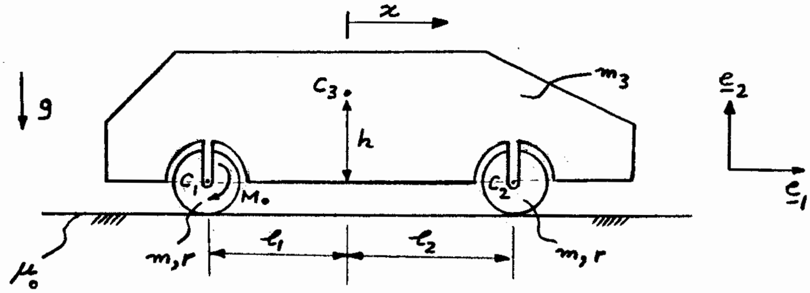
2.032 Dynamics

Fall 2004

Problem Set No. 5

Problem 1



(a) # DOF = 1

$\omega|_{\text{body of the car}} = 0$ $\qquad \omega|_{\text{each wheel}} = \omega$ $\qquad I|_{\text{each wheel}} = \frac{1}{2}mr^2$

Assuming pure rolling, $\quad v_{C3} = v_{C1} = v_{C2} = \dot{x} = r\omega$

Kinetic Energy: $T = \frac{1}{2}m_3 v_{c3}^2 + 4\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2\right]$

$$T = \frac{1}{2}(m_3 + 6m)\dot{x}^2$$

$\dot{T} = \frac{d}{dt}(\text{work of external forces}) = \text{power of external forces}$

There is no displacement in the direction of gravity and normal forces.

Velocity at the point of attack of friction force is $\underline{0}$.

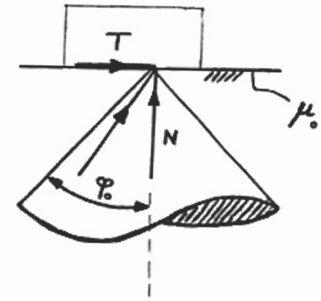
So the only thing that does work in this system is M_o.

$$\therefore \quad \left.\begin{aligned} \dot{T} &= \frac{d}{dt}(M_o\varphi) = M_o\omega = M_o\frac{\dot{x}}{r} \\ \dot{T} &= (m_3 + 6m)\dot{x}\ddot{x} \end{aligned}\right\} \Longrightarrow \boxed{\ddot{x} = \frac{M_o}{r(6m + m_3)}} \quad \text{acceleration of the car}$$

Problem 1

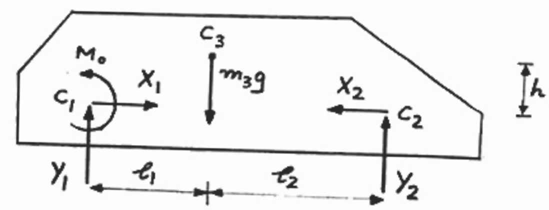
(b) One can show the no slip condition geometrically:

resultant reaction force must be within the cone.



$$\tan\varphi_0 = \mu_0$$

We need to find the forces acting on the rear wheel, then check whether the dynamical condition of rolling is satisfied.



FBD for the body of the car

Angular mom. about C_2:

$$\underline{M}_{C_2} = \dot{\underline{H}}_{C_2} + \underbrace{\cancel{\underline{v}_{C_2}} \times \underline{P}}_{=0\ (v_{C_2} \parallel P)}$$

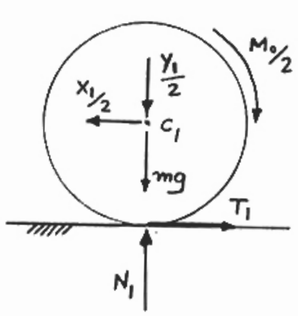
$$\underline{H}_{C_2} = \underline{H}_{C_3} + \underline{P} \times \underline{r}_{C_3C_2} = \underbrace{\cancel{\underline{\underline{I}}_{C_3}\omega}}_{=0\ (\omega=0)} + m_3 \underline{v}_{C_3} \times \underline{r}_{C_3C_2} = -m_3 h \dot{x}\, \underline{e}_3$$

$$\underline{M}_{C_2} = \left[-Y_1(\ell_1+\ell_2) + m_3 g \ell_2 + M_0\right] \underline{e}_3$$

$$\therefore \quad m_3 h \ddot{x} = Y_1(\ell_1+\ell_2) - m_3 g \ell_2 - M_0$$

$$\longrightarrow \quad Y_1 = \frac{m_3(h\ddot{x} + g\ell_2) + M_0}{\ell_1+\ell_2} = \frac{m_3 g \ell_2}{\ell_1+\ell_2} + M_0 \frac{6mr + m_3(r+h)}{r(\ell_1+\ell_2)(6m+m_3)} \qquad (1)$$

FBD for one of the rear wheels:



Linear mom. in vertical direction:

$$m \underbrace{\cancel{a_{C_1}|_y}}_{0} = N_1 - \frac{Y_1}{2} - mg \longrightarrow N_1 = mg + \frac{Y_1}{2} \qquad (2)$$

Angular mom. about C_1:

$$\left(T_1 r - \frac{M_0}{2}\right)\underline{e}_3 = \dot{\underline{H}}_{C_1} + \underbrace{\cancel{\underline{v}_{C_1}} \times \underline{P}}_{=0}$$

Problem 1

(b)

$$\underline{H}_{C_1} = \underline{\underline{I}}_{C_1}\underline{\omega} = -\left(\frac{1}{2}mr^2\right)\underbrace{\frac{\dot{x}}{r}}_{|\underline{v}_{C_1}|}\underline{e}_3 = -\frac{1}{2}mr\dot{x}\,\underline{e}_3$$

$$\therefore \quad mr\ddot{x} = -2T_1 r + M_0 \quad \longrightarrow \quad T_1 = \frac{M_0 - mr\ddot{x}}{2r} = M_0\frac{5m+m_3}{2r(6m+m_3)} \qquad (3)$$

We have so far assumed the kinematic condition of rolling. Rolling will, however, only hold as long as its dynamic condition is satisfied: $\frac{T_1}{N_1} \leqslant \mu_0$ (4)

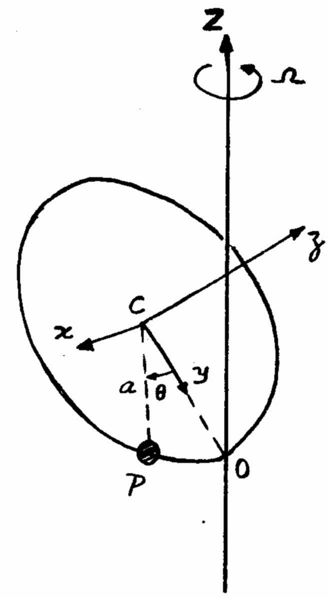
(1), (2), (3), (4) $\Longrightarrow$

$$M_0\frac{5m+m_3}{2r(6m+m_3)} \leqslant \mu_0 g\frac{2m\ell_1+(2m+m_3)\ell_2}{2(\ell_1+\ell_2)} + \mu_0 M_0\frac{6mr+m_3(r+h)}{2r(\ell_1+\ell_2)(6m+m_3)}$$

$$\longrightarrow \quad \underbrace{M_0 \leqslant \frac{\mu_0 rg(6m+m_3)\left[2m\ell_1+(2m+m_3)\ell_2\right]}{(5m+m_3)(\ell_1+\ell_2)-\mu_0\left[6mr+m_3(r+h)\right]}}_{\text{Largest } M_0 \text{ for which car does not slip.}}$$

Problem 2

Introducing xyz coordinate system fixed to the ring and as a result rotating with Ω about Z axis:

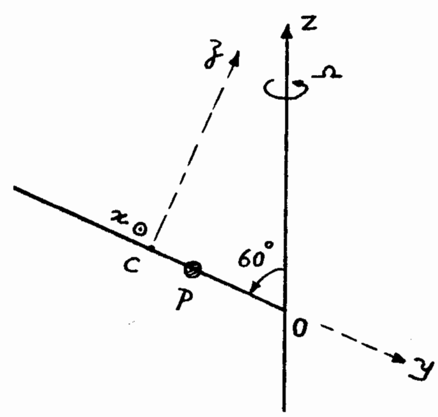


$$\underline{v}_P = \frac{d}{dt}\,\underline{OP} = \frac{d}{dt}\left(\underline{OC} + \underline{CP}\right) = \underline{v}_C + \underline{v}_P\big|_{\text{rel. to } C}$$

$$\underline{v}_C = \frac{d}{dt}\,\underline{OC} = \Omega\,\underline{e}_Z \times \underline{OC}$$

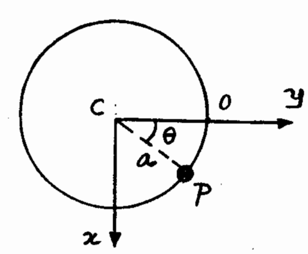
$$= \Omega\left(\frac{\sqrt{3}}{2}\,\underline{e}_z - \frac{1}{2}\,\underline{e}_y\right) \times \left(-a\,\underline{e}_y\right)$$

$$= a\Omega\frac{\sqrt{3}}{2}\,\underline{e}_x$$



$$\underline{v}_P\big|_{\text{rel. to } C} = \frac{d}{dt}\,\underline{CP} = \underline{\omega}\big|_{\underline{CP}} \times \underline{CP}$$

$$\underline{\omega}\big|_{\underline{CP}} = \Omega\,\underline{e}_Z - \dot\theta\,\underline{e}_z = -\frac{\Omega}{2}\,\underline{e}_y + \left(\Omega\frac{\sqrt{3}}{2} - \dot\theta\right)\underline{e}_z$$



$$\underline{CP} = a\sin\theta\,\underline{e}_x + a\cos\theta\,\underline{e}_y$$

$$\Longrightarrow \quad \underline{v}_P\big|_{\text{rel. to } C} = \left[-\frac{\Omega}{2}\,\underline{e}_y + \left(\Omega\frac{\sqrt{3}}{2} - \dot\theta\right)\underline{e}_z\right] \times \left[a\sin\theta\,\underline{e}_x + a\cos\theta\,\underline{e}_y\right]$$

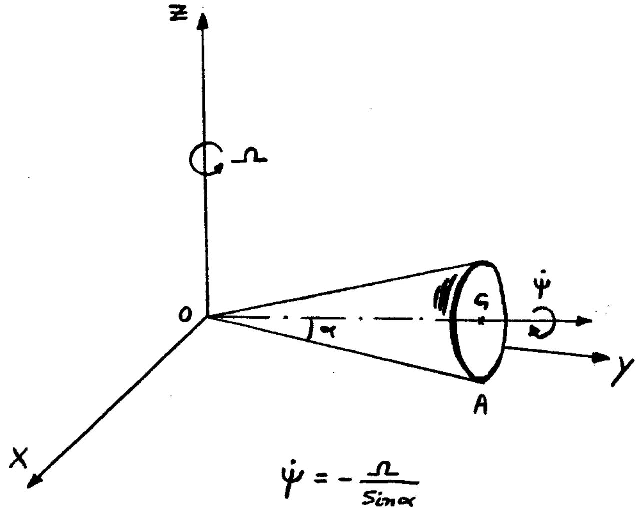
$$= \frac{\Omega}{2}a\sin\theta\,\underline{e}_z + a\sin\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot\theta\right)\underline{e}_y - a\cos\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot\theta\right)\underline{e}_x$$

$$\therefore \quad \underline{v}_P = \left[a\Omega\frac{\sqrt{3}}{2}\left(1 - \cos\theta\right) + a\dot\theta\cos\theta\right]\underline{e}_x + \left[a\sin\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot\theta\right)\right]\underline{e}_y + \left[a\frac{\Omega}{2}\sin\theta\right]\underline{e}_z$$

Velocity of P in terms of θ and $\dot\theta$

Problem 3

xyz rotates with Ω about Z axis.

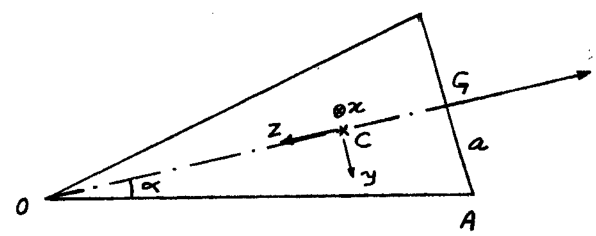


$$\dot\psi = -\frac{\Omega}{\sin\alpha}$$

$$\underline{\omega}_{cone} = -\Omega \cot\alpha \; \underline{e}_{OA}$$
$$= -\Omega \cot\alpha \left(-\cos\alpha \; \underline{e}_z + \sin\alpha \; \underline{e}_y\right)$$
$$= -\Omega \cos\alpha \; \underline{e}_y + \Omega \frac{\cos^2\alpha}{\sin\alpha} \underline{e}_z$$

(a)

Angular momentum of the cone about the tip O : $\quad \underline{H}_o = \underline{\underline{I}}_o \, \underline{\omega} \qquad (\underline{v}_o = \underline{0})$



$$a = \sqrt{\frac{10}{3}\frac{I_3}{M}}$$

$$\underline{\underline{I}}_o = \begin{bmatrix} I_1 & 0 & 0 \\ & I_1 & 0 \\ & & I_3 \end{bmatrix}$$

$$\underline{\omega} = \begin{Bmatrix} \omega_x = 0 \\ \omega_y = -\Omega \cos\alpha \\ \omega_z = \Omega \frac{\cos^2\alpha}{\sin\alpha} \end{Bmatrix}$$

$$\therefore \quad \boxed{\underline{H}_o = \underline{\underline{I}}_o \, \underline{\omega} = -I_1 \Omega \cos\alpha \; \underline{e}_y + I_3 \Omega \frac{\cos^2\alpha}{\sin\alpha} \underline{e}_z}$$

(b)

$$\underline{P} = M \underline{v}_c = M \left(\dot\theta \, \underline{e}_z \times \underline{OC}\right) = M \dot\theta \, \overbrace{|\underline{OC}|}^{\frac{3}{4}|\underline{OG}|} \cos\alpha \; \underline{e}_x = \frac{3}{4} M \Omega a \frac{\cos^2\alpha}{\sin\alpha} \underline{e}_x$$

$$\underline{F} = \dot{\underline{P}} = \frac{3}{4} M \Omega a \frac{\cos^2\alpha}{\sin\alpha} \frac{d\underline{e}_x}{dt}$$

$$\frac{d\underline{e}_x}{dt} = \Omega \underline{e}_z \times \underline{e}_x = \Omega \left(-\cos\alpha \; \underline{e}_y - \sin\alpha \; \underline{e}_z\right) \times \underline{e}_x$$
$$= \Omega \left(\cos\alpha \; \underline{e}_z - \sin\alpha \; \underline{e}_y\right) = -\Omega \, \underline{e}_{AO}$$

$$\boxed{\underline{F} = \frac{3}{4} M \Omega^2 a \frac{\cos^2\alpha}{\sin\alpha} \underline{e}_{AO}} \qquad \text{total required force}$$

Problem 3

(b)

$$\underline{M}_0 = \dot{\underline{H}}_0 \qquad (\underline{v}_0 = \underline{0})$$

$$\dot{\underline{H}}_0 = \overset{\circ}{\underline{H}}_0 + \underline{\omega}|_{xyz} \times \underline{H}_0 = \Omega\, \underline{e}_z \times \underline{H}_0$$

$$= \Omega\, \underline{e}_z \times \left(-I_1 \Omega \cos\alpha\, \underline{e}_y + I_3 \Omega \frac{\cos^2\alpha}{\sin\alpha}\, \underline{e}_z \right)$$

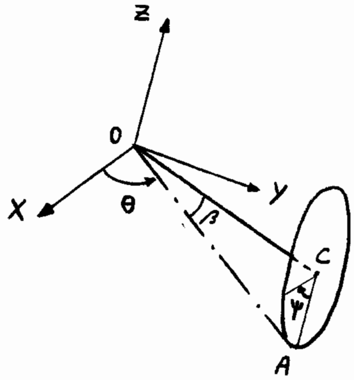
$$\Omega\, \underline{e}_Z \times \underline{e}_y = \Omega \left(-\cos\alpha\, \underline{e}_y - \sin\alpha\, \underline{e}_z \right) \times \underline{e}_y = \Omega \sin\alpha\, \underline{e}_x$$

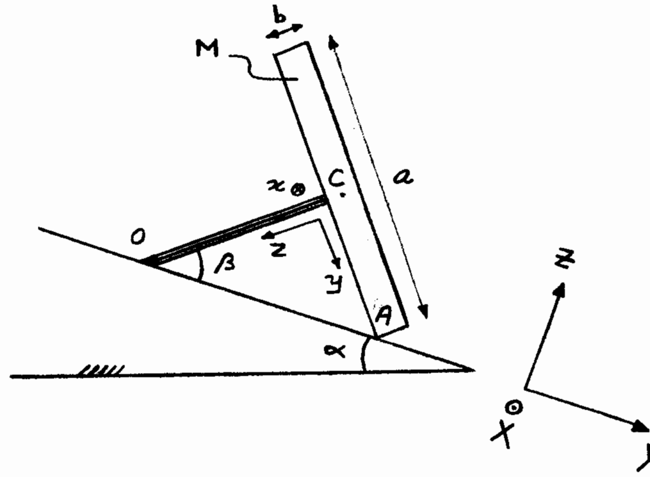
$$\Omega\, \underline{e}_Z \times \underline{e}_z = \Omega \left(-\cos\alpha\, \underline{e}_y - \sin\alpha\, \underline{e}_z \right) \times \underline{e}_z = -\Omega \cos\alpha\, \underline{e}_x$$

$$\therefore \quad \boxed{\underline{M}_0 = -\Omega^2 \left(I_1 \sin\alpha \cos\alpha + I_3 \frac{\cos^3\alpha}{\sin\alpha} \right) \underline{e}_x}$$

total required torque about point 0

Problem 4





$$I_1 = I_2 = I_{xx} = I_{yy} = \frac{Ma^2}{16}$$

$$I_3 = I_{zz} = \frac{Ma^2}{8}$$

$$\underline{\omega}_{disk} = \dot{\theta}\,\underline{e}_Z + \dot{\psi}\,\underline{e}_z \qquad \underline{e}_z = -\cos\beta\cos\theta\,\underline{e}_X - \cos\beta\sin\theta\,\underline{e}_Y - \sin\beta\,\underline{e}_Z$$

$$\underline{\omega}_{disk} = -\dot{\psi}\cos\beta\cos\theta\,\underline{e}_X - \dot{\psi}\cos\beta\sin\theta\,\underline{e}_Y + (\dot{\theta} - \dot{\psi}\sin\beta)\,\underline{e}_Z$$

No slip : $\underline{v}_A\big|_{disk} = \underline{0}$

$$\underline{v}_A\big|_{disk} = \overset{\underline{0}}{\underline{v}_o} + \underline{\omega}\big|_{disk} \times \underline{OA} = \underline{0} \qquad \underline{OA} = \frac{a}{2\sin\beta}\left(\cos\theta\,\underline{e}_X + \sin\theta\,\underline{e}_Y\right)$$

$$\underline{\omega}\big|_{disk} \times \underline{OA} = \underline{0} \Longrightarrow \dot{\psi} = \frac{\dot{\theta}}{\sin\beta} \longrightarrow \underline{\omega}_{disk} = -\dot{\theta}\cot\beta\,\underline{e}_{OA}$$

$$T = \frac{1}{2}M\,\underline{v}_c\cdot\underline{v}_c + \frac{1}{2}\,\underline{\omega}^T\,\underline{\underline{I}}_c\,\underline{\omega} = \frac{1}{2}M\,\underline{v}_c\cdot\underline{v}_c + \frac{1}{2}\left(I_1\omega_x^2 + I_1\omega_y^2 + I_3\omega_z^2\right)$$

$$\underline{v}_c = \dot{\theta}\,\underline{e}_Z \times \underline{OC} = \dot{\theta}\left(\frac{a}{2\tan\beta} + \frac{b}{2}\right)\cos\beta\,\underline{e}_x \longrightarrow \underline{v}_c\cdot\underline{v}_c = \dot{\theta}^2\cos^2\beta\left(\frac{a\cos\beta}{2\sin\beta} + \frac{b}{2}\right)^2$$

$$\underline{\omega}_{disk} = -\dot{\theta}\cot\beta\,\underline{e}_{OA} = -\dot{\theta}\cot\beta\left(-\cos\beta\,\underline{e}_z + \sin\beta\,\underline{e}_y\right) \longrightarrow \omega_x = 0, \quad \omega_y = -\dot{\theta}\cos\beta, \quad \omega_z = \dot{\theta}\frac{\cos^2\beta}{\sin\beta}$$

$$\therefore\ T = \frac{1}{8}M\dot{\theta}^2\cot^2\beta\left(a\cos\beta + b\sin\beta\right)^2 + \frac{1}{2}I_1\dot{\theta}^2\cos^2\beta + \frac{1}{2}I_3\dot{\theta}^2\frac{\cos^4\beta}{\sin^2\beta} = \boxed{\frac{\cos^2\beta}{32}Ma^2\dot{\theta}^2\left(1 + 6\cot^2\beta + 4\frac{b^2}{a^2} + 8\frac{b}{a}\cot\beta\right)}$$

kinetic energy of the disk