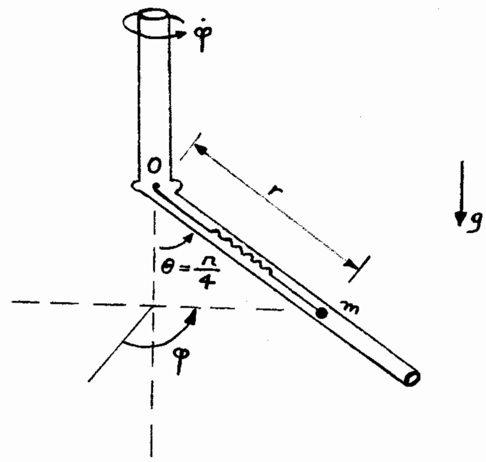
2.032 Dynamics

Fall 2004

Problem Set No. 10

Problem 1



(a)

$\#$ DOF = 2

$q_1 = r, \quad q_2 = \varphi$

$$T = \frac{1}{2} m v_m^2$$

$$\underset{\sim}{v}_m = \dot{r}\, \underset{\sim}{e}_r + r \sin\theta\, \dot{\varphi}\, \underset{\sim}{e}_\varphi = \dot{r}\, \underset{\sim}{e}_r + \frac{\sqrt{2}}{2} r \dot{\varphi}\, \underset{\sim}{e}_\varphi$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + \frac{1}{2} r^2 \dot{\varphi}^2 \right)$$

$$V = -mgr\cos\theta + \frac{1}{2} k (r - r_o)^2 = -mgr\frac{\sqrt{2}}{2} + \frac{1}{2} k (r - r_o)^2$$

$L = T - V \qquad\qquad Q_1 = Q_2 = 0$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) - \dfrac{\partial L}{\partial r} = 0 \longrightarrow m\ddot{r} - \dfrac{1}{2} m r \dot{\varphi}^2 - mg\dfrac{\sqrt{2}}{2} + k(r - r_o) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\varphi}}\right) - \cancelto{0}{\dfrac{\partial L}{\partial \varphi}} = 0 \longrightarrow \dfrac{\partial L}{\partial \dot{\varphi}} = \text{Const.} = p_\varphi \longrightarrow \dfrac{1}{2} m r^2 \dot{\varphi} = p_\varphi \end{cases}$$

$$\longrightarrow \dot{\varphi} = \frac{2 p_\varphi}{m r^2}$$

$$\Longrightarrow \quad \underline{m\ddot{r} - \frac{2 p_\varphi^2}{m r^3} - mg\frac{\sqrt{2}}{2} + k(r - r_o) = 0}$$

Single equation of motion for r.

(b) $L = T - V = \dfrac{1}{2} m \dot{r}^2 + \dfrac{p_\varphi^2}{m r^2} + mgr\dfrac{\sqrt{2}}{2} - \dfrac{1}{2} k (r - r_o)^2$ for the reduced system

at equilibria : $r = r_s = \text{Const.} \ (\dot{r} = 0)$, $\quad -\dfrac{\partial L}{\partial r}\Big|_{r=r_s} = \dfrac{\partial V^{(r)}}{\partial r}\Big|_{r=r_s} = 0$

Modified potential for the reduced system: $V^{(r)} = +\dfrac{p_\varphi^2}{m r^2} - mgr\dfrac{\sqrt{2}}{2} + \dfrac{1}{2} k (r - r_o)^2$

Problem 1

(b)

at equilibria : $\frac{\partial V(r)}{\partial r} = 0 \longrightarrow -\frac{2 P_\varphi^2}{mr^3} - mg\frac{\sqrt{2}}{2} + k(r-r_0) = 0$ (1)

$\downarrow$

$r = r_s \longrightarrow \dot{\varphi} = \frac{2P_\varphi}{mr_s^2} = \text{Const.} = \Omega \longrightarrow P_\varphi = \frac{mr_s^2\Omega}{2}$ (2)

(1), (2)

$\Longrightarrow -\frac{1}{2}mr_s\Omega^2 - mg\frac{\sqrt{2}}{2} + k(r_s - r_0) = 0$

$\longrightarrow r_s = \frac{mg\frac{\sqrt{2}}{2} + kr_0}{k - \frac{1}{2}m\Omega^2}$ $\quad (\Omega < \sqrt{\frac{2k}{m}})$

For stability :

$\left.\frac{\partial^2 V(r)}{\partial r^2}\right|_{r=r_s} = 6\frac{P_\varphi^2}{mr_s^4} + k > 0 \Longrightarrow$ r_s is stable.

(c)

equation of motion for r : $m\ddot{r} - \frac{2P_\varphi^2}{mr^3} - mg\frac{\sqrt{2}}{2} + k(r-r_0) = 0$

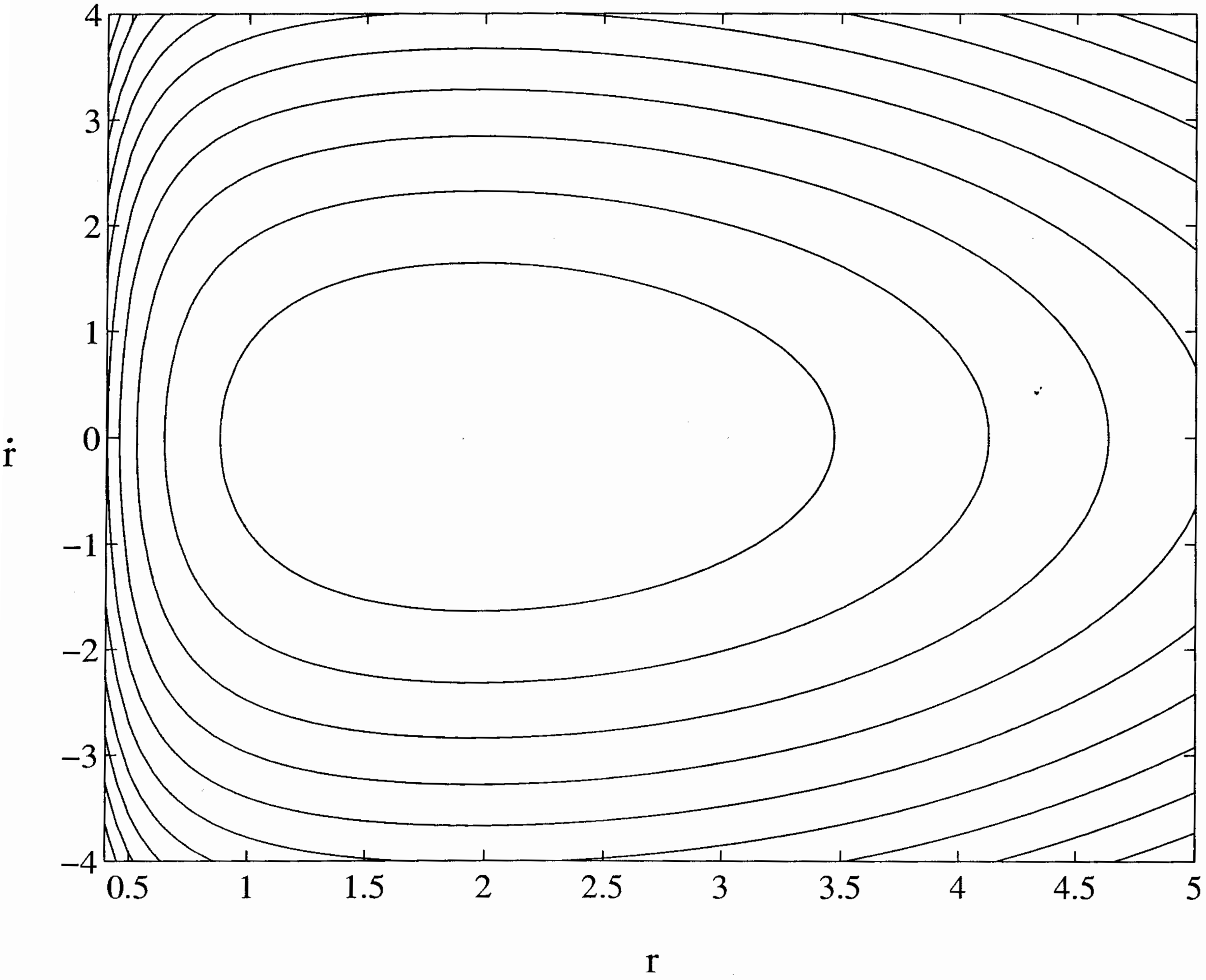
$\longrightarrow m\ddot{r}\dot{r} - \frac{2P_\varphi^2}{mr^3}\dot{r} - mg\frac{\sqrt{2}}{2}\dot{r} + k(r-r_0)\dot{r} = 0$

$\longrightarrow m\frac{\dot{r}^2}{2} + \frac{P_\varphi^2}{mr^2} - mg\frac{\sqrt{2}}{2}r + k\frac{r^2}{2} - kr_0 r = E_0$

$m = g = k = r_0 = P_\varphi = 1$

$\longrightarrow \dot{r}^2 + \frac{2}{r^2} - \sqrt{2}r + r^2 - 2r = 2E_0$

see the following page for the trajectories on the $(r, \dot{r})$ phase plane.


4
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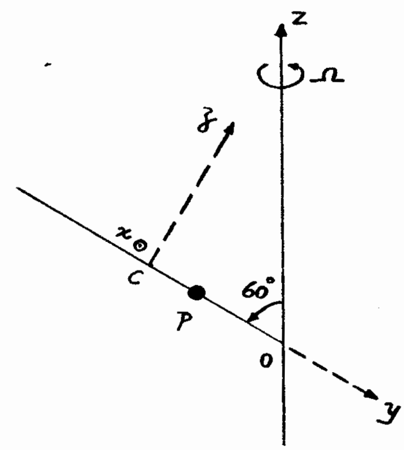
r

Problem 2

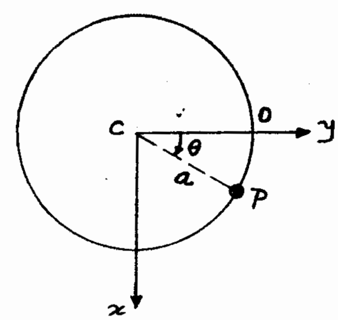
$$\underset{\sim}{v}_P = \left[a\Omega\frac{\sqrt{3}}{2}(1-\cos\theta) + a\dot{\theta}\cos\theta\right]\underline{e}_x$$
$$+\left[a\sin\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\right]\underline{e}_y$$
$$+\left[a\frac{\Omega}{2}\sin\theta\right]\underline{e}_z \qquad \text{(Problem 2 of PS 3)}$$



(i)

$$T = \frac{1}{2}m\,\underset{\sim}{v}_P\cdot\underset{\sim}{v}_P$$
$$= \frac{1}{2}ma^2\left\{\left[\Omega\frac{\sqrt{3}}{2}(1-\cos\theta) + \dot{\theta}\cos\theta\right]^2 + \left[\sin^2\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)^2\right] + \frac{\Omega^2}{4}\sin^2\theta\right\}$$

$$V = -mg\,a\cos\theta\,\frac{1}{2}$$



$$Q_\theta = 0$$

$$\mathcal{L} = T - V$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ma^2\left\{\frac{\sqrt{3}}{2}\Omega(\cos\theta - 1) + \dot{\theta}\right\}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = ma^2\left\{\frac{1}{4}\Omega^2\sin\theta(3+\cos\theta) - \frac{\sqrt{3}}{2}\Omega\dot{\theta}\sin\theta - \frac{g}{2a}\sin\theta\right\}$$

$$\frac{d}{dt}\left\{\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right\} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\ddot{\theta} - \frac{1}{4}\Omega^2\sin\theta(3+\cos\theta) + \frac{g}{2a}\sin\theta = 0 \qquad (1)$$

(ii)

θ is non-ignorable $\Longrightarrow$ steady motion $\theta = \theta_0$ (constant)

$$(1) \Longrightarrow \sin\theta_0\left[\frac{g}{2a} - \frac{\Omega^2}{4}(3+\cos\theta_0)\right] = 0$$

Problem 2

(ii)

Possible equilibrium positions :

A: $\sin\theta_0 = 0$, $\cos\theta_0 = 1$ $\Longrightarrow$ $\theta_0 = 0$

B: $\sin\theta_0 = 0$, $\cos\theta_0 = -1$ $\Longrightarrow$ $\theta_0 = \pm\pi$

C: $\frac{\Omega^2}{2}(3+\cos\theta_0) = \frac{g}{a}$ $\Longrightarrow$ $\cos\theta_0 = \frac{2g}{a\Omega^2} - 3$ $\longrightarrow$ $\theta_0 = \pm\cos^{-1}\left(\frac{2g}{a\Omega^2} - 3\right)$

possible only if $1 < \frac{g}{a\Omega^2} < 2$

$$\sqrt{\frac{g}{a}} > \Omega > \sqrt{\frac{g}{2a}}$$

check the stability assuming Ω stays constant:

$\theta = \theta_0 + \varepsilon(t)$

(1) $\Longrightarrow$ $\ddot{\varepsilon} - \frac{1}{4}\Omega^2(\sin\theta_0 + \varepsilon\cos\theta_0)(3+\cos\theta_0 - \varepsilon\sin\theta_0) + \frac{g}{2a}(\sin\theta_0 + \varepsilon\cos\theta_0) = 0$

$O(1)$ terms cancel. $\left(\sin\theta_0\left[\frac{g}{2a} - \frac{\Omega^2}{4}(3+\cos\theta_0)\right] = 0\right)$

$\longrightarrow$ $\ddot{\varepsilon} - \frac{\Omega^2}{4}\cos\theta_0(3+\cos\theta_0)\varepsilon + \frac{\Omega^2}{4}\sin^2\theta_0\,\varepsilon + \frac{g}{2a}\cos\theta_0\,\varepsilon = 0$

(A) $\ddot{\varepsilon} - \Omega^2\varepsilon + \frac{g}{2a}\varepsilon = 0$

$\ddot{\varepsilon} + \left(\frac{g}{2a} - \Omega^2\right)\varepsilon = 0$ unstable if $\frac{g}{2a} - \Omega^2 < 0$ $\longrightarrow$ $\Omega > \sqrt{\frac{g}{2a}}$

(B) $\ddot{\varepsilon} + \frac{2}{4}\Omega^2\varepsilon - \frac{g}{2a}\varepsilon = 0$

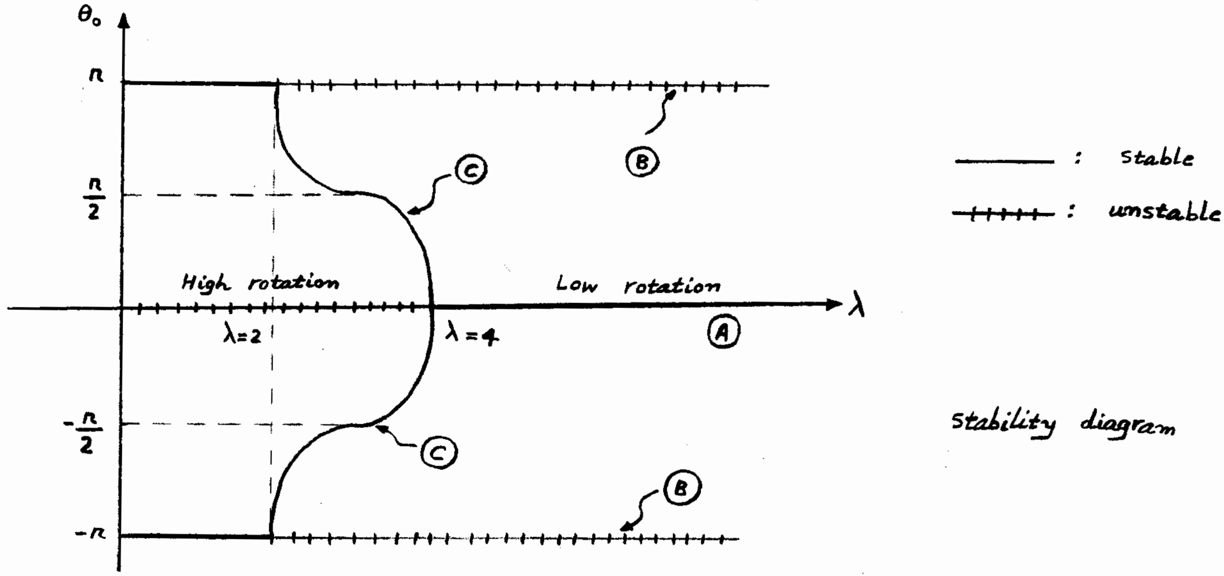
$\ddot{\varepsilon} + \left(\frac{\Omega^2}{2} - \frac{g}{2a}\right)\varepsilon = 0$ unstable if $\frac{\Omega^2}{2} - \frac{g}{2a} < 0$ $\longrightarrow$ $\Omega < \sqrt{\frac{g}{a}}$

(C) $\ddot{\varepsilon} + \frac{\Omega^2}{4}\sin^2\theta_0\,\varepsilon = 0$ stable

Problem 2

iii) Define parameter $\lambda = \frac{2g}{a\Omega^2}$ $(0<\lambda<\infty)$

$$\lambda_{crit} = 4, 2 \qquad \Omega_{crit} = \sqrt{\frac{g}{2a}}, \sqrt{\frac{g}{a}}$$



stability diagram

iv)

$$\ddot{\theta} - \frac{\Omega^2}{4}\sin\theta\,(3+\cos\theta) + \frac{g}{2a}\sin\theta = 0$$

Let $T = \sqrt{\frac{g}{a}}\,t \longrightarrow \ddot{\theta} - \frac{1}{2\lambda}\sin\theta\,(3+\cos\theta) + \frac{1}{2}\sin\theta = 0$

$$\longrightarrow \quad \ddot{\theta} + \frac{1}{2\lambda}(\lambda - 3 - \cos\theta)\sin\theta = 0$$

$$\begin{cases} \frac{d\theta}{dT} = y \\ \frac{dy}{dT} = \frac{1}{2\lambda}(\cos\theta + 3 - \lambda)\sin\theta \end{cases} \Longrightarrow \frac{dy}{d\theta} = \frac{1}{2\lambda y}(\cos\theta + 3 - \lambda)\sin\theta$$

$$\longrightarrow \quad y\,dy = \frac{1}{2\lambda}(\cos\theta + 3 - \lambda)\sin\theta\,d\theta$$

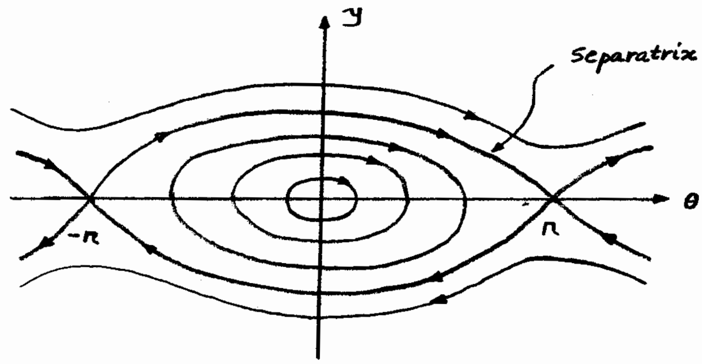
$$\longrightarrow \quad \frac{y^2}{2} = \frac{1}{2\lambda}\left[\frac{\sin^2\theta}{2} - (3-\lambda)\cos\theta\right] + C$$

iv)

Low rotation

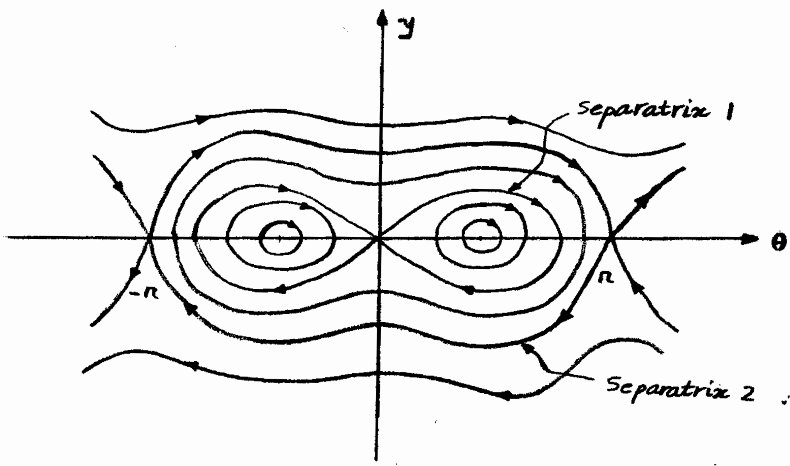
$\lambda > 4$

$\Omega < \sqrt{\frac{g}{2a}}$



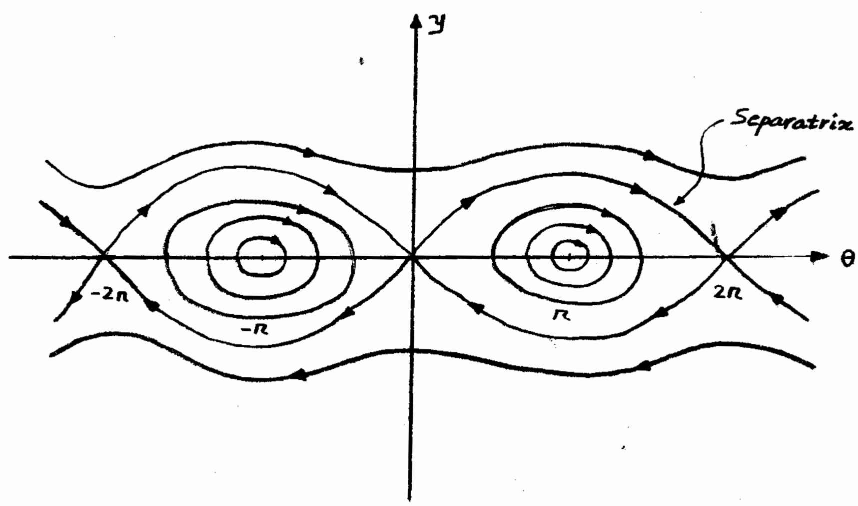
$4 > \lambda > 2$

$\sqrt{\frac{g}{a}} > \Omega > \sqrt{\frac{g}{2a}}$



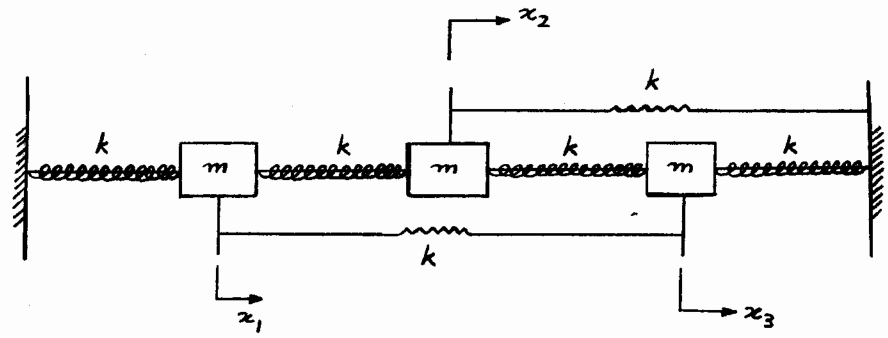
$0 < \lambda < 2$

$\Omega > \sqrt{\frac{g}{a}}$



v) If the ring is inclined at $120°$ to the vertical, just $\underline{\theta = 0}$ and $\underline{\theta = \pi}$ are equilibrium points, $\theta = 0$ being always unstable and $\theta = \pi$ being stable regardless of value of Ω. There are no equilibrium points between 0 and π in this case. The reason is that both gravity and rotation tend to move the bead toward $\theta = \pi$, whereas, in the $60°$ case, gravity tends to move the bead toward $\theta = 0$ while rotation tends to move it toward $\theta = \pi$.

Problem 3



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k (x_3 - x_1)^2$$

$$\mathcal{L} = T - V \qquad Q_1 = Q_2 = Q_3 = 0$$

equations of motion:

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{[M]} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \underbrace{\begin{bmatrix} 3k & -k & -k \\ & 3k & -k \\ \text{sym.} & & 3k \end{bmatrix}}_{[K]} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2 [M] + [K]\right) \{a\} = \{0\} \longrightarrow \left| -\omega^2 [M] + [K] \right| = 0$$

$$\begin{vmatrix} 3k - m\omega^2 & -k & -k \\ & 3k - m\omega^2 & -k \\ \text{sym.} & & 3k - m\omega^2 \end{vmatrix} = 0 \Longrightarrow (3k - m\omega^2)^3 - 2k^3 - 3k^2(3k - m\omega^2) = 0$$

$$\longrightarrow (m\omega^2 - k)\underbrace{(m^2\omega^4 - 8mk\omega^2 + 16k^2)}_{(m\omega^2 - 4k)^2} = 0 \Longrightarrow \begin{cases} \omega_1^2 = \frac{k}{m} \\ \omega_2^2 = \omega_3^2 = \frac{4k}{m} \end{cases}$$

Problem 3

$$\omega_1^2 = \frac{k}{m} \longrightarrow \begin{bmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Longrightarrow \quad \{a\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = \omega_3^2 = \frac{4k}{m} \longrightarrow \begin{bmatrix} -k & -k & -k \\ -k & -k & -k \\ -k & -k & -k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}_{2/3} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$-ka_1 - ka_2 - ka_3 = 0 \longrightarrow a_3 = -a_1 - a_2$$

$$\Longrightarrow \begin{Bmatrix} a_1 \\ a_2 \\ -a_1 - a_2 \end{Bmatrix} = a_1 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} + a_2 \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

$\longrightarrow$ Two independent eigenvectors: $\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$, $\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$

$$\{a\}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \quad \& \quad \{a\}_3 = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

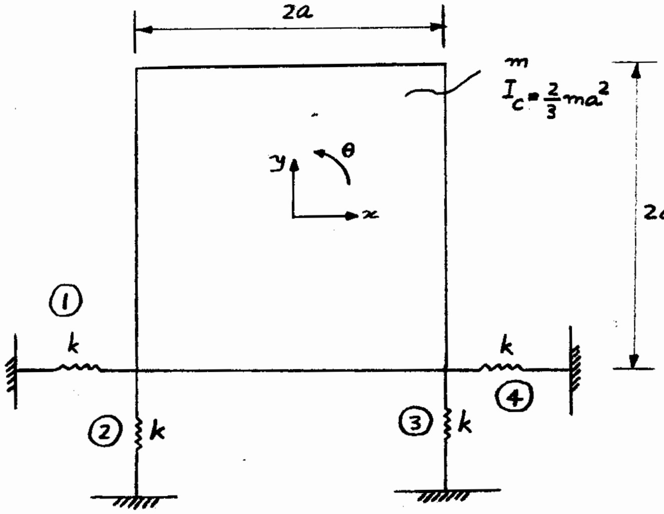
Problem 4

(a)

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \left(\frac{2}{3} m a^2\right) \dot{\theta}^2$$

$$V = \frac{1}{2} k \left(\Delta \ell_1^2 + \Delta \ell_2^2 + \Delta \ell_3^2 + \Delta \ell_4^2\right)$$

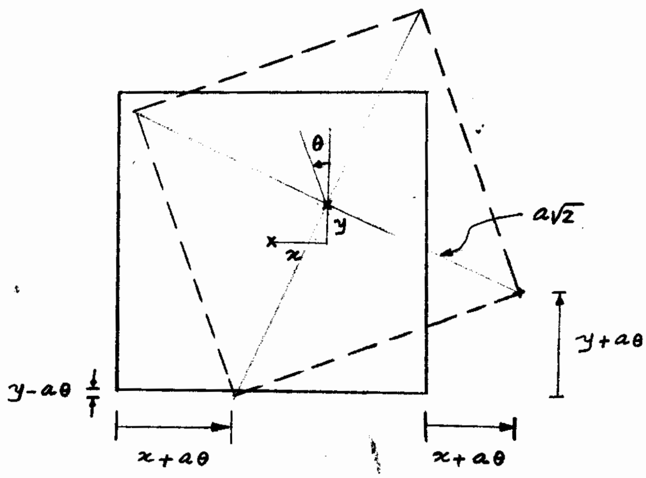


$$\Delta \ell_1 \approx x + (a\sqrt{2})\theta \, \frac{\sqrt{2}}{2} = x + a\theta$$

$$\Delta \ell_2 \approx y - a\theta$$

$$\Delta \ell_3 \approx y + a\theta \qquad (x, y, \theta \text{ small})$$

$$\Delta \ell_4 \approx x + a\theta$$



$$V = \frac{1}{2} k \left[2(x + a\theta)^2 + (y - a\theta)^2 + (y + a\theta)^2 \right]$$

$$\{x\} = \begin{Bmatrix} x \\ y \\ a\theta \end{Bmatrix}$$

$$\mathcal{L} = T - V$$

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{2}{3}m \end{bmatrix}}_{[M]} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ a\ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} 2k & 0 & 2k \\ 0 & 2k & 0 \\ 2k & 0 & 4k \end{bmatrix}}_{[K]} \begin{Bmatrix} x \\ y \\ a\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

matrix equations of motion

$$\left(-\omega^2 [M] + [K]\right) \{a\} = 0 \quad \longrightarrow \quad \left| -\omega^2 [M] + [K] \right| = 0$$

$$\longrightarrow \begin{vmatrix} 2k - m\omega^2 & 0 & 2k \\ 0 & 2k - m\omega^2 & 0 \\ 2k & 0 & 4k - \frac{2}{3} m\omega^2 \end{vmatrix} = 0 \quad \longrightarrow \quad (2k - m\omega^2)\left[(2k - m\omega^2)\left(4k - \frac{2}{3} m\omega^2\right) - 4k^2\right] = 0$$

Problem 4

$$\Rightarrow \begin{cases} \omega_1^2 = (4-\sqrt{10})\frac{k}{m} \\ \omega_2^2 = \frac{2k}{m} \\ \omega_3^2 = (4+\sqrt{10})\frac{k}{m} \end{cases} \Longrightarrow \begin{cases} \{a\}_1 = \begin{Bmatrix} 1 \\ 0 \\ 1-\frac{\sqrt{10}}{2} \end{Bmatrix} \\ \{a\}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \\ \{a\}_3 = \begin{Bmatrix} 1 \\ 0 \\ 1+\frac{\sqrt{10}}{2} \end{Bmatrix} \end{cases}$$

(b) $$[\varphi] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1-\frac{\sqrt{10}}{2} & 0 & 1+\frac{\sqrt{10}}{2} \end{bmatrix}, \quad [M] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}, \quad [K] = k\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$[M]_D = [\varphi]^t[M][\varphi] = \begin{bmatrix} \frac{10-2\sqrt{10}}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{10+2\sqrt{10}}{3} \end{bmatrix}(m)$$

$$[K]_D = [\varphi]^t[K][\varphi] = \begin{bmatrix} 10-3\sqrt{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10+3\sqrt{10} \end{bmatrix}(2k)$$

$$\begin{cases} \frac{k_1}{m_1} = \frac{10-3\sqrt{10}}{10-2\sqrt{10}} \cdot \frac{6k}{m} = \frac{(10-3\sqrt{10})(10+2\sqrt{10})}{60}\frac{6k}{m} = (4-\sqrt{10})\frac{k}{m} = \omega_1^2 \\ \frac{k_2}{m_2} = \frac{2k}{m} = \omega_2^2 \\ \frac{k_3}{m_3} = \frac{10+3\sqrt{10}}{10+2\sqrt{10}} \cdot \frac{6k}{m} = \frac{(10+3\sqrt{10})(10-2\sqrt{10})}{60}\frac{6k}{m} = (4+\sqrt{10})\frac{k}{m} = \omega_3^2 \end{cases}$$