[SQUEAKING]

[RUSTLING]

[CLICKING]

PAIGE BRIGHT: So today, in case you were thinking that there wasn't enough theorems on Tuesday, we're going to prove a lot of them today on the general metrics based theory. I like this notion of a theory of the spaces.

This is a notation that Dr. Casey Rodriguez taught me, loosely speaking, where you're just trying to encapsulate what are all the things we can say about the most fundamental objects in the space that you're considering? And essentially, a number of the theorems that we'll prove are essentially the same as what we would picture in Euclidean space, but we have to be slightly more careful sometimes. And all of this theory is going to be building up to understanding something that you probably haven't seen before if you haven't done 100B, known as compact sets.

So we're going to be building up to talking about compact sets. But first, we have to understand what are the fundamental tools in our tool belt with metrics based theory? And I do want to note one thing.

Even though last time, I said we can put a metric on every set, which is true, not every set has a nice metric on it. Have any of y'all taken topology before? No worries if not. It's a class that usually comes after real analysis, but in topology talk, not every topological space is metrizable-- i.e. there's not necessarily a metric on it that will make the topology nice.

And we'll talk about what topology is a little bit today. So in case you're interested in what that statement is, we'll come to it, but the basic idea is even though we can put a metric on every set, sometimes the trivial one is not that interesting. So all right.

Let's go ahead and jump into the metrics based theory for today. So first off, we're going to be talking about convergent sequences, just like we did in RN. So throughout this, let xn and yn be the sequences we're considering.

In the first statement theorem, suppose xn converges to x. What should we expect about the point that it converges to? Well, in Euclidean space, we knew that this limit point had to be unique. And in fact, we can prove that this must be the case.

So then x is unique. To prove a statement like this, we choose a different point y in our set to consider. So what we're going to show now is that convergent sequences have unique limit points.

So proof. Suppose there exists a y in your metric space x-- or I should say sequences in your metric space xd, so metric x with metric d. And suppose that there exists a y and x such that xn converges to y. Our end goal is to show then that x must be equal to y, in fact.

Now, to do this, what do we know about equality for points? Well, by the axioms of what we're defining a metric space to be, two points are going to be the same if the distance is precisely 0. It's an if and only if statement.

So that's what we want to prove. We want to prove that the distance between x and y is 0. So how can we do that? Well, firstly, let's write down what the definitions of them converging to x and y is respectively.

So xn converges to x. Means that for all epsilon bigger than 0, there exists an n in the natural numbers such that for all n bigger than or equal to n-- and I'll call this n1-- we have that the distance from xn to x is less than epsilon. This is the definition of convergent sequence that we're dealing with.

And similarly, xn converges to y if there exists an n2 in the natural numbers for this epsilon such that for n bigger than or equal to n2, the distance from xn to y is less than epsilon. So now, what do we want to do? We want to use this to show that the distance between x and y is small.

And to do this, we will apply the triangle inequality to understand the distance from x to y. So by the triangle inequality, this is less than or equal to the distance from xn to x plus the distance from xn to y, where here I'm applying symmetry so that I can move the terms around, right? And we know that for all epsilon bigger than 0, there exists an n-- let's say this is equal to the maximum of n1 and n2 in the natural numbers such that both of these terms are now less than epsilon.

In fact, you know what? To make it nicer, I'll make this epsilon over 2 both spots. So then this is less than epsilon over 2 plus epsilon over 2, which is equal to epsilon.

Now, does this imply immediately that the distance between them is 0? Essentially, we have to note one more thing, which is that distances are positive definite. So this is going to be bigger than or equal to 0.

So what this tells you is that because the distance between x and y gets arbitrarily small and arbitrarily small and close to 0, it has to be 0. This is from the real number theory that we already know. So this implies that the distance from x to y is 0. And therefore, x is equal to y, which is what we wanted to prove.

So notice that a lot of this used all of the axioms that we were talking about. At one point, we use symmetry, which wasn't too important, but was useful to note so that we can swap xn and x around and apply the triangle inequality. We applied the triangle inequality, and we had to use positive definiteness. So these are the bare bones of what we really need in the theory.

All right. So now what we're going to do is show that not only do limit points exist nicely, but the distances between limit points act nicely. So to state that more clearly, theorem-- let y be an x and xn converge to x. Then my claim is that the distance from xn to y will converge from the distance from x to y, which makes relative sense.

But notice here the statement is slightly different. Before, we were purely dealing with sequences of a metric space. And so we had to deal with the metric d itself. But when we're saying that this converges to d xy, what space am I considering these distances in? Anyone?

No worries. We're looking at it in r, right? Because for every single n, this is just a real number. And so when I say the distance from xn to y converges to the distance from x to y, I mean that this happens in Euclidean space.

And in fact, you can likely-- I don't want to state explicitly, but I'm pretty certain can prove this as well in a general setting, so between two metric spaces. OK, so let's prove this. To do this, ultimately, we want to show that for all epsilon bigger than 0, there exists an n in the natural numbers such that the distance from xn to y minus the distance from x to y is less than epsilon in absolute values.

And to do so, we can just find an upper and lower bound on the distance from xn to y. So let's do that. The first direction is nice.

The distance from xn to y is less than or equal to, by the triangle inequality, the distance from xn to x plus the distance from x to y. And we can make this term arbitrarily small. So I'm just going to choose the same n to be the one for this convergent sequence.

So this is less than epsilon plus the distance from x to y. So this gives us the upper bound that we're wanting on the distance from xn to y. Let's prove the lower bound. And the lower bound is very similar, but it's just a slightly different manipulation, where what we're then going to do-- so the distance from xn to y-- or actually, I'll look at the distance from x to y.

This is less than or equal to the distance from x to xn plus the distance from x to y. And then what we can do is subtract these terms. So specifically, we'll get that the distance from x to y minus the distance from x to xn is less than or equal to the distance from x to y.

And we can make this term arbitrarily small. So this is less than the distance from x to y minus epsilon. And notice that this implies the result because then we have that the distance from xn to y is less-- oh, shoot. Did I get this wrong?

No, no. OK, yeah. So then the distance from xn y is less than-- sorry, this should be the distance from x to y. There we go. The distance from xn to-- let me double-check this. Distance from xn to y plus xn to y.

There we go. Sorry about that. I mixed up what they were converging to.

So what this tells us is that the distance from xn to y minus the distance from x to y gets arbitrarily small, right? I can write that out, if y'all would prefer, but that's just the next step. You write it in terms of the absolute values. OK? Any questions?

So not only are limit points unique, but the distances between limit points are also going to be unique, at least in the real number sense, which is pretty nice. I'll remark. One proposition, which is pretty interesting-- you can study this for two convergent sequences at once.

I'll state this in two parts. Suppose xn converges to x and yn converges to y. Then the distance from xn to yn converges to the distance from x to y.

This relatively makes sense, but we can also state this slightly differently, or a different type of this theorem. Suppose xn and yn are Cauchy. So they're Cauchy sequences. So they get arbitrarily close together, but don't necessarily have a limit point.

Then we know that the distance from x to yn converges. Notice that I'm not saying what it converges to. I don't know that Cauchy sequences have limit points in my metric space. And we'll get back to that in a moment.

But nonetheless, the sequence itself in the real numbers will converge. And that uses the fact that the real numbers is Cauchy complete. Now, I'm not going to prove these two facts. These are facts that are on your problem set, but I do want to point out-- because this happened a lot last year-- you cannot assume that the limit points in the second part exist.

You can note that this implication implies the top one by uniqueness because convergent sequences are Cauchy, as we're going to prove right now. So if you prove the second one first, then you can make a small remark. You have to explain why the limit point is the distance from x to y, but that's not too bad.

So let's prove that theorem. Is that the next theorem I wanted to say? Yeah, I'll do that one now.

Cauchy sequences-- ope, sorry. Convergent sequences are Cauchy sequences. Suppose xn is the convergent sequence we're considering, and that xn converges to x.

Then we know that-- and I'm going to say this over and over again, just like we do in real analysis, but at a certain point, it just becomes second-hand nature-- so therefore, for all epsilon bigger than 0, there exists n in the natural numbers such that for all n bigger than or equal to n, the distance from xn to x is less than epsilon. So what does this tell us?

Well, what we're ultimately interested in is the distance from xn to xm, right? Let m be bigger than or equal to n. What we're interested in is the distance from xn to xm to state that it's a Cauchy sequence.

So what can we do? Anyone? Want y'all to think about it because we want to bound this and show that this is less than epsilon.

AUDIENCE: Triangle inequality.

PAIGE BRIGHT: Exactly. The triangle inequality. This is less than or equal to the distance from xn to x plus the distance from x to xm. And we can make both of these less than epsilon.

Now, if I was really careful, I could have made these epsilons over 2. And then this would have been exactly epsilon. Most of the time, if you show it's less than a constant times epsilon, you're good. You would just have to relabel things.

So all good things here. But yeah, exactly. You apply the triangle inequality.

Oh. And then this is precisely what we want for Cauchy sequence, right? This implies that it is, in fact, a Cauchy sequence. So we're done.

Now, the real question is, if Cauchy sequences are convergent sequences-- and as we know from the real numbers, from the real numbers, we know that there it's Cauchy complete, but here we don't. And what do I mean by Cauchy complete, which I've stated a few times? Definition-- a space is Cauchy complete if and only if-- i.e. the definition-- the Cauchy sequences are convergent.

So this is the definition of what Cauchy complete actually means. And this is a very powerful fact, right? We use Cauchy completeness everywhere in real analysis to state that limit points exist, to state that differentiability was nice. Things like that-- we needed Cauchy completeness there.

Now, not every space is Cauchy complete, unfortunately, but the ones that we really want them to be are, in fact. So proposition-- this is on your homework-- we have that yeah, the set of continuous functions on 0 to 1 is Cauchy complete. This is a statement on your homework that I've broken up into individual steps.

So this should be a little bit nicer. But yeah, a very useful proposition to show. OK.

Now, let me go back to what I meant to do slightly earlier. OK. We have a few more things to say about convergent sequences. And to do so, I'm going to define what it means for a set to be bounded, right?

When we are looking at real analysis, one of the most useful theorems we had in our tool belt was the Bolzano-Weierstrass theorem. Now, we're not going to have an analog of it, but we still have some statements akin to it that we want to show. And this will be especially important as we build up to compact sets.

So definition-- a sequence xn is bounded by, let's say, b bigger than 0 if, for all n in the natural numbers-- oh, sorry. I should say if there exists a point p in your metrics base such that for all n in the natural numbers the distance from xn to p is less than B-- which is what we would expect. We want the sequence itself to be-- the distance between the point and your sequence to be bounded. That's what it means for a sequence to be bounded.

And if you want to picture this, I always like to draw little pictures in my notes about this. If this is our metric space x, here's our point p. And what this is stating is that there exists a radius large enough of radius B such that it completely contains x. Now, granted, what does it mean to have stuff outside of x?

Nothing, really. So really, we just mean the region in the middle is contained in the ball of radius B. But that's a nice picture, if you prefer.

And not only can we say this about sequences in general, we can also define what this means for a set. A set A in x is bounded by B if, for all p in x-- oh, sorry. If there exists a p in x such that for all A in A the distance from A to p is less than B. Does the set theory notation that I'm using make sense, everyone?

There exists, and for all? Cool. So this is what it means for a sequence in a set to be bounded. Why would I bring this up?

Because convergent sequences are bounded sequences. We have this from real analysis. But I'm going to prove it now in metric spaces.

So proposition-- or I'll just write it out how I've been writing out the other ones. If xn converges to x, then xn is the bounded sequence. OK.

Let's start off with the proof, which will start off, as all of our statements have so far, with the statement of what convergence means. In the statement of what convergence means, we're going to choose the epsilon that I want so that I'm not dealing with all the possible epsilons.

So I'm going to note for epsilon equal to 1 bigger than 0, we know that there exists an n in the natural numbers such that for all n bigger than or equal to n, the distance from xn to x is less than epsilon, which again, I'm assuming is 1. Now, does this complete our statement?

The answer is no. Here we only have that this is true for all n bigger than or equal to capital N, right? But what we want for our statement is to show that for every single natural number, this sequence is bounded.

But this isn't an issue because what we can note is that there are finitely many terms less than n in the natural numbers, right? So what we can do is that B be the maximum of the finitely many terms of the distance from xi to x, or, let's say, 1 for i between 1 and capital N. So what I'm doing here is I'm noting that here I have my convergent sequence.

I know that most of them-- in fact, infinitely many of them-- are contained in a ball of radius 1. And I only have finitely many terms outside of that ball. So what I'm doing is I'm saying, OK, well, it's either in this ball, or in the next one, or the next one, or so on and so forth.

This is what the statement is that we wanted to show. And we know that this is, in fact, finite, as there is only finitely many terms. So this is the B that it's bounded by. I can write that out some more if y'all would prefer, but this is essentially the statement. Cool.

So I think that's mostly-- oh, there's one more thing we want to say about convergent sequences, which is that their subsequences act nicely, which will be essentially how we want it to be. So proposition-- let xn converge to x and xnk be a subsequence of xn. Then my claim is that xnk is convergent.

And in fact, to show that it's convergent, we're going to show that it converges to x, which is what we should expect of a subsequence. Does everyone here know what a subsequence is? Happy to redefine.

Cool. So let's go with this statement. So proof-- notice that my subsequence here is arbitrary. I'm not going to rewrite that out, but we could choose any subsequence of our sequence.

And what we're interested in is we want to show for all epsilon bigger than 0, there exists an n in the natural numbers such that for all nk now bigger than or equal to n, the distance from xnk to x is less than epsilon. And now, again, any guesses as to what we should do?

Triangle inequality. Precisely. Yeah. So we're considering the distance from xnk to x.

What we can do is apply the fact that we know that convergent sequences are Cauchy. So we can write this as less than or equal to the distance from xnk, let's say, to xm for m bigger than or equal to n plus the distance from xm to x. And now, I can choose a larger natural number, if necessary, such that this term gets less than epsilon over 2.

And this term, because it's Cauchy, is less than epsilon over 2. So without writing all those steps, this is less than epsilon. So this shows that the distance from xnk to x gets arbitrarily small. So therefore, it's convergent.

Yeah. This is fairly important. And it highlights one of the many ways in which Cauchy sequences are deeply important, as we'll talk about in the fifth lecture in the specific module of this class. Having things be Cauchy complete-- really, really helpful.

OK. Oh, one more note. This is something that I vaguely noted on Tuesday, but these are all of the major theorems and propositions that we're going to need for convergent sequences. You might be wondering why there aren't more.

We had a ton more in real analysis. In fact, it took a month of our time. The reason we don't have more is because the real numbers are a vector space. I can add two real numbers and get a real number.

I can multiply them and get a real number. Everything there was nicer, and we had addition defined. So one thing we could state in real analysis is that for instance, the sum of two convergent sequences is convergent. Here that doesn't make sense, necessarily.

You can't always add two points in a metric space and get a point in your metric space, let alone have addition be well defined. Similarly, we have the squeeze theorem in real analysis. We have a version of it based off of distances, but we don't have a squeeze theorem for points because we don't necessarily have an ordering on our set, right?

You can picture the complex numbers. The complex numbers don't have an ordering on them. You don't have a sense in which one complex number is bigger than the other. Yeah, I'm just going to leave it there.

If you've done complex analysis, then you'll know what I'm talking about. But yeah. So that's why we don't have more theorems as we used to.

So now, we're going to move on to the next major theorem that I've talked about, or the next major definition that I've talked about. We've talked about convergent sequences. We've talked a little bit about Cauchy sequences.

Cauchy sequences are helpful, but in terms of a metric space, we've mostly stated what we need for the moment being. Now, we're going to move on to open sets. So I'm going to recall what this definition is because it's the one that can be a little bit the weirdest.

So recall-- a set A contained in your metric space x is open if, for every single point in A, there exists an epsilon bigger than 0 such that the ball of radius epsilon around A is completely contained in your set. And this is the set of y in your metric space such that the distance from y to A is less than epsilon. Just to bring it up because sometimes, the notation comes up, this is also sometimes denoted the ball around A of radius epsilon.

So use whichever notation you prefer. Here-- I can lift it a little bit up. So use whichever notation you prefer. I prefer this one just because it means less commas, but whichever.

OK. So open sets are going to have a huge connection between topology, and continuity, and other definitions that are important. Now, the first thing that we're going to prove is topological properties of open sets. So because y'all haven't taken topology, this is, in fact, really, really helpful.

We're going to show three major properties of open sets in metric spaces. So a theorem-- and you can call this topological properties, if you want. Or I'll write it out because-- topological properties of open sets.

OK. Firstly, we have that the empty set and the entire metric space itself are open. Two, given Ai are open sets, then the union of all of these sets from i equals, let's say, 1 to infinity-- or I'll just use 0, why not-- is open. So in other words, the arbitrary union of open sets is open.

And finally, the finite intersections of open sets are open. In other words, if I intersect from i equals 0 to capital M of Ai, then this is open. The fact that we can't intersect infinitely many of them will become apparent in the proof.

So in the proof of these three properties, it will become apparent why we can't do infinitely many of them. But let's see this. One, so consider the empty set.

How do we know that the empty set is open? Well, it's vacuously true. It's true all the time because for every single point in the empty set, there are no points in the empty set. So as soon as that statement's false, we have that the empty set is open. So this is open vacuously.

Two, let's consider the set x. Well, for all epsilon-- or sorry, for every single point x in x, does there exist a ball around x in x? Yes, there is because just pick your favorite epsilon bigger than 0.

And then the ball of radius epsilon around that point x is, by definition, a subset of your metric space, right? Because the definition of a ball of radius epsilon is only the point in x. So we're not going to end up with some weird thing outside of x.

So this shows that x itself is open. Two, we want to show that the union of open sets is open. How do we do that?

i equals 0 to infinity is open. This is what we want to show. Well, as with the definition, we're just going to pick a point in the union. So pick arbitrary x in the union-- equals 0 to infinity of Ai. And we want to show that there exists a ball of radius epsilon around x that's contained in the set.

Well, because it's in the union, there has to exist at least 1 Ai that contains x. So therefore, there exists an Aj in your set of open sets Ai such that x is in Aj. And then we apply the fact that Aj itself is open. So therefore, there exists an epsilon bigger than 0 such that the ball of radius epsilon around x is contained in Aj, but this is a subset of the union.

So this shows that the ball of radius epsilon is contained in the infinite union, or potentially infinite union. This also works if it's finite, notice. But that's just a subcase. OK.

And can I squeeze it in here? No, I cannot. So I will move over here.

Any questions so far? Actually, let me move this. Any questions?

Open sets are weird in that when we learn about them in real analysis, I feel like they're pretty unintuitively important. Everyone says that they're important, but I never got why. And I'll explain why they are just in a moment.

But yeah, open sets are deeply important in real analysis and topology. And we're going to prove some more statements besides these topological properties that will be important and will relate to the major definitions in our toolbox. OK.

So now, we're interested in the third case, the intersection from i equals 0 to m of the Ai. Well, we're going to do exactly as we did before. If I have some x in the intersection of the Ai from i equals 0 to m, then what do I know? Before, we knew that it was in any of the Ai's.

But here we know even more information. Here we know that x is in Ai for every single i. So what can we do?

Well, we know that therefore, because each Ai is open, there exists an epsilon i bigger than 0 such that the ball of radius epsilon i around x is contained in Ai. The issue is that the entirety of Ai might not be in the intersection. So what can we do?

We have this for every single i. And we have finitely many of them. So choose epsilon to be equal to the minimum of the epsilon i's from i equals 0 or 0 to M-- capital M. This is bigger than 0, right?

Because we're only choosing finite many. This is where we're using the fact that there's the finite intersection. If you had an infinite intersection, you can picture taking more and more, smaller and smaller epsilon i's such that this infimum is 0. And that's an issue, but if it's finitely many, we can take the minimum of them.

And then what we know is that the ball of radius epsilon around x is a subset of Ai for all i because it's contained in the ball of radius epsilon i. Or I guess I should say that this is contained in the ball of radius epsilon i of x, which is contained in Ai for all i, which implies that the ball of radius epsilon around x is a subset of the finite intersection. All right?

So why is this called the topological properties of open sets? Because if you were to take 18901, which is a class that I would highly suggest you take at some point if you're interested in this sort of analysis, as we're going to see, open sets allow us to state things about continuity. You can define continuity in terms of open sets. You can define convergent sequences in terms of open sets.

Basically, everything we can redo in terms of open sets. And that is a topology. If I have a set of subsets such that these three properties hold, that's known as a topology. And in 18901, that's the basic toolbox that you're given. And then you go from there in terms of redefining everything.

So this is just an example of a more general thing than metric spaces. I'm going to talk a little bit about that in the final lecture, where I talk about where things go from here. But for now, this is the basic idea. All right.

Now what I'm going to state is some facts about closed sets. But what is a closed set, which I haven't defined before? So this is useful to know. A subset A in x is closed if the complement of A, which is known as the metric space x minus A, is open.

Now, if a set is closed, does that imply that it's not open? Yeah. The answer is no. How can we see that?

Well, let's consider the real numbers as an example. Notice-- what's the complement of the empty set in the real numbers? Anyone?

It's a good exercise in thinking about what the complement means. It's the real numbers minus the empty set. So it's just going to be all of the real numbers.

But this, as we've stated before in our first topological property-- this is open. So therefore, the empty set is closed by definition. But we also know, by our topological properties, again, that the empty set is open. What the heck?

Well, this is just one of those things in topology. It's weird. Open sets-- you can prove properties of closed sets instead, and everything's fine. I'll note-- there's a notion called connectedness, which I'll briefly define, but ultimately will be more of a problem set problem.

But if your metric space is connected-- let me state it that way. If your metric space x is connected, which I'll define in a moment, this is true if and only if the only open and closed sets-- so sets that are both open and closed-- are the empty set and the metric space itself. Now, what does it mean for your metric space to be connected?

We have a nice picture of it in our heads because connected is a pretty visual thing, but this is one definition. Once you prove this, this is a definition. How do you prove this?

Well, you prove it based off of the definition of disconnected. So definition-- x is disconnected by definition if there exist two open sets, U1 and U2, that are disjoint-- disjoint-- and nonempty such that the union of them is exactly x. This is what it means for your space to be disconnected.

And from here a set is connected if it is not disconnected. So it's circular reasoning. And then once you have that, you can prove this note, which I'll put in the third problem set, which is an interesting problem. But I just want to note-- showing something is closed does not necessarily imply that it's open.

I'll give one short example of why this is true. And then I'll move on to more properties of open sets. Anyone have questions? Cool.

Feel free to shout out questions if you have any, or interrupt. I'm more than happy to be interrupted. OK, so I'm just going to give one example of a metric space that is disconnected, which is one that makes relative sense, at least on the face of it.

Example-- I can take a unit of two open intervals, CR to 1 and 1 to 2, and give it the usual metric on R. What do I mean by usual metric? I mean the distance between two points in the set is just the distance normally in R-- absolute values.

This is an example of one that's disconnected, right? It is the union of two open sets that are disjoint. And you can notice here the interval 0, 1 is both open and closed because the complement of it is just 1, 2, and that's open.

So this is an interesting example to think through. This is known as the subspace metric or the subspace topology, technically. But yeah. OK.

Let's go back to proving things about the general theory of open sets. Proposition-- I'm first going to point out that closed sets are very similar to open sets. In fact, we can define everything in terms of closed sets, if we wanted to. Properties of closed sets.

Here we have that the empty set in x are closed. Two, the potentially infinite intersection of closed sets, Ai closed, is closed. And three, the finite union of closed sets is closed.

Nearly identical to the topological properties of open sets. How do we prove this? It's not too bad. I'm not going to actually do it.

The fact that the empty set and x are closed is just the proof that I did above for R, right? You can replace R with just any metric space. And we have that this first property holds.

How about the other two? Well, for the other two, I'm just going to quickly note what are known as De Morgan's laws. So this is a nice lemma from set theory.

These are known as De Morgan's laws. I'm going to switch which chalk I'm using. This states that the complement of a union is the intersection of the complements-- so i am A. And similarly, the complement of an intersection-- Ui complement i and A-- is equal to the union of the complements.

These are known as De Morgan's laws. They make relative sense? You can prove them for two sets as opposed to infinitely many of them. But once you have these two properties, then you can rewrite these two properties in terms of open sets. That's why there's this duality, right? I can write the intersection of closed sets as the union of open sets.

That's how you apply De Morgan's laws. And I'm not going to prove that, or I'm not going to prove these topological properties. This is in the lecture notes, but I do want to note-- you could do everything in terms of closed sets if you really truly wanted to.

We just don't because open sets are nice. OK. We'll show why some of those things are actually nice.

What more things do I want to note? OK, cool. Now, I want to state some specific examples of open sets in metric spaces, which we can already do quite a bit of.

Example-- did I erase it already? Oh, wait, no. I did it here. Where did I do it?

Oh, I did at the very top. The ball of radius epsilon around A is sometimes referred to as an open ball, but you should prove that the ball is, in fact, open, which will be our first example. So given some point x in your metric space x and epsilon bigger than 0, we have that the ball of radius epsilon around x is open.

How do we do this? Well, we choose an arbitrary point in your ball of radius epsilon, right? So proof-- choose some y in the ball of radius epsilon around x.

We want to find a ball of radius, let's say, delta around y contained in the ball of radius epsilon round x. That's the definition of open set. Well, notice that the distance from x to y is less than epsilon by definition, right?

Because we're assuming that the distance between points in our ball is less than epsilon. So let delta be the distance from x to y over 2. Why does this work? Because then we'll notice that the ball of radius delta around y has to be contained in the ball of radius epsilon around x.

Why is this true? Essentially, the triangle inequality, right? We have a ball of radius epsilon centered at x.

We have a point y here. And I'm noting that the distance from here to here is less than epsilon, by definition. And I'm just choosing a ball around y to be 1/2 of that distance. And then you can apply the triangle inequality to show that this ball must actually be contained in the bigger one.

Anyone want me to work through that? I'm happy to. Oh, wait a moment. Let me write it up.

You know what? I'll write it out on the back board. So let z be in the ball of radius delta around y. Then we have that the distance from x to z is less than or equal to, by the triangle inequality, the distance from x to y plus the distance from y to z-- distance from x to y.

And the distance from y to z is less than the distance from x to y over 2. Is this what I wanted to do? Oh, I should say, no. This is not what I wanted to do.

Let R be the distance from x to y minus epsilon. Then what we know is that the distance from y to z is less than R. What am I trying to say?

Distance from y to z is less than-- or sorry, this should be epsilon minus the distance. There we go. And this is then less than epsilon.

So this is in the lecture notes. I would highly suggest drawing up the picture and working through it separately. But loosely speaking, it's just this diagram on the right-hand side. It's an application of the triangle inequality.

OK. So the ball of radius epsilon is an open ball. In fact, you can state much more than this. This is an optional problem on your second problem set.

And for the record, I do suggest y'all look at the optional problems because they will potentially give you more intuition, even if you don't solve them. But yeah, this is just one of those cases where this isn't a topology class, so I'm not going to ask that y'all do a bunch of topology, but it's pretty nice.

OK. So just a small note. Any open set U contained in your metric space can be written as a union, a potentially infinite union, of open balls. I'll outline briefly how to do this right now.

So if y'all want to do it later, you can do it. But the idea is that for any single point in your open set, there exists a ball of radius epsilon around that point. And then you can just take the union of all those balls of radius epsilon. The epsilon might change for every single point, but that doesn't matter, right?

Because it's an infinite union. So I'll let y'all work through those details, but this is just a proposition you can show. OK, let's also just prove an example of a closed set, which will come important for next time.

Just an example. Oh, sorry. I should say let x be a point in your metric space. Then the set x is closed.

So this is just a singular point in your metric space. How are we going to prove this? Well, we only have one definition of closed. We have that the complement is open, right?

So proof-- consider the complement of x, which is x not including x, and let y be a point here. What we want to show is that there's a ball of radius epsilon-- want to show-- around y-- or not in-- a subset of x not including x. Let's draw a picture of what this should actually look like, right?

Well, I didn't want to do that. Here's our metric space. Here's our singular point x. And here's our point, let's say, y.

We want to choose a ball around y that doesn't contain x. How can we do it? Anyone?

You could choose whatever epsilon you want, so it's a question of what epsilon you want to choose. We can choose epsilon to be 1/2 the distance between x and y. I think that's what I was mixing up on the other proof, but it's in the lecture notes, so I'll let y'all do the other one.

But choose epsilon to be the distance from x to y over 2. And then pictorially, we have the rest of the problem, but let's actually work through the details. So then let z be in your ball of radius epsilon around y.

Then we want to show that z cannot be x, right? Well, notice-- this implies that the distance from y to z has to be less than epsilon, right? This is by construction, but can we get a lower bound on this?

We can. We can note that the distance from-- I want to make sure I'm getting this right. Actually, you know what? I'll do this directly, or I'll do this by contradiction.

Suppose, for the sake of contradiction, x was in the ball of radius epsilon around y. Then this would imply that the distance from x to y is less than epsilon, but this can't be the case because again, we're assuming that epsilon is the distance from x to y over 2. And the distance from x to y cannot be less than the distance from x to y over 2. So that is our contradiction.

I'll make one more small note here, which will become important for Tuesday, which is, as opposed to looking at a singular point, you can do this with a finite union of points, right? And the argument would be essentially the same as our finite intersection of open sets argument. You have a finite Union of open sets-- or sorry, a finite union of closed things. So it should be closed.

In fact, you could use that to prove it immediately. But you can also prove it directly, like this. OK.

So now, what we're going to do is talk about how this definition of open sets relates to convergent sequences and to continuity. Proposition-- let xn-- to do this, I'm going to do it in the real numbers, but note that everything can be done in your metric space. I just want to use the notation from the real numbers.

Let xn be a sequence-- am I writing it backwards? I am-- in R that converges to x. So then this is true, I claim, if and only if, for all epsilon bigger than 0, all but finitely many xi are in x minus epsilon to x plus epsilon. So this is the ball of radius epsilon around x, right?

How do we prove this if and only if? Proof-- well, OK. I should say convergent sequence. Let's prove the forward direction first.

So if xn is convergent, then for all epsilon bigger than 0, there exists n in the natural numbers such that the ball of radius epsilon around x-- I should say such that for all n bigger than or equal to n, xn is in the ball of radius epsilon, x plus epsilon. This is true because the distance from xn to x is less than epsilon. This is just the open set way to view it, right?

And this capital N-- notice that there are only finitely many terms less than it. So this shows that infinitely many of the terms are in this open set. And in fact, only finitely many are outside of it. So that proves the forward direction.

Let's prove the opposite direction. I should write-- suppose, for all epsilon bigger than 0, all but finitely many xi are in x minus epsilon to x plus epsilon. I want to choose a sequence that will converge to x then, right? Well, how can we do that?

We can construct a subsequence that is going to be convergent. And remember, a subsequence, if it's convergent, converges to what the original sequence converges to, right? So this is where we're using a fact from earlier that we proved.

OK. So suppose epsilon is 1 over n for n in the natural numbers, or I'll say m. Then choose x and m in the ball of radius epsilon-- so 1 over m to x plus 1 over m. What does this tell us?

I claim that this convergent sequence converges to x. Well, to show that, we need to write down the definition. When Professor Rodriguez was erasing the board, he'd always tell some joke, but I don't have any jokes right now, unfortunately. Though he would do them two at a time-- so he could actually have time to actually say the joke before he finished.

OK. So we want to show that x and m converges to x, right? Proof-- for all epsilon bigger than 0, we can choose M large enough-- or I should say capital M-- such that for all nm bigger than or equal to M, we have that 1 over nm is less than epsilon, right? Because epsilon is bigger than 0, I can find a large enough natural number such that 1 over that natural number is less than epsilon.

That's a fact from real analysis. So what this tells us is that we know, then, that x and M is in x minus epsilon to x plus epsilon because this is from x over 1 over nm to x plus 1 over nm, which is a subset of the ball of radius epsilon, by construction, right? Because 1 over nm is less than epsilon.

So what this tells you is that the distance from x nm minus x is less than epsilon, which implies that the convergent sequence or the convergent subsequence converges to x. So to reiterate how the proof went-- we had a convergent sequence. We had a subsequence that we showed converges to x.

And by what we showed earlier, x must be the limit point of the original convergence sequence, then. So we're done. All right. I won't write out the analogous statement for metric spaces.

You just change x minus epsilon to x plus epsilon with the ball of radius epsilon. I just stated it in real numbers because I prefer the interval notation to think about it. And it's a fact that not everyone fully sees in real analysis because sometimes, you don't know what an open set is in 100A. It depends on who's teaching it, realistically.

OK. We're going to prove one more-- oh, wait. What do I want to say? Yeah, OK.

This was just how I wanted to show that it's related to convergent sequences. Now, what I'm going to do is show that it's related to continuous functions. And I'm going to recall what that definition is because that can, again, be a little bit weird between metric spaces.

So a function f from a metric space x to a metric space y, let's say, with metric dx and dy, is continuous if, for all epsilon bigger than 0, there exists a delta bigger than 0 such that if the distance between two points x and y is less than delta, then the distance in the metric space y of the images of x and y is less than epsilon. This is the definition of continuous in the metric space setting. And so now what we're going to do is prove a statement akin to convergent sequences, but for continuity.

We can write the definition of continuity in terms of open sets. And not only can we do that, we can write continuity in terms of convergence of sequences. So I'll do that right now.

Theorem-- under all the same assumptions here I'm going to let x and y be metric spaces with distances dx and dy. And yeah, OK. Claim-- f from x to y is continuous at a point c in x if and only if xn is a sequence that converges to x. Then f of xn converges to f of x. This is my statement.

When I say continuous at the point c, I mean just replace y with c here. That means that this is continuous at every point, continuous at x in x. And it's continuous if it's continuous everywhere, right?

This is the same as in real analysis. We had continuity at a point and continuity everywhere. This is just continuity at a point. OK.

Proof-- I always prefer going the downwards direction first. So let f be continuous at c. Then by definition, for all epsilon bigger than 0, there exists a delta such that this implication holds. What does this actually tell us?

Well, I'm going to suppose the thing in the second hypothesis. Suppose xn is a sequence in x that converges to x. Then what can I say?

Well, we know that there exists-- or sorry, I should say to c. Then by definition of convergent sequence, we know that for all epsilon bigger than 0, there exists an n in the natural numbers such that for all n bigger than or equal to n, the distance from xn to c is less than epsilon. But we know that for all epsilon bigger than 0, furthermore, there exists a delta bigger than 0 such that the distance from f of xn to f of c is less than delta over 2-- or sorry, not delta over 2, just delta. No.

I should have relabeled this. I should have said for all delta bigger than 0, there exists one such that this is less than delta. And then there exists an epsilon bigger than 0 such that-- no, no. I said it right the first time.

For all epsiolon bigger than 0, there exists a delta bigger than 0 such that this term is less than delta. And furthermore, the distance between the images is less than epsilon. There we go. That's what I meant to say.

And so what this tells you is that f of xn must converge to f of c, right? Because there exists an n in the natural numbers such that the distance between the two points is less than epsilon, which is the definition of convergent sequence. And then to prove the other direction, I'm just going to state it because-- are we running out of time?

No, I'll write it out. I'll write it out over here. To prove the other direction, we're going to use it by contradiction. We're going to assume that it's not continuous at c, and then choose a sequence that doesn't converge, or such that the images don't converge.

OK. If and only if. Suppose f is not continuous. This is the proof of the upwards direction. Suppose it's not continuous at c.

What does this tell you? Well, saying something is not continuous this is a little bit weird, but I'll state out what that definition means. Given some epsilon bigger than 0, we know that there exists an xn in the sequence such that the distance from xn to c gets small. Do I want to say over n?

No, I'll just say less than-- given epsilon bigger than 0, there exists a delta bigger than 0 in an xn in your sequence such that the distances get small-- so let's say 1 over n. But the distances between the images of them, f of c, is bigger than epsilon, right? Because the opposite would be if this was less than or equal to epsilon, right?

And so then what you can show is that therefore, xn converges to c, but f of xn does not converge to f of c. How do we know that it doesn't converge to f of c? Because we just proved up here that if a sequence is to converge all but finitely many of them, i.e. infinitely many of them-- or that's not quite the same-- but all but finitely many of them must exist in any ball of radius epsilon, right?

And we just showed here that there are, in fact, infinitely many that don't exist in a ball of radius epsilon. So that proves the other direction. This is known as sequential continuity, sometimes. If you want to study this in more generality, as opposed to assuming continuous, you can assume this definition on a topological space.

And that's slightly different. But for metric spaces, the two definitions are the same, which will be the case a lot of the time. A lot of the time in this class, a definition that's specific to metric spaces will imply other very useful definitions that might not hold in more generality. OK.

I just have one more statement to show, which is how continuity relates to open sets. Let me write out what this lemma will be. Oh, sorry.

This is poor board manners. I should have raised this and lowered this. OK, OK. Just one more lemma.

f is continuous at c if and only if-- oh, I need to state a definition first. I'll state it over here. Definition-- a neighborhood of a point y is simply an open set that contains y-- containing y.

For our purposes, though, you can just think of this open neighborhood as a ball, right? Because I've stated that every open set can be written as a union of infinitely many balls. So this is what a definition of a neighborhood is, and that lets us state the lemma, which is for every open neighborhood of f of c-- this will be our y-- then we have that the inverse image, f inverse of-- oh, sorry.

I should say every open neighborhood U of f of c, f inverse of U is an open neighborhood of f of c-- oh, sorry, of c because we're in the inverse image. OK. Has anyone seen this definition before regarding open sets? Yeah.

It doesn't come up all the time because it is essentially topology. If you take a topology class, you would see this definition of-- this would be the definition of continuity in a topological class. But here we're going to actually prove it for a metric space. And because not all of y'all have seen it before, we'll just go through that proof.

So proof-- we'll prove the downwards direction. So we have that f is continuous at c. And I'm going to let U be an open neighborhood of f of c.

And so then I want to show that f inverse of c-- oh, sorry, f inverse of U-- is an open set of c, open neighborhood of c, right? What I'm going to do instead-- instead of considering this entire open neighborhood, I'm going to consider a ball of radius epsilon around f of c. So because U is open, we know that there exists an epsilon bigger than 0 such that the ball of radius epsilon with metric dy around f of c is contained in U, right?

We can find a ball around f of c such that is contained in U because it's open. What does this tell you? Well, by continuity, furthermore, there exists a delta bigger than 0 such that if the distance from x to c is less than delta, then the distance of f of x to f of c is less than epsilon. What does this tell you?

Well, this tells you that the image of the ball of radius delta around c is contained in the ball of radius epsilon around f of c, right? Because if I look at the images of these points, I'm going to get that the distance is less than epsilon around f of c. And what you can check-- I won't state this right now, or I won't prove it-- but you can check that if I do f inverse here, it's going to be contained in f inverse here.

Why is this helpful? Because this is a subset of f inverse of U. So what we've done is we've gone from an open neighborhood of f of c to an open neighborhood of c, which is what we wanted to show, right? We wanted to show that f inverse of U is an open neighborhood of c.

OK. So we now need to prove the other direction, which is sometimes a little bit harder. Only a little bit harder, but should be able to fit right here. OK, so now we're going to suppose that let epsilon be bigger than 0.

And consider the open neighborhood of f of c given by the ball-- and consider the ball of radius epsilon around f of c. Then we know, by assumption, that f inverse of this ball of radius epsilon of f of c-- 1, 2, 3-- is an open neighborhood of c. But because this is an open set around c, there exists a delta bigger than 0 such that the ball of radius delta around c is contained in f inverse of the ball of radius epsilon of f of c, right?

Because this is an open set. And therefore, we know that there exists a radius such that the ball of radius delta of that radius is contained in the open set. And then we're done, right?

Because then we can apply f to both sides. And we get that f is a ball of radius delta around c is contained in the ball of radius epsilon of f of c, which is exactly the statement of continuity. You can just write this out in terms of the metrics, right?

Because then the distance from x to c being less than delta implies that the distance from f of x to f of c is less than epsilon. So this concludes the proof. OK. This was the general theory of metric spaces.

We've talked about convergent sequences, which was essentially just like the real numbers. We talked about open sets-- specifically, the topological properties of them, which is important for topology. And then we related those two concepts to continuity, right?

We talked about how continuity is sequentially continuous at a point c if it's continuous on the metric space. And we've talked about how continuity can be defined in terms of these open sets. This is by no means a simple connection. It can take a while to feel comfortable with these ideas, but I highly suggest, if you're finding this a little bit confusing, to try and draw out the pictures.

Why does this conclusion hold? Why can I apply f to both sides and have things to be fine? It's just a practice in set theory, but it is helpful to do at some point.

OK? Starting next time, we're going to start talking about something you might not have seen before at all known as compact sets. Until then have a great weekend. Remember that the p set is due on the 13th, OK? All right, have a great day.