PAIGE BRIGHT: So welcome to the last lecture of the class. Today we're going to talk about where all of this material particularly goes from here-- so for instance, how this material applies to 18.102, a little bit of 18.101, 901, for sure, and a little bit of 152. If those numbers don't mean anything to you right now, that's totally fair. I'm going to write out what the number and the title of it is. But loosely speaking, that's the general plan for today.

But before we do that, I want to introduce just a little bit of brief history to metric spaces in the first place because I think that there's this weird notion that all of these mathematical tools just pop up. But in fact, everyone knew each other in the field, which I think is really cool. So firstly, the main thing to know is that, in the early 1900s, mathematics was far less axiomatized. It was far more localized in each field, like-- oh, in this field who's studying this type of function? We can study what this means for the entire space. What does being Lipschitz mean what? Is a vector space, things like that?

But because of this, a bunch of different people had all different types of convergence. They just weren't unified. And now, this is fine, but it makes it difficult to see if it's something that's true specifically about the space or if it's something true in higher generality. So in 1906, Fréchet invented metric spaces. The reason I think this is particularly interesting is because this concept is one that he introduced in his PhD dissertation. So he went throughout his PhD course load and then created metric spaces in his dissertation. It's a very, very powerful tool.

And as we've seen in this class so far, it unified all these different types of convergence at once. No longer did people have to state convergence in their field and then prove all the properties. They can simply state the distance on it and then state what convergence means, if, it is, in fact, a metric. So this, in particular, really unified notions of convergence and other types of open sets and things like that. So in particular, this lets you prove facts about metric spaces and then state it about your space. That's the main idea.

And then later on, in 1914, a mathematician known as Hausdorff, which will be a familiar name if you've done topology, popularized metric spaces. So in particular, he wrote this book called *Principles of Set Theory.* And in this book, he was able to define, or-- this book was very, very popular, and in it, he includes metric spaces. So it made it more and more popular.

But in fact, he did quite a bit more than that in this book. In this book, he introduced topological spaces. And topological spaces are, of course, what is studied in Introduction to Topology, 18.901. Now, throughout the day, I'm going to use this diagram, where what I'm going to talk about is this idea that topological spaces are more general than metric spaces. Let me write this down. Topological spaces-- so a way to see this is that the definition of a topology, which we'll get to in a moment, is simply a definition of what it means to be open. And of course, we have a notion of what it means to be open under the metric. You can just cover it with epsilon balls and everything's fine. And so topological spaces are far more general than metric spaces, and there's quite a bit to study here.

So let me introduce what the definition is of a topological space. So a topological space is simply a set that has a topology on it. And I'll define what it means to be a topology in a moment. A topology T on a set x is a collection of subsets of x such that, one, the empty set and the entire met-- or not metric space-- the entire set is in your topology. Two, if I let Ti be a set or a collection of subsets of T, then the arbitrary union of them-- Ti is nT. And three-- the finite intersection of open sets-- Ti-- are nT.

So again, just to reiterate, a topology is just a collection of subsets of the metric space X-- or not metric space-- the set X such that the empty set and the entirety of the set is in the topology. Arbitrary unions of points of the topology are in the topology. And finite intersections are as well. And notice that these are the same as the topological properties of open sets we defined in metric spaces, which is a good thing because this is what we would want to have if, in fact, metric spaces are more specific than topological spaces. And what is a topological space? Well, a topological space is simply a set with a topology on it. So I'll just write, with T, where T is the topology.

So with these three properties, we have notions of openness, in particular. In fact, one can define a topology-- this is more of a terminology thing. A set is open-- let's say a set A inside a topological space, X, is open if A is in the topology, and closed if X minus A is in the topology, which implies that the complement of a closed set is open. So that's an equivalent definition. But yeah, these are properties of what are known as open sets in this much, much more general setting.

Now, you might be asking yourself, why should we particularly care? This is a very abstract definition. But I'll remind you is that when we first started talking about metric spaces, we had a very, very general, somewhat abstract definition. It's one that, at first, seems not too bad. It's just the topological properties of open sets we've already discussed. But with this generality, we can prove quite a bit more.

Yeah, I already noted the topology on a metric space. It's simply-- simply unions of epsilon balls. If you're in a topology class, this would say-- this would be the same as having a basis of epsilon balls for the topology. But I'll leave it there for now. And in fact, when does-- when is a topological space, in fact, a metric space? That is known as a metrizable space. So a topological space, X, is metrizable if there exists-- if there exists a metric, d, inducing the topology on x-- topology.

So here, the topology on a metric space is induced by these epsilon balls. In more generality, a topological space is metrizable if there exists a metric inducing that topology. And now this notion of inducing is one that I'll leave for 18.901. But I just bring it up now because, as I mentioned on our second day, not every topological space is metrizable. Not every set that you're considering is going to be as nice as a metric space. Even though we can put a metric on any space, it doesn't mean that it gives us actually good information. So having metrizability is deeply important.

So let's take this time real quick to define the notions of convergence and open sets and continuity in terms of the topology, just like we did on our first day. We defined what a metric was, and then we defined all of the terms we were used to in terms of that metric. And so we're going to do the same now. One-- a neighborhood-- I should write this out the first time. A neighborhood of a point x inside your set, X, is simply an open set-- U containing x. Why do I bring this up? Because our definition of a convergence and of continuity will depend on this definition of a neighborhood.

So two, a sequence xn converges to a point x if for every neighborhood of x, all but finitely many xi-- many-- xi are not in the neighborhood. Now, recall that this is the same definition that we had for metric spaces. We had the dilemma that a sequence converges to a point x if and only if for every epsilon ball around it, all but finitely many terms are outside of it, which was a pretty direct lemma. So this is how we define convergence for topological space.

Three, we also state things about continuity. If I have a function f from a topological space x to y-- and I'll say that the topologies are on-- that the topologies are called Tx and Ty, then f is continuous. Or I should write this out-- continuous if, for every single open set, T and Ty-- f-inverse of T is in Tx. In other words, the inverse images of open sets-- the inverse images of open sets remain open, which is as we've shown, as well, the same definition of continuity we can have for metric spaces. So all of this is still very, very related to what we've been talking about.

So you might wonder why we even study metric spaces at all, then, right because topological spaces are much, much more general. If I can prove a fact about a topological space, I will have proven it for my metric space as well. And the short answer to that is, in my opinion-- the same answer to why would you study real analysis after calculus. In theory, real analysis shows all the facts you need to know about calculus. It proves it in more generality. But you gain a lot of intuition from dealing with calculus in the first place. Before you prove the mean value theorem abstractly, it makes sense to have an idea of what's happening before you even do so.

So very similarly here, having intuition about metrics and metric spaces provides a lot of intuition about topological spaces, even though most topological spaces are not metric spaces. It still gives you the framework to move forward. For this reason, that's essentially why, whenever I could, I would draw pictures of what's happening as a blob because then you get the intuition of, oh, how can I start to problem solve via a diagram? It just gives you that right frame of mind to move forward.

OK, now that being said, metric spaces, because they're much more specific than topological spaces, are a current area of research. Topological spaces are mostly done being researched. Topology is not a close field, but point set topology-- the basic framework-- is not as much of an active area of research. And metric spaces, on the other hand, very much so are. So that is yet another answer to the question of, why should I care about metric spaces if I could talk about topological spaces?

OK, so now we're going to talk about 18.102 unless you have some questions on the material we've talked about today. Cool. Did I write down 18.901? I didn't. I meant to do so. So this is the material that's covered in one of the first lectures of 18.901-- Introduction to Topology.

OK, 18.102-- Functional Analysis, or Intro to Functional Analysis. Functional analysis is all about studying what are known as normed spaces, and we've already talked about them before. Normed spaces are just a subset of metric spaces. We talked about this in lecture 3. It's slightly more specific than metric spaces. Normed spaces.

So again, recall a normed space is simply a set with a norm on it, or a vector space with a norm on it, denoted with absolute value bars. And specifically here, the three properties we want again are a positive definiteness. We want the norm to be bigger than 0 or, if it's equal to 0, for the point P 0 itself. Positive definiteness-- absolute homogeneity and the triangle inequality. These are the three properties we want on normed spaces. And on the homework, you've already shown that a metric induced by the norm is, in fact, a metric. So normed spaces are a generalization of metric spaces.

OK, now, with this definition of norm, we can, yet again, redefine all of our notions of convergence and neighborhoods and continuity as we've done before, so I'll quickly do so. The way to do it is just write it in terms of the metric. A sequence xn converges to x in a normed space if, for all epsilon bigger than 0, there exists an n in the natural numbers such that for all n bigger than or equal to n, the distance between xn and x is less than epsilon. This is the definition in terms of the metric space.

But recall that we can define this metric in terms of the norm. So if you're in a class like Functional Analysis, you'll probably just use this notation as opposed to going to the metric one. But this is the definition of convergence in a normed space. And again, we want all these definitions to be compatible. So if it feels like repetition, there's a good reason why it does. It's just-- we want a new definition but one that still works with the broader setting of metric spaces so that we don't have to redo all of our work.

Two-- of course, you can find Cauchy sequences in the same way-- simply ones such that-- for all epsilon bigger than zero, there exists an n to the the natural numbers such that for all n and m bigger than or equal to n, the distance between xn and xm is less than epsilon. So this is the definition of Cauchy sequences. And finally, we define a set to be open if and only if we can find a ball of radius epsilon around each point. So three-- a subset A inside of x is open if, for all x and A, there exists an epsilon bigger than 0, such that the ball of radius epsilon around x is contained in A. And in terms of the norm, this is the set of y such that the norm distance between x minus y is less than epsilon. So precisely the same as in a metric space. Now, to this point, as I was talking about last time-- so again, a set is open if and only if, for all x in A, there exists a ball of radius epsilon that's contained in A.

So just to continue off where we left off last time in terms of lecture 5, we started introducing completions of metric spaces because Cauchy completeness is very, very important, so much so, in the setting of functional analysis, that we have a name for it. A Banach space is a normed space that's Cauchy complete, essentially. The only difference is that it's complete with respect to the metric because, again, we have a notion of Cauchy completeness in terms of metric spaces, so we want it to be compatible. And as we talked about last time, you can take the completion of a normed space to get a Banach space. So Banach spaces are slightly more general-- or, sorry, slightly more specific than a normed space, but the tools that we use are very, very important.

So in particular, as an example of a Banach space, you can show that C infinity of a metric space M-- or even a normed space-- is a Banach space. We talked about this last time, but the proof is essentially the same as in this sense of continuity, where, again, this is a set of functions, f, that are continuous. And this n bounded-- so the supremum over m and m-- of f of m-- sorry, f of m-- is less than infinity. So you can show that this is a Banach space.

There's a few other classic examples, like Rn, Cn, instead of continuous spaces, or C0 AB. But most of them are based off of these four. These are the few key ones to have in mind. And these are so useful that it makes sense to use these as our main sense of intuition. In fact, we even have weirder examples of Banach spaces, and I'll introduce one of them now, where, specifically, you're interested in the dual of the vector space. Have you heard of the notion of dual before? No worries. Yeah, it's totally fair. In the case of norm spaces it's not too bad. It's the set of functionals. A functional is a linear map-- let's say T-- from your normed space, X, into the real numbers, or complex numbers if you prefer. So it's just a linear map from your normed space into the real numbers or the complex numbers. And in fact, this is the definition of a dual of a vector space in general. Just replace X with the vector space.

And in fact, you can show that the set of functionals are a Banach space. But to do so, we need to introduce a norm. In order to even have the possibility of defining or showing that a space is a Banach space, you need to have a notion of a norm. So the norm on it-- the norm on T, known as the operator norm, is simply the supremum over x in-- x and capital X with the norm of x equal to 1 of T of x. So it's just the largest image- sorry, the supremum of points on the ball of radius 1 around x. That's what the definition of the operator norm is. And you can show that this, in fact, in a norm. Homogeneity is not too bad. The triangle inequality is where things get worse, as always, but homogeneity and positive definiteness are nearly immediate.

So with this norm, then you can show that the set of functionals is a Banach space. And why is this important? Because it turns out that studying your norm space is pretty much analogous to studying your dual space. So studying this at a functional-- is you can redefine continuity. You can redefine open sets, things like that. You can show all of these properties. But for functionals, as a specific example, and the fact that operators are a Banach space is particularly really helpful.

OK, now, I want to note one more example of a norm space before I move-- or a Banach. Yeah, I want to give one more example of a norm space before I move on, which is one that I briefly talked about before. I'll write up here because we're done with topologies. Specifically, you can study what are known as inner product spaces, which is something-- which is a title that I-- which is a word I introduced last time. But I'll define right now-- it's basically the same concept as a normed space and a metric space. You just introduce an inner product on your set. And your inner product-- you can think of like a dot product on rn. Having the dot product was very, very helpful in rn because that lets you define things like magnitude, which is part of the reason why we introduced norms in the first place, is to introduce the notion of magnitude.

So an inner product space is a set or a vector space, X, with an inner product defined on it. And specifically, we want this inner product, as usual, to have three main properties. The three properties are-- and I'm going to assume-- I'm going to assume that the end product takes in two points in X and spits out a real number. So the three properties are symmetry. The inner product of x and y should be the same as y and x. Two, you want linearity. The inner product of ax plus by-- inner product with z is the same as ax inner product, C, plus by inner product C. So this is linearity.

And lastly, we want positive definiteness, but it's slightly weirder. If x is not 0, then the inner product of x with itself is bigger than 0. It's not an if and only if statement because, of course, we have notions of orthogonality from real-- sorry, from rn. But if it's non-zero, the inner product should be non-zero.

These are the three properties of an inner product space. And the reason I bring this up right now is because you can show-- consider-- just as we did in calculus, you can consider the inner product of x and x to the one half. This makes sense because it's positive, so taking the square root is totally fine. What you can show is that this induces a norm on x. The way that you show this-- so, in other words, induces norm.

So what you would want to show is that, given the inner product space, if I look at the square root of x inner product itself, you would want to show that this implies you have the three properties you want for a norm space, just as we did in the proof of metric spaces. And this is really important. And this shows up all the time, in fact, in quantum mechanics, if that's some part of math and physics that you're interested in. And in fact, I should say we can Cauchy complete it, as usual.

Once we have that our inner product induces a norm, we know that it then induces a metric. The one thing we don't know is if it's a bonding space or not. And that's really fine. We can just Cauchy complete it and get what is known as a Hilbert space. A Hilbert space is a Cauchy complete inner product space. So you can view it as a completion. You can also view it as just the definition. And here, Cauchy completeness with respect to the metric that's induced.

I'm going to rearrange this diagram real quick because not every product space is a Banach space, but it is a subset. So specifically, you can have something like this. Inner product spaces are a subset of norm spaces. And Hilbert spaces are inner product spaces that are Cauchy complete-- so therefore are Banach spaces. This is the end of my diagram. No more drawing squares all the way down. But I just wanted to show all these ideas are deeply related, and each one of them is interesting to study in its own respect. Now, granted, all of the center squares-- norm spaces, Banach spaces, and inner product spaces-- these are mostly talked about in functional analysis, which roughly makes sense because you need vector spaces in order to move forward. But yeah, because, again, functional analysis is done on normed spaces, and more specifically, usually, Banach spaces.

OK, I want to note one small application to 18.101, and then I'm going to talk about the application to differential equations. In 18.101, Analysis of Manifolds, and you're studying, of course, manifolds. Now, what is a manifold? A manifold is just some smooth enough blob-- and you can picture it in Euclidean space. To be even more specific, picture you have an orange. The main property that we like about an orange or a sphere is that, locally, it looks flat, just like it does in calculus. A function, locally, looks like a line. In higher generality, a manifold is just a space that, locally, looks flat.

The reason I bring this up is because, in the definition of a manifold, you assume, or at least you define a manifold-- you assume it is metrizable. It's a space with a metric on it, or at least a metric that induces the norm. So why is this so important? Why do we study metrizability in our manifold? Well, we want there to be a notion of distance on our manifold, which makes sense. The reason why metrizability I bring up here is because, in reality, when you're studying metrizable spaces, you don't often care what the metric is. Sometimes you do. Sometimes you want to say, oh, once there exists a metric, then x, y, and z. But most of the time, you just use the properties of the open sets.

So most of the time you just want to use the fact that balls of radius epsilon are open. And doing so lets you define smooth functions. It lets you define integration. It lets you define-- I had one more thing. I mentioned it. You can define integration, and you can define vector fields, as you might have seen in 1802. All of this is just letting us do calculus on weirder shapes, doing calculus on a sphere, doing calculus on a smooth enough tree, things like that. But really, all we really need is that the balls of radius epsilon are open to start doing so.

So sometimes, in 18.101, you won't see how metrics come up, in particular, but the intuition is still there. You still want to have a notion of distance on your manifold. So yeah, that's all I'll say about 18.101 because, of course, it takes a month to actually get into meaningful theory because you're redefining multivariable calculus. So I'll leave that there for now.

Yeah, the last example I want to talk about today is specifically the application to differential equations. And we already briefly talked about this. Last time, we talked about integral operators, specifically as an application of the Banach Fixed Point Theorem. So let me just write this down. Last time, we considered ODEs of the form. What was it? g of x plus the integral from a to b. kxy f of x f of y dy.

So the last time, we considered PDE that leads to this nice integral operator. If this is unfamiliar, that's totally fair. But I'm just bringing up the fact that we've already seen one more application of the Banach Fixed Point Theorem to differential equations. But in fact, you can use differential equations to motivate the notion of compact sets. I should have written 18.152-- Intro to Partial Differential Equations. So we can use the notion-- or we can use ODE or PDE to motivate compact sets.

And let me just briefly explain why this is the case. So picture some subset, omega. Subset, omega, of R2. And picture this as your metal sheet. So I just have some subset out here-- omega. I'm going to assume it's nice and connected. And I want to consider it as a metal sheet. Why do I do that? Because, from here-- let's say I just heat up one tiny portion of it. Let's say I take a little blowtorch and I heat it up right here. We want to know how the temperature is affected by this, in particular.

Let's say that u of xy is the temperature at xy and omega. What you can derive, using physics or using just differential-- looking at it from a differential viewpoint-- you can define-- you can show that the heat equation that you'll get from heating up this tiny spot is of the form, derivative of u with respect to x twice, plus derivative of u with respect to y twice. This is known as the heat equation.

And in fact, if it reaches-- if we let this metal sheet reach equilibrium-- if we let the blow torch heat dissipate to all over the metal sheet, then you'll get that this equation is equal to 0, which is the differential equation that we're particularly interested in. If I have a solution of this form, what do I know? This is, in fact, so important-- this operator-- that we just call it the Laplacian of u, if you've heard of that before, where the Laplacian is just the derivatives squared applied each time, and something over them.

OK, so the question is, what if I only know the temperature at the boundary? So here, the boundary is just-- we could just draw it out pictorially. It's the point that's just along the edge. Suppose, just along the edge, I knew the temperature and that's all I knew. So f equals u on boundary. And we denote the boundary as partial omega. This is just how we define. This is just the notation that's used. The question is, given this equation is true and given that I know the temperature values along the boundary, does there exist-- or I should just write out-- exist a u satisfying this? So specifically, does there exist a u such that the Laplacian view is 0 and such that used the function when restricted to the boundary? So this will be what I call Q1.

The question is quite a bit difficult to answer, at least immediately. In fact, that's how most PDE questions go. It's particularly difficult to show existence of a solution, but we can perhaps think about it more physically. What if, instead of viewing it in terms of its temperature, which can be a little bit weird, we try to minimize the energy? If we minimize the energy, then maybe, then, we'll have reached thermal equilibrium. And this is exactly what mathematicians did at the time. I'm sorry, one second.

Oh, I should note this question is known as the Dirichlet problem. You would have saw-- or you've likely seen-- a discrete version of this if you've done 18.701. On the first problem set-- sometimes professors include it. Sometimes they don't. It's totally fair if not. But yeah, the first question is, does there exist T satisfying this? And mathematicians, at the time, just tried to study the energy. So define the energy of a function, u, to be one half times the integral over omega of the gradient of u squared dA, where A's the area differentiable on A, on omega. This is known as the energy of the function u. You can think about it as the heat energy.

But now the question is, if I minimize this, is it a solution to the differential equation? And that's what mathematicians, in particular, did. They said the question-- question two-- does there exist a function u-- specifically, u in C2-- too because we want to be able to differentiate it twice-- such that energy of u is equal to the information of the energy? So it is equal to E inf, where E inf is defined as the infimum of the energy. So just take the infimum of this. Assume that I know that, in the end, the energy should be, insert value here. Does there exist a u that minimizes that energy?

And I'll come back to these questions in a moment. But the thing I want to know is that what mathematicians did is they said, oh, if there exists a function u that satisfies this, then it will solve the problem. But the issue here is that-- sorry, let me just read through this one more time-- what I'm trying to say. The issue here is that we still have to find this function. The question of existence is finding one that works. And what was done is they were taking sequences of functions-- so what was done is you would study functions un converging to u.

So you would construct un such that un would convert to the function, u, that you wanted. And once this was done, you would get that the energy of un converges to the energy of u. The proof of this fact just follows from taking the limit right here and applying uniform continuity. But yeah, this was the idea, and this idea did not work, weirdly. Why didn't it work? Because even if there exists a sequence of these functions in C2 that converges to u, how do we know that as C2? Question-- is u in C2? The answer is not directly, yes, because the C2 functions are not complete or they're not a Banach space-- or yeah, they're not a Banach space.

The way that you can see this-- so not necessarily-- not compact. The way that you see that it's non-compact is you just consider the sequence x to the n as your un. Each of these is differentiable twice, but un we'll converge, pointwise, to 1-- or, sorry, to u of x equal to 1 when x is not equal-- when x is in 0 to 1 and 0 otherwise, i.e. if it's equal to 0. And this is not twice differentiable. It's not differentiable at the point 0. It's discontinuous. So the issue is that this function wasn't in C2, necessarily. The way that you see that this isn't compact recall is by the fact that this would imply that it's not sequentially compact.

Now, what else could we do? Well, mathematicians then also showed that if the sequence of un in instead in C1, does that work? And what they were able to show is that, if the sequence is in C1-- sorry, if un was in C1 and this limit existed, then you can show that U was in C2, so retroactively, we'd be done. But the issue is, again-- sorry, not that u is in C2 but that each of the uns be in C2. The issue is that, again, C1 is not compact, so we're still not done.

Now, in the end, we were able to solve this problem with techniques that are quite beyond the scope of this class. It's sometimes talked about in 18.102. So C 18.102. Sometimes it's talked about there. It's talked about in the lecture notes that are on a CW, so if you want to read more about that there. But that's the very, very last lecture, so it's difficult, yes, but interesting nonetheless. But it's an interesting question. How should we be approaching these problems? Should we be approaching them physically, or should we be approaching them super rigorously?

And the answer is unclear. There's no direct answer to this question. This gave us the framework to think about compactness. Even though the solution didn't work, it motivated the development of compact sets for metric spaces. We have the C1 and C2 are metric spaces. We want to understand if limit points are in your set. And even though it wasn't the case for C1 and C2, it still developed all this terminology that, as you've already seen, is very, very important. So yeah, they were able to solve this problem eventually, just using much different techniques.

So that's all I've prepared for today. This was mostly just a broad overview of where the material goes from here. The intuition that comes from metric spaces shows up all the time. For instance, in FOIA analysis you want to understand if FOIA theories converge to your function and in which sense. And in that way, it's related to metric spaces. We've talked about how it applies to manifold theory. We've talked about functional analysis, topology, differential equations. These are five major applications of the material.

But the intuition that you gain from metric spaces will continue to work throughout your time at MIT. So for instance, in general, proving things axiomatically are functions that satisfy certain properties, like norms, inner products and metrics, is a useful skill. But even more in particular, I think a class like this and a class like Real Analysis is interesting because of all of these different subsets and bigger subsets-- or bigger subsets-- is the terminology.

We started off with studying Euclidean space, which was just a tiny little dot in the set of metric spaces. But there's obviously much more to be done here. We talked about metric spaces. But it makes sense to start right in the middle and then work your way out-- work your way out towards topological spaces and then work your-- or potentially also work your way towards norm spaces and Banach spaces in whichever order you choose. But the main important thing-- the main reason I've been teaching this class for two years-- is to highlight the fact that-- is to give you the tools to be able to move forward into studying topological spaces and norm spaces with some amount of intuition, as much as that's possible. OK, so unless you have any questions, we'll end 30 or 25 minutes early.