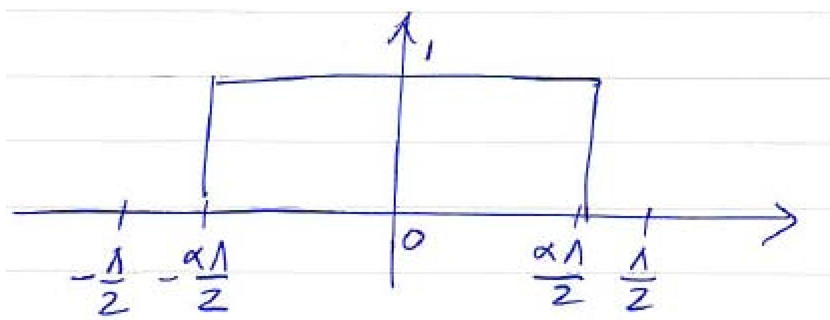
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering
2.71/2.710 Optics
Spring 2009

Supplement to Lecture 19A
Binary amplitude grating, arbitrary duty cycle α



$$g_t(x) = \sum_{q=-\infty}^{\infty} C_q e^{i2\pi q \frac{x}{\Lambda}}$$

where

$$C_q = \frac{1}{\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} g_t(x) e^{-i2\pi q \frac{x}{\Lambda}} dx = \frac{1}{\Lambda}\int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{\alpha\Lambda}\right) e^{-i2\pi q \frac{x}{\Lambda}} dx$$
$$= \frac{1}{\Lambda}\cdot \alpha\Lambda \text{sinc}\left(\alpha\Lambda \cdot \frac{q}{\Lambda}\right) = \alpha \text{sinc}(\alpha q) = \frac{\sin(\pi\alpha q)}{\pi q}$$

$$g_t(x) = \sum_{q=-\infty}^{\infty} \alpha \text{sinc}(\alpha q) e^{i2\pi q \frac{x}{\Lambda}} = \alpha + 2\sum_{q=1}^{\infty} \frac{\sin(\pi\alpha q)}{\pi q} \cos\left(2\pi q \frac{x}{\Lambda}\right)$$

$$\eta_q = \frac{\sin^2(\pi\alpha q)}{\pi^2 q^2} = \alpha^2 \text{sinc}^2(\alpha q)$$

$$\left.\begin{array}{l} \lambda = 0.5\ \mu\text{m} \\ f = 20\ \text{cm} \\ \Lambda = 10\ \mu\text{m} \end{array}\right\} \rightarrow \frac{\lambda f}{\Lambda} = \frac{0.5\times 20}{10}\ \text{cm} = 1\ \text{cm}$$

$$\text{sinc}\left(\frac{q}{3}\right) = \frac{\sin(\frac{\pi q}{3})}{\frac{\pi q}{3}} \quad \rightarrow \quad \frac{1}{3}\text{sinc}\left(\frac{q}{3}\right) = \frac{1}{\pi q}\sin\left(\frac{\pi q}{3}\right)$$

q	C_q
0	$\frac{1}{3}\text{sinc}(0) = \frac{1}{3}$
1	$\frac{1}{\pi}\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2\pi}$
2	$\frac{1}{2\pi}\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2\pi}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4\pi}$
3	$\frac{1}{3\pi}\sin\left(\frac{3\pi}{3}\right) = 0$

$$g_t(x) = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}(\alpha q) e^{i2\pi q \frac{x}{\Lambda}}$$
$$= \frac{1}{3} + \frac{\sqrt{3}}{2\pi} e^{i2\pi \frac{x}{\Lambda}} + \frac{\sqrt{3}}{2\pi} e^{-i2\pi \frac{x}{\Lambda}} + \frac{\sqrt{3}}{4\pi} e^{i2\pi \frac{2x}{\Lambda}} + \frac{\sqrt{3}}{4\pi} e^{-i2\pi \frac{2x}{\Lambda}} + \cdots$$

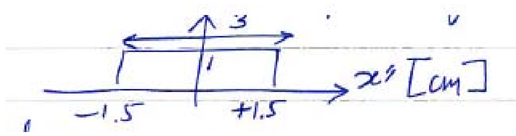
$$G_t(u) = \frac{1}{3}\delta(u) + \frac{\sqrt{3}}{2\pi}\delta(u - \frac{1}{\Lambda}) + \frac{\sqrt{3}}{2\pi}\delta(u + \frac{1}{\Lambda}) + \frac{\sqrt{3}}{4\pi}\delta(u - \frac{2}{\Lambda}) + \frac{\sqrt{3}}{4\pi}\delta(u + \frac{2}{\Lambda}) + \cdots$$

Field to the left of pupil plane:

$$g_{\text{pp-}}(x'') = G_t\left(\frac{x''}{\lambda f}\right) \text{ where } \lambda = 0.5\ \mu\text{m},\ \Lambda = 10\ \mu\text{m},\ \text{f} = 20\ \text{cm}$$

$$\rightarrow g_{\text{pp-}}(x'') = \frac{1}{3}\delta(x'') + \frac{\sqrt{3}}{2\pi}\delta(x'' - 1\ \text{cm}) + \frac{\sqrt{3}}{2\pi}\delta(x'' + 1\ \text{cm})$$
$$+ \frac{\sqrt{3}}{4\pi}\delta(x'' - 2\ \text{cm}) + \frac{\sqrt{3}}{4\pi}(x'' + 2\ \text{cm}) + \cdots$$

1. Low-pass filter: orders 0, ± 1 are passing



$$g_{\text{pm}}(x'') = \text{rect}\left(\frac{x''}{3\ \text{cm}}\right)$$

Field to the right of pupil plane:

$$g_{\text{pp+}}(x'') = g_{\text{pp-}}(x'') \times g_{\text{pm}}(x'') = \frac{1}{3}\delta(x'') + \frac{\sqrt{3}}{2\pi}\delta(x'' - 1\ \text{cm}) + \frac{\sqrt{3}}{2\pi}\delta(x'' + 1\ \text{cm})$$

$$G_{\text{pp+}}(u) = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} e^{i2\pi u \cdot 1\ \text{cm}} + \frac{\sqrt{3}}{2\pi} e^{-2\pi u \cdot 1\ \text{cm}}$$

Field at output plane:

$$g_{\text{out}}(x') = G_{\text{pp+}} \overbrace{\left(\frac{x'}{\lambda f}\right)}^{u = \frac{x'}{\lambda f}} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} e^{i2\pi \frac{x'}{10\ \mu\text{m}}} + \frac{\sqrt{3}}{2\pi} e^{-i2\pi \frac{x'}{10\ \mu\text{m}}} = \frac{1}{3} + \frac{\sqrt{3}}{\pi} \cos\left(2\pi \frac{x'}{10\ \mu\text{m}}\right)$$

Intensity at output plane:

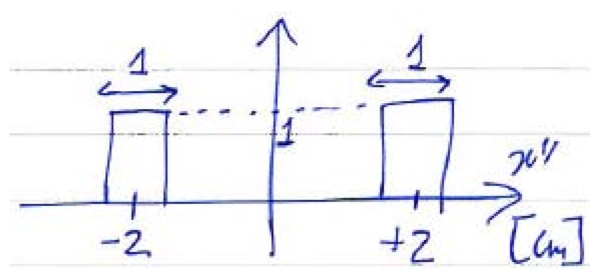
$$I_{\text{out}}(x') = |g_{\text{out}}(x')|^2 = \left(\frac{1}{3}\right)^2 + \frac{3}{\pi^2}\cos^2\left(2\pi \frac{x'}{10\ \mu\text{m}}\right) + \frac{2}{\sqrt{3}\pi}\cos\left(2\pi \frac{x'}{10\ \mu\text{m}}\right)$$
$$= \left(\frac{1}{3}\right)^2 + \frac{3}{2\pi^2} + \frac{2}{\sqrt{3}\pi}\cos\left(2\pi \frac{x'}{10\ \mu\text{m}}\right) + \frac{3}{2\pi^2}\cos\left(2\pi \frac{2x'}{10\ \mu\text{m}}\right)$$

2. Round-pass filter: orders ±1 are passing

$$
\begin{aligned}
g_{\text{pm}}(x'') &= \text{rect}\left(\frac{x''+1\text{ cm}}{1\text{ cm}}\right) + \text{rect}\left(\frac{x''-1\text{ cm}}{1\text{ cm}}\right) \\
g_{\text{pp+}}(x'') &= g_{\text{pp-}}(x'')g_{\text{pm}}(x'') \\
&= \frac{\sqrt{3}}{2\pi}\delta(x''-1) + \frac{\sqrt{3}}{2\pi}\delta(x''+1) \\
G_{\text{pp+}}(u) &= \frac{\sqrt{3}}{2\pi}e^{i2\pi u} + \frac{\sqrt{3}}{2\pi}e^{-i2\pi u} \\
g_{\text{out}}(x') &= \frac{\sqrt{3}}{2\pi}e^{i2\pi\frac{x''}{10\ \mu\text{m}}} + \frac{\sqrt{3}}{2\pi}e^{-i2\pi\frac{x''}{10\ \mu\text{m}}} \\
&= \frac{\sqrt{3}}{\pi}\cos\left(2\pi\frac{x''}{10\ \mu\text{m}}\right) \\
I_{\text{out}}(x') &= |g_{\text{out}}(x')|^2 = \frac{3}{\pi^2}\cos^2\left(2\pi\frac{x''}{10\ \mu\text{m}}\right) \\
&= \frac{3}{2\pi^2}\left[1+\cos\left(2\pi\frac{2x''}{10\ \mu\text{m}}\right)\right] = \frac{3}{2\pi^2} + \frac{3}{2\pi^2}\cos\left(2\pi\frac{x''}{5\ \mu\text{m}}\right)
\end{aligned}
$$

Compare with the previous example.

3. Band-pass filter: orders ±2 are passing



$$
\begin{aligned}
g_{\text{pm}}(x'') &= \text{rect}\left(\frac{x''+2\text{cm}}{1\text{cm}}\right) + \text{rect}\left(\frac{x''-2\text{cm}}{1\text{cm}}\right) \\
g_{pp+}(x'') &= \frac{1}{4\pi}\delta(x-2) + \frac{1}{4\pi}\delta(x+2) \\
G_{pp+}(u) &= \frac{\sqrt{3}}{4\pi}e^{+i2\pi 2u} + \frac{\sqrt{3}}{4\pi}e^{-i2\pi 2u} = \frac{\sqrt{3}}{2\pi}\cos(2\pi 2u) \\
g_{\text{out}}(x') &= \frac{\sqrt{3}}{2\pi}\cos\left(2\pi\frac{2x'}{10\mu\text{m}}\right) = \frac{\sqrt{3}}{2\pi}\cos\left(2\pi\frac{x'}{5\mu\text{m}}\right) \\
I_{\text{out}}(x') &= |g_{\text{out}}(x')|^2 = \frac{3}{4\pi^2}\cos^2\left(2\pi\frac{x'}{5\mu\text{m}}\right) = \frac{3}{4\pi^2} + \frac{3}{4\pi^2}\cos\left(2\pi\frac{x'}{2.5\mu\text{m}}\right)
\end{aligned}
$$

Compare with the previous examples.

4. Phase pupil mask

$$
g_{\text{pm}}(x'') = \begin{cases} 0 & \text{if} \quad |x''| > 1.5\text{cm} \\ +1 & \text{if} \quad 1.5\text{cm} < |x''| < 0.5\text{cm} \\ e^{i\phi} & \text{if} \quad |x''| < 0.5\text{cm} \end{cases}
$$

where $e^{i\phi} = e^{i\frac{\pi}{2}} = i$.

$$g_{\text{pm}}(x'') = \text{rect}\left(\frac{x''}{3\text{cm}}\right) + (i-1)\text{rect}\left(\frac{x''}{1\text{cm}}\right)$$

Field before pupil plane

$$g_{pp-}(x'') = \frac{1}{3}\delta(0) + \frac{\sqrt{3}}{2\pi}\delta(x''-1\text{cm}) + \frac{\sqrt{3}}{2\pi}\delta(x''+1\text{cm}) + \frac{\sqrt{3}}{4\pi}\delta(x''-2\text{cm}) + \frac{\sqrt{3}}{4\pi}\delta(x''+2\text{cm}) + \cdots$$

Field after pupil plane

$$\begin{aligned}
g_{pp+}(x'') &= g_{pp-}(x'') \times g_{\text{pm}}(x'') \\
&= \frac{i}{3}\delta(0) + \frac{\sqrt{3}}{2\pi}\delta(x''-1\text{cm}) + \frac{\sqrt{3}}{2\pi}\delta(x''+1\text{cm}) \\
G_{pp+}(u) &= \frac{i}{3} + \frac{\sqrt{3}}{2\pi}e^{i2\pi u\cdot 1\text{cm}} + \frac{\sqrt{3}}{2\pi}e^{-i2\pi u\cdot 1\text{cm}} \\
&= \frac{i}{3} + \frac{\sqrt{3}}{\pi}\cos\left(2\pi u\cdot 1\text{cm}\right) \\
g_{\text{out}}(x') &= \frac{i}{3} + \frac{\sqrt{3}}{\pi}\overbrace{\cos(2\pi\frac{x'}{10\mu\text{m}})}^{u=x'/\lambda f_2} && \text{Field at output plane} \\
I_{\text{out}}(x') &= |g_{\text{out}}(x')|^2 = \frac{1}{9} + \frac{3}{\pi^2}\cos^2(2\pi\frac{x'}{10\mu\text{m}}) && \text{Intensity at output plane} \\
&= \frac{1}{9} + \frac{3}{2\pi^2} + \frac{3}{2\pi^2}\cos(2\pi\frac{2x'}{10\mu\text{m}})
\end{aligned}$$

Compare this result with example 1.

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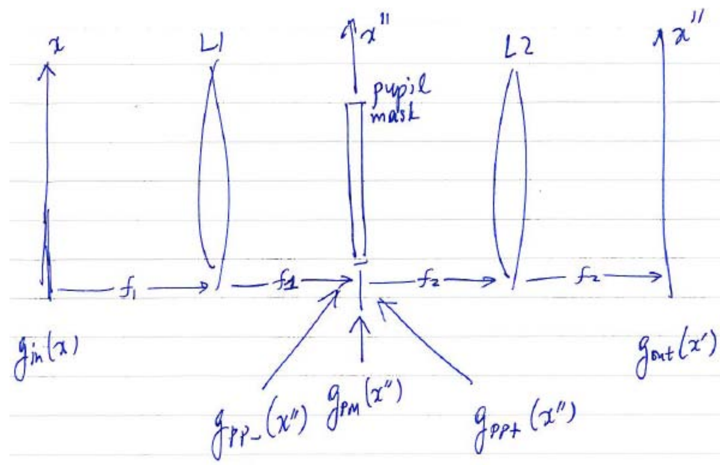
Supplement to Lecture 19B
Derivation of 4F system PSF and ATF



$$g_{pp-}(x'',y'') = \frac{e^{i2\pi 2f_1/\lambda}}{i\lambda f_1} G_{\rm in}\left(\frac{x''}{\lambda f_1}, \frac{y''}{\lambda f_1}\right) \qquad \text{[Fourier transform property of } L_1\text{]}$$

$$g_{pp+}(x'',y'') = g_{pp-}(x'',y'') \times g_{\rm pm}(x'',y'') = \frac{e^{i2\pi 2f_1/\lambda}}{i\lambda f_1} G_{\rm in}\left(\frac{x''}{\lambda f_1}, \frac{y''}{\lambda f_2}\right) g_{\rm pm}(x'',y'')$$

$$g_{\rm out}(x',y') = \frac{e^{i2\pi 2f_2/\lambda}}{i\lambda f_2} G_{pp+}\left(\frac{x'}{\lambda f_2}, \frac{y'}{\lambda f_2}\right)$$

$$= \frac{e^{i2\pi 2(f_1+f_2)/\lambda}}{-\lambda^2 f_1 f_2} \mathcal{F}\left[G_{\rm in}\left(\frac{x''}{\lambda f_1}, \frac{y''}{\lambda f_1}\right)\right]_{(x'/(\lambda f_2),\ y'/(\lambda f_2))} * \mathcal{F}[g_{\rm pm}(x'',y'')]_{(x'/(\lambda f_2),\ y'/(\lambda f_2))}$$

$$\mathcal{F}[G_{\rm in}(u,v)] = g_{\rm in}(-x',y') \quad \overset{\substack{\text{scaling}\\ \text{(similarity)}\\ \text{theorem}}}{\Longrightarrow} \quad \mathcal{F}\left[G_{\rm in}(\frac{x''}{\lambda f_1}, \frac{y''}{\lambda f_1})\right]_{(u,v)}$$

$$= (\lambda f_1)^2 g_{\rm in}(-\lambda f_1 u, -\lambda f_1 v) \Rightarrow \mathcal{F}\left[G_{\rm in}(\frac{x''}{\lambda f_1}, \frac{y''}{\lambda f_1})\right]_{(\frac{x'}{\lambda f_2}, \frac{y'}{\lambda f_2})}$$

$$= (\lambda f_1)^2 g_{\rm in}\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right)$$

$$g_{\rm out}(x',y') = \frac{e^{i2\pi 2(f_1+f_2)/\lambda}}{-\lambda^2 f_1 f_2} (\lambda f_1)^2 g_{\rm in}\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right) * G_{\rm PM}\left(\frac{x'}{\lambda f_2}, \frac{y'}{\lambda f_2}\right)$$

$$= e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_1}{f_2}\right) \iint g_{\rm in}\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right)\ G_{\rm PM}\left(\frac{x'-x_1'}{\lambda f_2}, \frac{y'-y_1'}{\lambda f_2}\right) dx'dy'$$

Set $x_0 = -\frac{f_1}{f_2}x_1', y_0 = -\frac{f_1}{f_2}y_1' \Rightarrow \begin{cases} x_1' = -\frac{f_2}{f_1}x_0 & y_1' = -\frac{f_2}{f_1}y_0 \\ dx_1' = (-\frac{f_2}{f_1})dx_0 & dy_1' = (-\frac{f_2}{f_1})dy_0 \end{cases}$

$$g_{\rm out}(x',y') = -e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_1}{f_2}\right)\iint g_{\rm in}(x_0,y_0)G_{\rm PM}\left(\frac{x'+\frac{f_2}{f_1}x_0}{\lambda f_2},\frac{y'+\frac{f_2}{f_1}y_0}{\lambda f_2}\right)\left(\frac{f_2}{f_1}\right)^2 dx_0dy_0$$

$$= e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)\iint g_{\rm in}(x,y)G_{\rm PM}\left(\frac{x'+\frac{f_2}{f_1}x}{\lambda f_2},\frac{y'+\frac{f_2}{f_1}y}{\lambda f_2}\right)dx\ dy$$

In the last line, we have dropped the 0 subscript from the dummy variables.

Special case: $g_{\rm in}(x,y) = \delta(x-x_1, y-y_1)$, then

$$g_{\rm out}(x',y') = -e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)\iint \delta(x-x_1,y-y_1)G_{\rm PM}\left(\frac{x'+\frac{f_2}{f_1}x}{\lambda f_2},\frac{y'+\frac{f_2}{f_1}y}{\lambda f_2}\right)dx\ dy$$

$$= -e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)G_{\rm PM}\left(\frac{x'+\frac{f_2}{f_1}x_1}{\lambda f_2},\frac{y'+\frac{f_2}{f_1}y_1}{\lambda f_2}\right)$$

If $(x_1,y_1) = (0,0)$ then $g_{\rm out}(x',y') \equiv h(x',y')$

$$= -e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)G_{\rm PM}\left(\frac{x'}{\lambda f_2},\frac{y'}{\lambda f_2}\right)$$

More generally, the PSF is ($a \equiv -\frac{f_2}{f_1}e^{i2\pi 2(f_1+f_2)/\lambda}$):

$$h(x',y',x,y) = -\ e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)G_{\rm PM}\left(\frac{x'+\frac{f_2}{f_1}x}{\lambda f_2},\frac{y'+\frac{f_2}{f_1}y}{\lambda f_2}\right)$$

$$g_{\rm out}(x',y') = -\ e^{i2\pi 2(f_1+f_2)/\lambda}\left(\frac{f_2}{f_1}\right)\iint dx\ dy\ \iint G_{\rm in}(u,v)e^{i2\pi(ux+vy)}du\ dv$$

$$\times \iint g_{\rm PM}(x'',y'')e^{-i2\pi[\frac{x'+\frac{f_2}{f_1}x}{\lambda f_2}x''+\frac{y'+\frac{f_2}{f_1}y}{\lambda f_2}y'']}dx''dy''$$

$$= a\iint du\ dv\ \iint dx''dy''\iint dx\ dy\ G_{\rm in}(u,v)g_{\rm PM}(x'',y'')$$

$$\times\ e^{i2\pi[(u-\frac{x''}{\lambda f_2})x+(v-\frac{y''}{\lambda f_1})y]}e^{-i2\pi(\frac{x'x''}{\lambda f_2},\frac{y'y''}{\lambda f_2})}$$

$$= a\iint du\ dv\ \iint dx''dy''G_{\rm in}(u,v)g_{\rm PM}(x'',y'')$$

$$\times\ \hat{o}\left(u-\frac{x''}{\lambda f_1}, v-\frac{y''}{\lambda f_1}\right)e^{-i2\pi(x'x''+y'y'')/(\lambda f_2)}$$

$$= a\iint du\ dv\ G_{\rm in}(u,v)g_{\rm PM}(\lambda f_1 u,\lambda f_1 v)e^{+i2\pi[u(-\frac{f_1}{f_2}x')+v(-\frac{f_1}{f_2}y')]}$$

In the "reduced" coordinates ${x_0}' = -\frac{f_1}{f_2}x'$, ${y_0}' = -\frac{f_1}{f_2}y'$ there is no inversion and the magnification equals one. The output field, resampled on the reduced coordinates, is:

$$g^0_{\rm out}({x_0}',{y_0}') = a\iint du\ dv\ G_{\rm in}(u,v)g_{\rm PM}(\lambda f_1 u,\lambda f_1 v)e^{i2\pi(u{x_0}'+v{y_0}')}$$

By definition of the Fourier integral,

$$g^0_{\rm out}({x_0}',{y_0}') = \iint du\ dv\ G^0_{\rm out}(u,v)e^{i2\pi(ux_0'+vy_0'')}$$

$$\Rightarrow G^0_{\rm out}(u,v) = aG_{\rm in}(u,v)g_{\rm PM}(\lambda f_1 u,\lambda f_1 v) \equiv G_{\rm in}(u,v)H^0(u,v)$$

where $H^0(u,v) \equiv a g_{\mathrm{PM}}(\lambda f_1 u, \lambda f_1 v)$ is the Amplitude Transfer Function.

To relate to the original (unreduced) coordinates we use the scaling theorem:

$$\begin{aligned}
G_{\mathrm{out}}(u,v) &= \left(\frac{f_2}{f_1}\right)^2 G^0_{\mathrm{out}}\left(-\frac{f_2}{f_1}u, -\frac{f_2}{f_1}v\right) \\
&= \left(\frac{f_2}{f_1}\right)^2 a G_{\mathrm{in}}\left(-\frac{f_2}{f_1}u, -\frac{f_2}{f_1}v\right) g_{\mathrm{PM}}(-\lambda f_2 u, -\lambda f_2 v) \\
&= \left(\frac{f_2}{f_1}\right)^2 G_{\mathrm{in}}\left(-\frac{f_2}{f_1}u, -\frac{f_2}{f_1}v\right) H^0\left(-\frac{f_2}{f_1}u, -\frac{f_2}{f_1}v\right)
\end{aligned}$$

The ATF of optical systems by convention is quoted with respect to the input (equivalently, the unreduced output) coordinates, and omitting the scaling factors, as:

$$H(u,v) = g_{\mathrm{PM}}(\lambda f_1 u, \lambda f_1 v)$$