set of differential equations that link the displacement to the momentum-- that is the velocity. And again, the derivative of the displacement back to the-- I'm sorry, the derivative of the velocity, the derivative of the momentum, back to the displacement itself.

So if you look at it, if you omit this middle part, if you look at it like this, these are simply-- as I said, these are the equations of motion. If you look at this part over here, this is of course a more general result. That's why I bothered to do it. And this is where the connection to optics will come from next week. This set of equations is called Hamiltonian.

And I will put in brackets "canonical." I don't particularly like this term. But anyway, some people use it. Hamiltonian canonical equations. So the point, then, which I will say just this, and then I will stop and continue next time, the point is that this is now a much more general property. Even though I showed that it is true for the specific case of a mass spring system without damper-- I showed that to be true. But this is a more general property for any dynamical system that has a conserved quantity.

So in the case of the mechanical system, the conserved quantity is energy. If this conserved quantity can be written as a function of the dynamical variables, in this case, of the mechanical system position and momentum-- OK. So I started a long sentence. Let me close my sentence.

So if a dynamical system has a conserved quantity, then I can write down a set of Hamiltonian equations for this dynamical system in terms of the gradients of this conserved quantity. And that will be the equations of motion.

OK. So what we will do then next time is we will actually show, in almost one to one analogy, that-- oh, and by the way, this conserved quantity that we called it energy in the case of the mechanical system, in general it's called the Hamiltonian. So H itself is the Hamiltonian.

So what we'll do next is we will actually show we can derive a Hamiltonian for rays for an optical system. And of course, when we write equations of motion for rays, we'll get the ray trajectories. Therefore, we will be able to ray tracing.

And the reason this is interesting because the potential actually will turn out to be the index of refraction. And in the same way, the potential here, if you go back to my equation-- the potential turned out to be a function of position. In the gradient index optics that we mentioned just now, the index of refraction is a function of position.

So we will see that in the case of the GRIN element, the index of refraction actually has the role of a potential. And it is this potential that draws the-- very similar to gravity. The potential that draws the sun-- I mean, that draws the Earth to rotate around the sun and not the other way around. The same way the potential draws the Earth to rotate around the sun, it is a similar, or I should say analogous, kind of potential that draws rays inwards in a GRIN element and causes them to focus.

So we'll pick up on that next time. And obviously, don't worry about this for the quiz. So I'm not crazy. So yeah. So we'll properly conclude this topic next time. It will be next week's homework.