2.032 Dynamics

Fall 2004

Problem Set No. 2

Problem 1

$$\underline{\underline{A}} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}$$ 3x3 skew-symmetric matrix $\underline{\underline{A}}$

$$\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$ three-dimensional vector $\underline{x}$

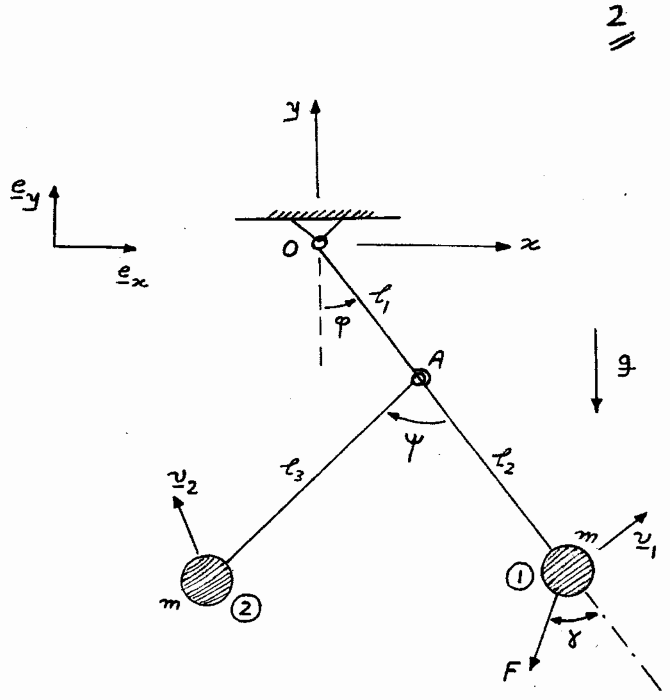
$$\underline{\omega} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$ three-dimensional vector $\underline{\omega}$

$$\underline{\omega} \times \underline{x} = \begin{Bmatrix} \omega_2 x_3 - \omega_3 x_2 \\ \omega_3 x_1 - \omega_1 x_3 \\ \omega_1 x_2 - \omega_2 x_1 \end{Bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\omega} \times \underline{x} \implies \begin{cases} \omega_1 = -a_{23} \\ \omega_2 = a_{13} \\ \omega_3 = -a_{12} \end{cases} \longrightarrow \underline{\omega} = \begin{Bmatrix} -a_{23} \\ a_{13} \\ -a_{12} \end{Bmatrix}$$

Problem 2



- Constraints :

$$\ell_1 + \ell_2 = \text{Const.}$$

$$\ell_3 = \text{Const.}$$

- $\# \text{ DOF} = \underbrace{2 \times 2}_{\text{point mass in 2D}} - 1 - 1 = \underline{\underline{2}}$

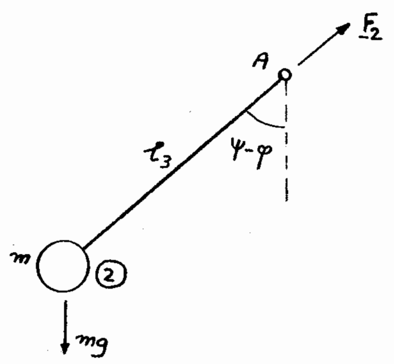
- Equations of motion for ψ and φ :

$$\begin{cases} \underline{r}_1 = (\ell_1 + \ell_2)\left(\sin\varphi\, \underline{e}_x - \cos\varphi\, \underline{e}_y\right) \\ \underline{r}_2 = \left(\ell_1 \sin\varphi - \ell_3 \sin(\psi - \varphi)\right)\underline{e}_x - \left(\ell_1 \cos\varphi + \ell_3 \cos(\psi-\varphi)\right)\underline{e}_y \end{cases}$$

$$\begin{cases} \underline{v}_1 = (\ell_1 + \ell_2)\,\dot\varphi\left(\cos\varphi\, \underline{e}_x + \sin\varphi\, \underline{e}_y\right) \\ \underline{v}_2 = \left(\ell_1\dot\varphi \cos\varphi - \ell_3 \cos(\psi-\varphi)(\dot\psi - \dot\varphi)\right)\underline{e}_x + \left(\ell_1\dot\varphi \sin\varphi + \ell_3 \sin(\psi-\varphi)(\dot\psi-\dot\varphi)\right)\underline{e}_y \end{cases}$$

Angular momentum about point A :

$$\underline{M}_A = \underline{\dot H}_A + \underline{v}_A \times \underline{P}$$



FBD

$$\underline{M}_A = mg\,\ell_3 \sin(\psi - \varphi)\, \underline{e}_z$$

$$\underline{v}_A = \ell_1 \dot\varphi \left(\cos\varphi\, \underline{e}_x + \sin\varphi\, \underline{e}_y\right)$$

$$\underline{P} = m\, \underline{v}_2$$

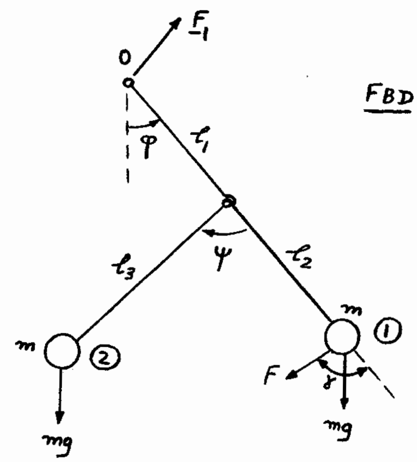
$$\underline{H}_A = \left(-\ell_3 \sin(\psi-\varphi)\, \underline{e}_x - \ell_3 \cos(\psi-\varphi)\, \underline{e}_y\right) \times m\, \underline{v}_2$$

$$= m\,\ell_3\left(\ell_1 \dot\varphi \cos\psi - \ell_3(\dot\psi - \dot\varphi)\right)\underline{e}_z$$

Problem 2

$$\therefore \quad mg\ell_3 \sin(\psi-\varphi) = m\ell_1\ell_3\left(\ddot{\varphi}\cos\psi - \sin\psi\,\dot{\psi}\dot{\varphi}\right) - m\ell_3^2(\ddot{\psi}-\ddot{\varphi}) + m\ell_1\ell_3\dot{\varphi}(\dot{\psi}-\dot{\varphi})\sin\psi$$

$$\longrightarrow \quad \boxed{g\sin(\psi-\varphi) = (\ell_1\cos\psi+\ell_3)\ddot{\varphi} - \ell_3\ddot{\psi} - \ell_1\dot{\varphi}^2\sin\psi} \qquad (1)$$



Angular momentum for the system about O:

$$\underline{M}_0 = \dot{\underline{H}}_0 + \overset{\underline{0}}{\underline{v}_0} \times \underline{P} = \dot{\underline{H}}_0$$

$$\underline{M}_0 = \underline{r}_1 \times \left(-mg\,\underline{e}_y + F\sin(\varphi-\gamma)\,\underline{e}_x - F\cos(\varphi-\gamma)\,\underline{e}_y\right)$$

$$+\ \underline{r}_2 \times -mg\,\underline{e}_y$$

$$= \left[-(\ell_1+\ell_2)\sin\varphi\left(mg + F\cos(\varphi-\gamma)\right) + (\ell_1+\ell_2)F\cos\varphi\sin(\varphi-\gamma) - mg\ell_1\sin\varphi + mg\ell_3\sin(\psi-\varphi)\right]\underline{e}_z$$

$$\underline{H}_0 = \underline{r}_1 \times m\underline{v}_1 + \underline{r}_2 \times m\underline{v}_2$$

$$= m\left\{\left[(\ell_1+\ell_2)^2+\ell_1^2\right]\dot{\varphi} - \ell_1\ell_3\dot{\psi}\cos\psi + 2\ell_1\ell_3\dot{\varphi}\cos\psi - \ell_3^2(\dot{\psi}-\dot{\varphi})\right\}\underline{e}_z$$

$$\therefore \quad -mg\sin\varphi\,(2\ell_1+\ell_2) + mg\ell_3\sin(\psi-\varphi) - (\ell_1+\ell_2)F\sin\varphi\cos(\varphi-\gamma) + (\ell_1+\ell_2)F\cos\varphi\sin(\varphi-\gamma)$$

$$= m\left\{\left[(\ell_1+\ell_2)^2+\ell_1^2\right]\ddot{\varphi} - \ell_1\ell_3\left(\ddot{\psi}\cos\psi - \dot{\psi}^2\sin\psi\right) + 2\ell_1\ell_3\left(\ddot{\varphi}\cos\psi - \dot{\varphi}\dot{\psi}\sin\psi\right) - \ell_3^2(\ddot{\psi}-\ddot{\varphi})\right\}$$

$$\longrightarrow \quad -mg\sin\varphi\,(2\ell_1+\ell_2) + mg\ell_3\sin(\psi-\varphi) - (\ell_1+\ell_2)F\sin\gamma$$

$$= m\left[(\ell_1+\ell_2)^2+\ell_1^2+\ell_3^2+2\ell_1\ell_3\cos\psi\right]\ddot{\varphi} - m\left(\ell_1\ell_3\cos\psi+\ell_3^2\right)\ddot{\psi} + m\ell_1\ell_3\dot{\psi}\sin\psi\,(\dot{\psi}-2\dot{\varphi})$$

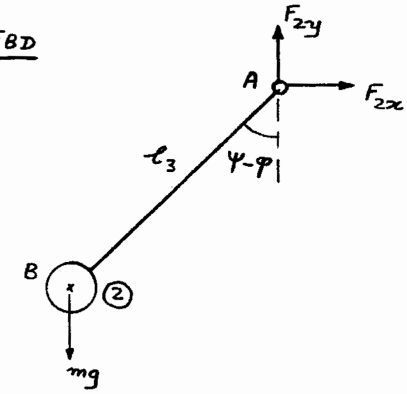
using (1) $\Longrightarrow$

$$\boxed{-g\sin\varphi\,(2\ell_1+\ell_2) - \frac{F}{m}(\ell_1+\ell_2)\sin\gamma = \left[(\ell_1+\ell_2)^2+\ell_1^2+\ell_1\ell_3\cos\psi\right]\ddot{\varphi} - \ell_1\ell_3\cos\psi\,\ddot{\psi} + \ell_1\ell_3\sin\psi\,(\dot{\psi}-\dot{\varphi})^2} \qquad (2)$$

Problem 2

- The constraint forces:

FBD



Use linear momentum in x direction:

$$\dot{\underline{P}} = \underline{F}^{ext}$$

$$F_{2x} = m\dot{\underline{v}}_2\big|_x$$

$$= m\left[\ell_1\left(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi\right) - \ell_3\left(\ddot{\psi}-\ddot{\varphi}\right)\cos(\psi-\varphi) + \ell_3\left(\dot{\psi}-\dot{\varphi}\right)^2\sin(\psi-\varphi)\right]$$

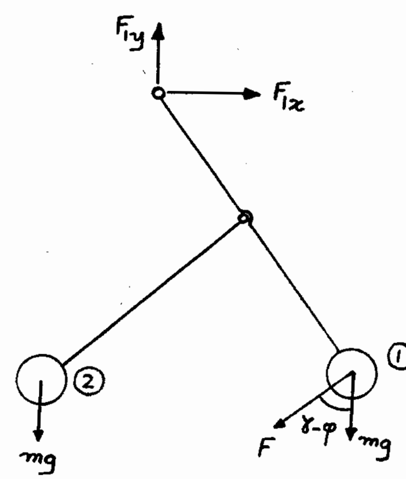
Angular momentum about point B:

$$\underline{M}_B = \overbrace{\dot{\underline{H}}_B}^{0} + \overbrace{\underline{v}_B \times \underline{P}}^{0} = \underline{0} \quad \Longrightarrow \quad F_{2x}\ell_3\cos(\psi-\varphi) - F_{2y}\ell_3\sin(\psi-\varphi) = 0$$

$$\longrightarrow \quad F_{2y} = F_{2x}\cot(\psi-\varphi)$$

$\therefore$ Constraint force $\underline{F}_2$ is along the rod.

FBD



Linear momentum in x direction:

$$F_{1x} - F\sin(\gamma-\varphi) = m\dot{\underline{v}}_1\big|_x + \overbrace{m\dot{\underline{v}}_2\big|_x}^{F_{2x}}$$

$$F_{1x} = m(\ell_1+\ell_2)\left(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi\right) + F\sin(\gamma-\varphi) + F_{2x}$$

in y direction:

$$F_{1y} - 2mg - F\cos(\gamma-\varphi) = m\dot{\underline{v}}_1\big|_y + m\dot{\underline{v}}_2\big|_y$$

$$F_{1y} = 2mg + F\cos(\gamma-\varphi) + m(2\ell_1+\ell_2)\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right) + \ell_3\left(\ddot{\psi}-\ddot{\varphi}\right)\sin(\psi-\varphi) + \ell_3\left(\dot{\psi}-\dot{\varphi}\right)^2\cos(\psi-\varphi)$$

Note that force $\underline{F}_1$ is not along the rod.

Problem 2

- Is the system conservative?

If just potential forces work and non-potential forces (if any) do not work at all, the system is conservative.

$\underline{F}_1$ do not work since $\underline{v}_0 = \underline{0}$.

$\underline{F}_2$ is internal force.

$m\underline{g}$ is potential.

$\underline{F}$ has a component $F\sin\gamma$ along $\underline{v}_1$ so $\underline{F}$ does work for $\gamma \neq 0$. If $\underline{F}$ is potential, one can find V such that $\underline{F} = -\nabla V$ $\left(\underline{F} = F_x \underline{e}_x + F_y \underline{e}_y ,\quad F_x = -\frac{\partial V}{\partial x} \;\&\; F_y = -\frac{\partial V}{\partial y} \right)$ and we have $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial x \partial y}$.

Here, $\underline{F} = F\sin(\varphi - \gamma)\, \underline{e}_x - F\cos(\varphi - \gamma)\, \underline{e}_y$

$$\varphi = -\tan^{-1}\left(\frac{x}{y}\right)$$

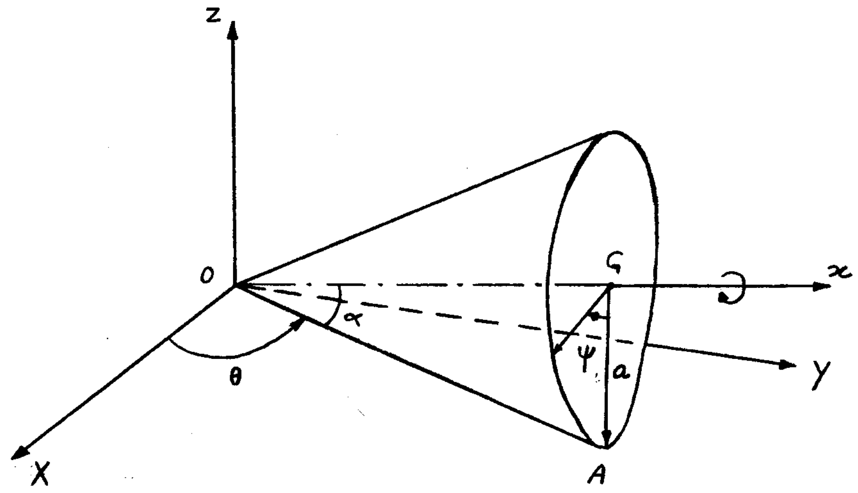
$$\begin{cases} \dfrac{\partial F_x}{\partial y} = \dfrac{\partial}{\partial y} F\sin(\varphi-\gamma) = F\cos(\varphi-\gamma)\,\dfrac{x}{x^2+y^2} = \dfrac{x^2\sin\gamma - xy\cos\gamma}{(x^2+y^2)\sqrt{x^2+y^2}}\,F \\[2ex] \dfrac{\partial F_y}{\partial x} = \dfrac{-\partial}{\partial x} F\cos(\varphi-\gamma) = F\sin(\varphi-\gamma)\,\dfrac{-y}{x^2+y^2} = -\dfrac{xy\cos\gamma + y^2\sin\gamma}{(x^2+y^2)\sqrt{x^2+y^2}}\,F \end{cases}$$

$\frac{\partial F_x}{\partial y} \neq \frac{\partial F_y}{\partial x} \implies \underline{F}$ is not a potential force.

$\therefore$ System is <u>not</u> conservative for $\gamma \neq 0$ and is conservative for $\gamma = 0$.

Problem 3

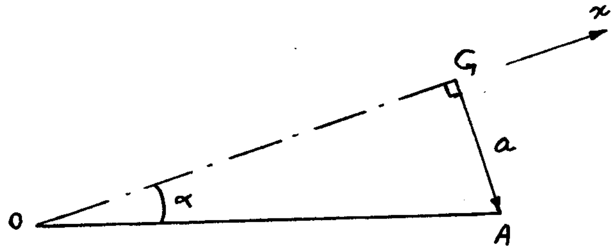
$$\underline{\omega}_{cone} = \dot{\theta}\,\underline{e}_Z - \dot{\psi}\,\underline{e}_x$$



Impose no slip : $\quad \underline{v}_A\big|_{cone} = \underline{0}$

$$\underline{v}_A\big|_{cone} = \underline{v}_O\big|_{cone} + \underline{\omega}\big|_{cone} \times \underline{OA} = \underline{0}$$

$$\underline{v}_O\big|_{cone} = \underline{0}$$



$$\underline{OA} = \frac{a}{\sin\alpha}\left(\cos\theta\,\underline{e}_X + \sin\theta\,\underline{e}_Y\right)$$

$$\underline{e}_x = \cos\alpha\cos\theta\,\underline{e}_X + \cos\alpha\sin\theta\,\underline{e}_Y + \sin\alpha\,\underline{e}_Z$$

$$\therefore \quad \underline{\omega}\big|_{cone} \times \underline{OA} = \underline{0} = \left[-\dot{\psi}\cos\alpha\cos\theta\,\underline{e}_X - \dot{\psi}\cos\alpha\sin\theta\,\underline{e}_Y + (\dot{\theta} - \dot{\psi}\sin\alpha)\,\underline{e}_Z\right] \times \left[\frac{a}{\sin\alpha}\cos\theta\,\underline{e}_X + \frac{a}{\sin\alpha}\sin\theta\,\underline{e}_Y\right]$$

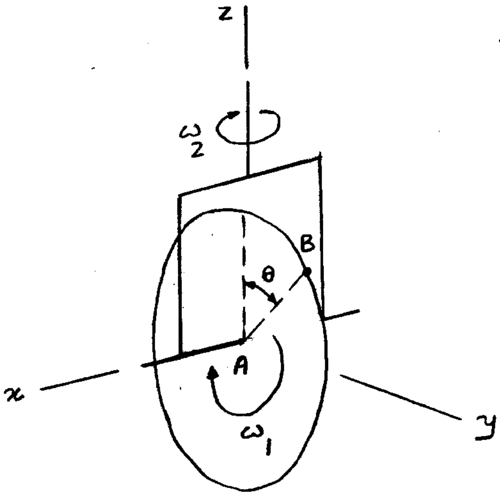
$$= -\underline{e}_X\left(a\dot{\theta}\frac{\sin\theta}{\sin\alpha} - a\dot{\psi}\sin\theta\right) + \underline{e}_Y\left(a\dot{\theta}\frac{\cos\theta}{\sin\alpha} - \dot{\psi}a\cos\theta\right) = \underline{0}$$

$$\Longrightarrow \quad \dot{\psi} = \frac{\dot{\theta}}{\sin\alpha}$$

Finally, $\quad \underline{\omega}_{cone} = \dot{\theta}\,\underline{e}_Z - \dfrac{\dot{\theta}}{\sin\alpha}\,\underline{e}_x = -\dot{\theta}\,\mathrm{Cotg}\,\alpha\left(\cos\theta\,\underline{e}_X + \sin\theta\,\underline{e}_Y\right) = -\dot{\theta}\,\mathrm{Cotg}\,\alpha\,\underline{e}_{OA}$

$\underline{\omega}_{cone}$ and axis of rotation are along OA.

Problem 4



Radius of the disk = R

$\underline{v}_A = \underline{0}$

$$\underline{v}_B = \underline{v}_A + \underline{\omega}_{disk} \times \underline{AB} = \underline{\omega}_{disk} \times \underline{AB}$$

$$\underline{AB} = R\left(\cos\theta\, \underline{e}_z + \sin\theta\, \underline{e}_y\right)$$

$$\underline{\omega}_{disk} = -\omega_1 \underline{e}_x - \omega_2 \underline{e}_z$$

$$\therefore \quad \underline{v}_B = -\left(\omega_1 \underline{e}_x + \omega_2 \underline{e}_z\right) \times R\left(\cos\theta\, \underline{e}_z + \sin\theta\, \underline{e}_y\right)$$

$$\underline{v}_B = R\omega_2 \sin\theta\, \underline{e}_x + R\omega_1 \cos\theta\, \underline{e}_y - R\omega_1 \sin\theta\, \underline{e}_z$$ velocity of point B

Note that xy are rotating about z with ω_2.

$$\underline{a}_B = \frac{d\underline{v}_B}{dt} = \frac{d\underline{\omega}_{disk}}{dt} \times \underline{AB} + \underline{\omega}_{disk} \times \frac{d\,\underline{AB}}{dt}$$

$$\frac{d\,\underline{AB}}{dt} = \underline{\omega}_{disk} \times \underline{AB} \qquad \frac{d\underline{e}_x}{dt} = -\omega_2 \underline{e}_y \qquad \frac{d\underline{e}_y}{dt} = \omega_2 \underline{e}_x$$

$$\frac{d\underline{\omega}_{disk}}{dt} = \omega_1\omega_2 \underline{e}_y \qquad \frac{d\,\underline{AB}}{dt} = R\omega_2 \sin\theta\, \underline{e}_x + R\omega_1 \cos\theta\, \underline{e}_y - R\omega_1 \sin\theta\, \underline{e}_z$$

$$\therefore \quad \underline{a}_B = \left(\omega_1\omega_2 \underline{e}_y\right) \times R\left(\cos\theta\, \underline{e}_z + \sin\theta\, \underline{e}_y\right) + \left(-\omega_1 \underline{e}_x - \omega_2 \underline{e}_z\right) \times R\left(\omega_2 \sin\theta\, \underline{e}_x + \omega_1 \cos\theta\, \underline{e}_y - \omega_1 \sin\theta\, \underline{e}_z\right)$$

$$\underline{a}_B = 2R\omega_1\omega_2 \cos\theta\, \underline{e}_x - R\left(\omega_1^2 + \omega_2^2\right)\sin\theta\, \underline{e}_y - R\omega_1^2 \cos\theta\, \underline{e}_z$$ acceleration of point B

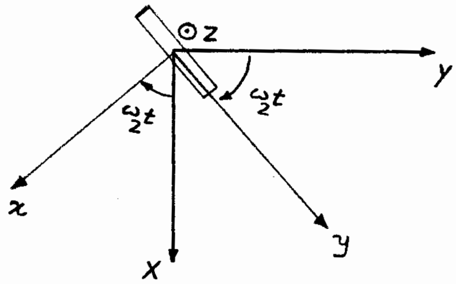
Problem 4

An alternative solution :

One can express the vectors in terms of fixed unit vectors $\underline{e}_X$ and $\underline{e}_Y$:

XY are fixed while xy rotate about z with ω_2.

$$\begin{cases} \underline{e}_x = \cos\omega_2 t \, \underline{e}_X - \sin\omega_2 t \, \underline{e}_Y \\ \underline{e}_y = \sin\omega_2 t \, \underline{e}_X + \cos\omega_2 t \, \underline{e}_Y \end{cases}$$



$$\underline{AB} = R\left(\sin\theta \sin\omega_2 t \, \underline{e}_X + \sin\theta \cos\omega_2 t \, \underline{e}_Y + \cos\theta \, \underline{e}_z\right)$$

$$\dot{\theta} = \omega_1$$

$$\underline{v}_B = \frac{d}{dt}\underline{AB} = R\left(\omega_1 \cos\theta \sin\omega_2 t + \omega_2 \sin\theta \cos\omega_2 t\right)\underline{e}_X$$
$$+ R\left(\omega_1 \cos\theta \cos\omega_2 t - \omega_2 \sin\theta \sin\omega_2 t\right)\underline{e}_Y - R\omega_1 \sin\theta \, \underline{e}_z$$

$$\therefore \quad \underline{v}_B = R\omega_2 \sin\theta \, \underline{e}_x + R\omega_1 \cos\theta \, \underline{e}_y - R\omega_1 \sin\theta \, \underline{e}_z$$

$$\underline{a}_B = \frac{d^2}{dt^2}\underline{AB} = R\left(-\omega_1^2 \sin\theta \sin\omega_2 t - \omega_2^2 \sin\theta \sin\omega_2 t + 2\omega_1\omega_2 \cos\theta \cos\omega_2 t\right)\underline{e}_X$$
$$+ R\left(-\omega_1^2 \sin\theta \cos\omega_2 t - \omega_2^2 \sin\theta \cos\omega_2 t - 2\omega_1\omega_2 \cos\theta \sin\omega_2 t\right)\underline{e}_Y - R\omega_1^2 \cos\theta \, \underline{e}_z$$

$$\therefore \quad \underline{a}_B = 2R\omega_1\omega_2 \cos\theta \, \underline{e}_x - R\sin\theta\left(\omega_1^2 + \omega_2^2\right)\underline{e}_y - R\omega_1^2 \cos\theta \, \underline{e}_z$$