[SQUEAKING]

[RUSTLING]

[CLICKING]

PAIGE BRIGHT: Let's just go ahead and get started. So as a recap of what we've been up to, the first day, we talked about what a metric space is and went through a ton of examples, which was mostly what that day was supposed to be for.

The next day, we went through some of the general theory, which was helpful, I believe. But it's really short, all things considered, right? It's just a single lecture of most of what real analysis took up for us, right-- what are sequences, what are Cauchy sequences, et cetera. That's what the beauty of metric spaces is to cover now, because because we have done all of that work in real analysis, now we can simply state and prove the theorems that we know and love for metric spaces because they're very, very much analogous.

The one thing that wasn't analogous, though, was compact metric spaces. If you've never seen compact metric spaces, that's totally fair because if you only studied it-- if you only studied real analysis on Euclidean space, they're just closed and bounded sets. And those you can define without compactness.

But there's clearly a ton of important statements you can make about compactness. We had the four-part huge theorem last time, where we showed that topologically compact is the same as sequentially compact, which was the same as totally bounded and Cauchy complete, and another property with the finite intersection property.

But the main thing I want to talk about today is complete metric spaces because there's a ton can say about metric spaces once you know that they're complete. In particular, I want to talk about two specific examples-- one, the Banach fixed point theorem, which is a very useful theorem for differential equations. But secondly, I'm going to then talk about completions of metric spaces, which show up all of the time. So, yeah, let's go ahead and jump right in.

So to motivate the conversation of the Banach fixed point theorem, I'm going to start by introducing what are known as Lipschitz functions-- Lipschitz.

So a Lipschitz function is one that is continuous. It's pretty nice. It's a function f such that-- it's a function f from a metric space x to y such that there exists a K in the real numbers such that the distance between the image points, f of x, f of y, is bounded by K times the distance between x and y.

Now, let's think to ourselves what values of k are actually interesting here for a moment. One, if k is negative, then positive definiteness tells us that everything is just 0, which is particularly nice, right? But if it's positive, then we have to do a little bit more work.

But, yeah, this is the definition of Lipschitz, or K-Lipschitz, which is that there exists a K such that the distance between the image points is less than or equal to K times the distance of the original points. All right. So I claim that this is continuous, or functions like this are continuous. And that fact is not too hard to show.

So proposition-- "Lipschitz implies continuous. The proof of this is not too bad because we're going to let epsilon be bigger than 0. And we want to choose a delta such that the distance between the image points is less than epsilon, where the distance between these two points is less than delta.

But we can choose delta to be equal to epsilon over K, where K is the specific constant that works for our Lipschitz function. So then what we'll have is that the distance between the image points, f of x, f of y, will be less than or equal to K times the distance between x and y.

But this, under assumption, is going to be less than K times delta, which is equal to epsilon. So this shows that the distance between points in the image gets arbitrarily small once you choose a small enough region around x. So this implies continuous.

But, in fact, it implies a little bit more than continuous. Does anyone know what it implies, the stronger thing? Totally fair if not-- it's something you might not have seen before.

Yeah, it's uniform continuity. So, in fact, Lipschitz implies uniform continuity because it doesn't depend on x at all. It only depends on f and epsilon.

So that's our next section of discussion. I'm going to redefine what uniform continuity means for metric spaces. But most of the theory will still hold up.

So definition, uniform continuity-- a function f from metric space x to metric space y is uniformly continuous if, for all epsilon bigger than 0, there exists a delta bigger than 0 such that if the distance between two points x and y in x is less than delta, then the distance between the image points in y, f of x, f of y, is less than epsilon.

And the key difference between this definition and the definition of continuity is that this delta does not depend on the point x. The definition of continuity is that for every x, there exists this statement.

OK. So let's prove some nice statements about uniform continuity, including but not limited to how it is affected by compactness. So proposition-- suppose f is a function from x to y continuous, and x is compact.

Then what we can show is that, in fact, f is uniformly continuous, which is something that we should expect, that we had continuity on bounded intervals implying uniform continuity-- so let's keep going-- uniform continuity on the open interval. So then, f is uniformly continuous.

The proof of this, I think, is particularly interesting because it's going to introduce something that we talked about a few lectures ago-- or reintroduce something. We're going to use the little big number lemma.

So proof-- first we're going to start by covering our metric space x by a bunch of balls, just like we would do if we're doing a proof using compactness. So for all c in x, there exists a delta perhaps depending on c, which I'll note was delta c, bigger than 0, such that the distance between x and c is less than delta c.

Let epsilon be bigger than 0, I should say. So then this implies that the distance between the image points is less than epsilon. So here epsilon is going to be fixed throughout this proof.

In fact, while I'm at it, I'm going to say that this is less than epsilon over 2 because we're going to use the triangle inequality later on in this proof. So what we know, then, is that these balls of radius delta c are going to cover our set x because we're just taking a ball centered at every point.

So the balls of radius delta c around c cover x. So what we know is by the Lesbegue number lemma, which was true for sequentially compact spaces, but we know that sequentially compact is the same as topologically compact. So by the Lesbegue lemma, there exists a delta bigger than 0 such that for all x in x, there exists a c in x such that the ball of radius delta around x is fully contained in the ball of radius delta c around c.

Now, does this make sense to everyone? I know that this was a huge statement with a bunch of quantifiers. But this is what the Lesbegue numbers lemma says. We are covering our set x, this compact set x, with these balls. And we can choose a delta such that for any point in x, I can squeeze this ball of radius delta into one of these sets in our open cover. That's what the Lesbegue number lemma says.

Now, why is this so helpful? Because we know that to do-- then we know that if the distance between x and y is less than delta, then, in fact, y is already in the ball of radius delta c, which lets us know that the distance between the image point of f of y and f of c is less than epsilon over 2.

And so now we can directly imply the triangle inequality, where here we'll have that the distance in y between f of x and f of y-- when the distance between x and y is less than delta, then this distance is less than the distance between-- or sorry - equal to the distance between f of x and f of c plus the distance between f of c and f of y.

But both of these are less than epsilon over 2. So this delta, remember, from the Lesbegue number lemma, does not depend on x. So we've proven uniform continuity. Everyone follow? Cool.

I think that this is a very interesting statement. And let's look at an example of how we can actually use this. It's going to be a very powerful application. Specifically, it's an application to integral operators, which the name of it will become apparent by the proposition itself.

So proposition-- let f be a function from the interval a, b cross c, d to R continuous. So, in fact, what we know right now is that this implies that it's uniformly continuous, which is great. That would be helpful.

But two, what we're going to look at is the function g of y being equal to the integral from a to b of f of x, y dx. So, in other words, we're considering the function g of y where I plug in y into this integral.

And now what I claim is that this function g-- so consider g such that this is true. I claim that g is continuous. And the proof of this is going to be relatively short because we're going to use uniform continuity for the integral.

Now, if you haven't seen this before, it is very powerful. It's a powerful statement. And I'll say what the real analysis statement is that we're using in just a moment. But let's just go through the proof.

Let yn be a sequence in cd such that yn converges to y. To show continuity by our second lecture on the general theory, we want to show that g of xy-- sorry, yn-- goes to g of y as n goes to infinity because this will imply that g is continuous, right? This goes with our statement of continuity, otherwise known as sequential continuity.

The proof of this is not going to be so bad because we're just going to write out what the limit is. We're going to have the limit as n goes to infinity of g of yn. Well, this is the limit as n goes to infinity of the integral a to b, f of x yn dx by definition.

But by uniform continuity, for real analysis, we know that we can swap the limit and the integral, right? Has everyone seen this theorem before? It's very, very useful. If you need to relearn it, I would highly suggest doing so. But it's one that is mostly used with Lesbegue integration.

Or, specifically, the lemma uses Riemannian integration, which makes it kind of annoying to prove. So I will not do so here. But the point is that we can switch these two limits as n goes to infinity of f of x, yn.

But by the fact that f is continuous, we know that this is going to converge to f of x, y. So this is integral from a to b of f of x, y dx, which is g of y. So we're done with the proof.

It was a very simple proof. But we highlighted the importance of uniform continuity. We can simply state that we can swap the limits using uniform continuity. And then our job is nearly done.

But this integral operator is, in fact, going to come up a little bit later today. There's one on the face of it that is pretty straightforward but has very useful applications to differential equations.

All right. So now we're turning back to the definition of Lipschitz functions. I'm going to define what a contraction is, which is going to lead us to the Banach fixed point theorem, which is our main goal of this first section.

OK. Definition, contraction-- a function f from a metric space x to itself is a contraction if it is k-Lipschitz for k between 0 and 1, for k bigger than or equal to 0 and less than 1.

In other words, the distance between two points f of x and f of y is less than or equal to k times the distance in x of x to y, where here k, small k, is between 0 and 1.

Now, why is this called a contraction? Well, we can just sort of draw a picture of what's happening. If this is my metric space x, and this is my initial points x and y, I know that the distance between the images of them must be less than k times the distance from x to y.

So if this is the distance from x to y, then for k really small, this will be the distance between f of x and f of y. And we can continue applying this over and over and over again.

So what is this-- what do you think will actually happen in the end? Well, these points are going to get really close together. So our question is, does there exist a point such that its image is the same as its initial value? And I'll state that more explicitly in a moment.

Specifically, we want to know if there exists a fixed point. I should have probably drawn that bigger, but that's fine-- a fixed point. If f is, again, a function from x to itself, x is a fixed point if f of x equals x.

All right. So our claim for the Banach fixed point theorem, as you can probably guess from this image that I've drawn, is going to be that when I have a contraction on a Cauchy complete metric space, there is, in fact, going to be a fixed point. And let me write that out explicitly. Actually, this is more of a theorem-- sorry, theorem.

It's called the Banach fixed point theorem but also sometimes called the contraction mapping theorem-- so whichever name you prefer to use. I usually use contraction fixed point-- sorry, contraction mapping theorem in the notes just because it's slightly shorter.

So let x be a Cauchy complete metric space and f-- oh, sorry. It also needs to be nonempty-- because otherwise, if it's empty, then what does having a fixed point even mean? How can a point be fixed if there is no points in x?-- and then that f be a map from x to x, a contraction.

Then the claim is that there exists a fixed point. In fact, it's going to be unique, which will be an interesting part of this proof, an interesting yet short part of this proof. Now let's restate what we already just so that we can make sure that we're all on the same page.

We know that by definition, a contraction is a k-Lipschitz map. And as we've shown already up above, Lipschitz functions are uniformly continuous. Are we going to use this in our proof? Not explicitly.

But it is helpful to just keep in mind, like, what are we actually doing here? We have a continuous map that just so happens to be a contraction, and we're going to show that there exists a fixed point.

All right. The proof of this is relatively short. We're going to utilize Cauchy completeness. That's the main thing that we're going to utilize in this proof.

We want to construct a function, or we want to construct a sequence that is Cauchy complete and therefore will have a limit. And that limit will be the fixed point. And to do so, we're just going to pick-- oh, proof-- pick arbitrary x0 big X.

And this is where we use that it's nonempty. The fact that we can pick any x0 X has to do with the fact that it's nonempty. The fact that it's nonempty is relatively a small moot point. But I do want to iterate that this assumption is necessary.

And then what we're going to do is define xn plus 1 to be f of xn. So our sequence will be x0, f of x0, f of f of x0, and so on. This is going to be our sequence. And our hope is that this is going to converge, i.e., hopefully, it will be Cauchy.

And the proof of this is pretty straightforward. Well, we'll note that the distance between xi plus 1 and xi-- what can we state about this? Well, firstly, we can plug in what the definitions of them are. This is the distance between f of xi and the distance between f of xi minus 1.

And what we know is by the fact that it's a contraction is that this is less than or equal to k times the distance of xi to xi minus 1. And I can continue reiterating this over and over and over again, where here I'll get less than or equal to k to the i of the distance from-- i or xi? Yeah, i-- distance from x1 to x0.

Everyone follow? And we're going to use this to consider-- to show that our sequence is in fact Cauchy, we're going to look at the distance between xn and xm. So I'll do that here.

We're considering the distance from xn to xm, and we want to imply this simple fact between the distances between two nearby points, two neighbor points. Well, to do that, we can just look at the triangle inequality. We can write this as less than or equal to the distance from xi plus 1 to xi, where i goes from n to m minus 1.

Does everyone see how I did this step? You can expand it out to see that it is precisely the triangle inequality. This is less than or equal to the distance from x to xn plus 1, xn plus 1 to xn plus 2, so on and so forth until we get to m. And here I'm assuming that m is bigger than n because it doesn't particularly matter.

OK. Well, then, this is less than or equal to the sum of k to the i times the distance from x1 to x0, which is great. Notice already that this distance is just fixed. It doesn't depend on i at all.

But now we want to figure out how we deal with the sum from n to mx1. And to do so, we can apply a pretty nice trick, which is just multiply by kn. This will be less than or equal to kn-- and I'll pull out the distance x1, x0-- of the sum from i equals 0 to m minus 1 minus n-- right, minus n? Yeah-- of k to the i.

So here I'm just multiplying by k to the n such that when I plug it back in, I get the same exact sequence. And so what can we do? This is then just a geometric series.

This is less than or equal to k to the n times the distance from x1 to x0 of the sum from i equals 0 to infinity of ki, which is simply k to the n over 1 minus k of the distance from x1 to x0, where here I'm applying the geometric series formula to this geometric series.

And now we're going to use the fact that-- now we're going to use the fact that k is less than 1. Or, specifically, it's between 0 and 1, right? I can choose n large enough-- large enough so that this term is less than epsilon.

I'm not going to state explicitly what this n is. You can figure this out independently. But we know that one minus exist, because k to the n is going to be going to 0.

So does this complete our proof? Not fully. We have that this is a Cauchy sequence. So we know it converges.

We need to, one, show that the point it converges to is our fixed point, and two, show it's unique. So let's go ahead and do that.

OK. So we want to show that this is, in fact, our limit point. And to do so-- or specifically, first I'll state we know there exists an x and X such that xn converges to x. How do we know this? Because it's Cauchy complete.

I'll reiterate this point. When you're going through proofs of theorems, it's important to double-check that you're using the statements of your theorem in every possible-- at least once in your proof, or at least the implications of your statements.

So we know that there exists an x such that xn converges to X by Cauchy completeness. And we want to show that x is its own fixed point. But how do we do that?

Well, x is the limit as n goes to infinity of f of xn. This is how we literally defined what the fixed point was. Or this is how we defined our sequence.

So what can we do? We can imply the fact that f is continuous because it's Lipschitz. So this is the same as f of the limit as n goes to infinity of xn, which is f of x.

So we've shown that it's a fixed point. It's not too bad of a proof. But it does utilize the fact that it's continuous. So that's why I talked about that earlier.

Secondly, we want to show that this fixed point is, in fact, unique. And the way to do so is very similar to what we've done before. So, two, let y be an x such that y is equal to f of y.

Then what will we do? Well, we will have that the distance from x to y will be equal to the distance from x to f of x, f of y. And then we can apply the fact that it's a contraction, right? This will be less than or equal to k times the distance from f-- sorry, of f x to y.

And we can move this term over. This will be 1 minus k times the distance from x to y. It's less than or equal to 0.

Now, what does this tell us? Well, 1 minus k is going to be nonzero, right? k is between 0 and 1. And it doesn't include 1. So what this tells us is then that the distance between x and y is 0, which, by our metric space theory, implies that x equals 5.

Now, I'm going to quote the book Lebl for a moment here because I think that he states it very well. "Not only does this proof tell us that there exists a fixed point, but this proof is constructive." It tells you how to find the fixed point itself. You just reiterate the process of applying the function to itself over and over and over again, which, if you're in the math lecture series, is something that Professor Staffilani talked about in the conversation of the nonlinear Schwartz equation. I think it's very interesting and, in fact, has something to do with differential operators.

So, to that point, I'm going to go through a proposition that is very directly tied to differential operators, or differential equations. So an example-- let lambda be in the real numbers and f and g be continuous on a to b.

I'm going to define one more function. I'm going to let k be continuous on the interval a, b cross a, b. So k here is going to depend on two parameters

OK. So then my question is, for which lambda is Tf of x equal to g of x plus the integral from a to b of k x, y f of x dx-- for which lambda-- sorry, lambda-- is this a contraction?

I know it's a long statement. And I know I haven't largely motivated it. And I'll get to why the motivation is at the end of the proof.

But let's think about this for a moment. I have this integral operator where what it's doing is it's taking in a function f and spitting out g of x plus lambda times f applied to this integral operator. This is pretty directly applied to differential equations.

The second thing I want to know is that in order for it to be a contraction, by definition, it has to be a map from the set to itself, right? A contraction is a map from x to itself. So the first question we should ask ourselves is that if it takes in a function that's continuous, does it spit out something that's continuous?

The answer will be yes. How do we see that? Well, g of x is continuous. So we know that that's all good.

But secondly, we've already shown that the integral from a to b of a continuous function is going to be continuous when I plug in the parameter y. That was our-- I think I might have erased it. Yeah, I did. But that was one of our statements was that this is, in fact, going to be continuous.

So we know that this is a map from continuous functions to itself. And that's what makes this question so interesting because now we're looking at specific functions. We're not looking at the distance between x and y. We're looking at the distance between functions f and g.

I want to reiterate this point. Metric spaces give us the ability to view functions as single points. There are things that we can manipulate and plug in and figure out the distances between. Now we're no longer viewing functions as things that take in inputs and spit out outputs, but rather things that we can manipulate.

And this viewpoint-- deeply influential for all parts of mathematics moving forward. But the question still stands. For which lambda is this a contraction? Well, to go about proving for which lambda it's a contraction, let's look at the distance between Tf of f1 of x and Tf2 of x, where, again, what I'm changing here is f. It's not x that we're changing. It's f. All right?

Well, then this is equal to, by definition, the difference of these two functions. So this will be the integral from a to b k x, y f1 of x. And while I'm at it, I'm going to combine this with f2 of x just because I'm running out of room on this board. So it's going to be the integral of k applied to f1 minus the integral of k applied to f2.

And we want to make this small. How do we do that? Well, firstly, let's note one very useful fact, which is that k is a continuous function on a bounded interval, or on a bounded interval across itself. So this tells us is that, in fact, k is bounded.

Why don't we want to apply the fact that the difference between f1 and f2 is bounded? Well, because we want to write this in terms of the distance between f1 and f2. So we don't want to do that.

So how do we do that? Well, first we apply the triangle inequality to this integral. So this is less than or equal to b of k of x, y times f1 of x minus f2 of x. And now we want to get in somehow the distance, or the metric, on the continuous functions.

To do so, I can just move in the supremum here. I know that this integral will be less than or equal to the term k of x, y times the supremum of the distance between f of 1 minus f of 2. So then this will be left equal to an integral from a to b k x, y times the supremum over x of f1 of x minus f2 of x dx.

All right? So how are we going to actually use this? Well, notice right now this is just a constant. So we can pull this term out. Oh, sorry. I should have had a lambda here this entire time-- lambda, lambda, lambda.

OK. So this term is a con-- this term is just the distance between f1 and f2. How do we bound this term? Well, we're going to use the fact that k of x, y is bounded because it's continuous on a compact set.

So suppose that it's bounded by c where c is just a real number. Then what we'll have-- doo-doo-doo-- is that the distance between T of f1 minus T of f2 is less than or equal to lambda times the distance in c0 of f1 to f2 times integral from a to b of k x, y dx.

But, again, this is less than some constant c. So I can write this as c times the interval a to b-- so b minus a. Everyone see how I did that?

So for which lambda is this going to be a contraction? Well, there will be the lambda such that lambda is less than 1 over c times b minus a, right? Because once this is less than 1 over c times b minus a, then this suddenly states that the distance between f1 and f2-- i.e., the supremum of that distance, will be less than or equal to a constant that's less than 1 times the distance between f1 and f2.

So this is now a contraction. Why is this so helpful? Why did we do this? This is the one part of today's lecture that I was having a hard time motivating specifically, or figuring out how to state. Why did we do this?

Well, it's deeply important for differential equations on the first hand, which I'll just state without argument because that would require a study of partial differential equations. But the second thing I want to know is that then notice that by the Banach fixed point theorem, there exists a function f such that it's a fixed point. Not only that, but we know that it's unique.

What does this actually tell us? This tells us that there exists an f for lambda less than 1 over c times b minus a such that T of f, which is equal to g of x plus lambda integral a to b k x, y f of x dx.

This tells us that Tf is, in fact, equal to f. We know this by the Banach fixed point thorem, where here we're using the fact that continuous functions are Cauchy complete. This was on your second problem set, which is why I had y'all do it.

So we know that f of x is equal to this right-hand side, which, if you differentiate it or-- I'll make a smaller mark-- remark. If g is, in fact, differentiable-- so c1 a to b-- what does this tell us? Well, now, all of a sudden, I can differentiate the right-hand side. I'll get that f of x, or the derivative of f of x, is equal to-- how do I want to say this? Let me be more careful.

One, we know that there exists an f that is continuous such that this is true. If g is, in fact, differentiable, you can think of it as a constant. If g is differentiable, then f must be differentiable, because then it's a differentiable function plus something that's differentiable because it's an integral.

So this tells us that, in fact, f is in c1, which is important in differential equations, right? This tells us a way to prove that there exists a unique solution to the differential equation that defines this integral operator. Now, of course, not every differential equation will result in an integral operator like this. But a huge class of them can't be, which is very important.

If you're interested in this question, I would highly suggest looking at section 7.6.2 on Picard iteration in Lebl's book. It's a very useful book to have. Or it's a very useful section to read into.

It just uses an application that I think will take up the rest of our time. And I want to talk about completions of metric spaces. So any questions before I move on?

Totally fair. OK, so next section-- ah how do i put it? - 2 completions of metric spaces.

Has anyone heard of this before, completion of the metric spaces? Well, one way to motivate completions of metric spaces is through an example that you all have already seen. Example-- the real numbers are a completion of the rationals.

So remember how we started our journey in real analysis, right? We started by looking at rational numbers. We stated things like they're a field, and we stated things like there aren't every real number-- or, sorry, every real number is not a rational one, right? The square root of 2 is an example.

And then we said, well, hold on. Let's just throw in the square root of 2, right? We completed the rationals by filling in all of the holes. And doing so, this method, is known as a completion.

Now, there's a few ways you might have seen this for the rational numbers. And I'll state them right now. One, you might have seen a proof of it using Dedekind cuts. This is how Rudin uses it in his book, Baby Rudin, Dedekind cuts. If you're interested in how this method works, it's in his Appendix 1 of his first chapter.

This one is not super used. It's sort of difficult to get around. But you could have viewed it this way.

Two, you also could have viewed it by using the least upper bound property. Has anyone seen this method before? Totally fair if not-- I'll simply state what it is if not.

For this method, what we do is we define the real numbers as the smallest set that contains the rational numbers such that every bounded set, its upper bound is in the set. Great. Does this make sense verbally? I can also write it up more explicitly. But this is another way you might have seen it.

Three, you might have learned it using equivalence classes of Cauchy sequences. Did anyone see it this way? Or, in fact, let me just ask, what ways have y'all seen it, the completion of the rationals?

The second one, the least upper bound property? What about you? It's really fair. It was a while ago.

[INAUDIBLE student question].

Oh, interesting-- the equivalence class has the Cauchy sequences. Well, it was at least a full semester ago, if not longer. But the third one is the one I'm going to iterate on today, all right?

Where, today, we're looking at Cauchy complete metric spaces, it makes sense to look at Cauchy sequences as our starting point. But the equivalence classes is not too bad to motivate. The idea is we say a Cauchy sequence am is equivalent-- or I'll use this sym, note symbol.

So here an and bn are Cauchy sequences. I'm going to say that they're equivalent if the distance between an and bn goes to 0 as n goes to infinity. This is how we define equivalence classes of Cauchy sequences.

What this let us do is define all of our limit points to be the same. The issue-- let me say this differently. The idea is we want to take sequences of rational numbers that will converge to the real numbers we want. We want them to converge to the square root of 2. We want them to converge to the square root of 3, things like this.

The issue is we don't want to overdefine our space. We don't want multiple sequences to go-- or we don't want multiple square roots of 2. We only want one specific one.

If we use equivalence classes, let's just define the limit points as being the same. So this is what lets us do it. We have equivalence classes of real number-- of rational numbers. And then what we say is that R is the set of equivalence classes.

The notion of an equivalence relationship, I've put on the lecture notes. So I would highly suggest looking at that on Wikipedia if you haven't seen it before. But the fact that this is true, you can simply write out and prove the fact that this is an equivalence relation.

In fact, I'll just state what the three properties are now, the three properties you would have to show, because there are ones that, at least on the face of them, we should be able to think through a little bit. Doo-doo-doo.

OK. So we have this set of equivalence classes. What we want to show is that this is, in fact, an equivalence relationship. So we want to show first that the sequence an, if this is Cauchy, this is equivalent to an.

But how do we know that this is true? Well, it's simply by the fact that an minus an equals 0 for every single n. So it's definitely going to tend to 0.

Secondly, you want to show that if an is equivalent to bn, this is true if and only if bn is equivalent to an. Why is this going to be true? Well, we know that absolute values are symmetric. So we can just swap those around. All good on that front.

And three, we want to show that if an is equivalent to bn and bn is equivalent to cn, then an is equivalent to cn. This last one, I'll iterate on. I'll expand upon this.

How do we do that? Well, we look at the distance from an to cn, and we know that this is less than or equal to the distance from an to bn plus the distance from bn to cn. And we can make both of these terms arbitrarily small because the distances between the points goes to 0.

So then this is less than epsilon, which tells us that this is going to converge to 0. This is less than epsilon for all epsilon bigger than 0. There exists an n such that this is true.

OK. So this tells us that, in fact, the distance between an and cn are going to tend to 0. And notice that all of the things I've stated so far, everything in this equivalence relationship, only uses the metric, right? Here we're just using the metric defined by absolute values. But you can picture replacing this with any metric you care to use, which is why we can view completions of metric spaces as a whole topic for today.

So notice we have the same equivalence classes notion for metric spaces. Why is this helpful? Well I'll give one short example, at least verbally, which is this metric is not the only one that can exist on the rational numbers.

We can impose it with a bunch of other metrics if you wanted to and then look at the equivalence classes of that. And that leads to what is known as p-adic numbers. I wish I had more time to talk about that today, but it's one of the examples that shows up in number theory. So I'll leave that for you to look at.

But how do we add in these limit points? That's the whole question, right? Well, there's a theorem that tells us that we can simply do so. And let me state what that theorem is.

This theorem, I'll just write as completions because it's what tells us that there exists a completion.

OK. Let M, d be a metric space. And it doesn't need to be Cauchy complete, right? The idea is that we want to complete it so that it is Cauchy complete.

Then there exists an M bar such that, one, M is a subset of M bar; two-- doo-doo-doo-- the distance on M bar-- I should note that this is a metric space. And I'll call the metric on it d bar for simplicity.

The distance on M bar restricts to the regular distance on M. So what this tells you is that we have the subspace metric on our space, right? The distance between two points and M is going to stay the same in this new metric d bar.

And three, M bar is Cauchy complete. And, finally, four, the closure of M is M bar, which is where this notation comes from.

If you've ever seen the definition of closure before in a topology book or in Lebl's book, it's M bar. Does everyone remember what the definition of closure is? It's the smallest closed set that contains your smaller set. So the smallest closed set that contains M is, in fact, M bar. You can think of it as just including all the boundary points as the main thing. Doo-doo-doo

The reason why that's the right notion to consider is because we know that closed sets contain all of your limit points. Specifically, the issue are the ones at the boundary. But if you add those points in, then we're all good.

Before we go into this proof, however, I need to simply state one small lemma. I won't go through the proof, because it's very similar to what's on your homework, but I'll definitely talk through it, which is the set C at lower infinity of M, which is the set of functions f, which are continuous, with the supremum of m and M of f of m bounded-- so this is less than infinity-- as a metric space.

So what this is is the set of continuous bounded functions. And denoted it's c infinity of M. Notice it's in the small part because it's not smooth. Smooth goes in the upper part.

So this is a metric space, specifically with metric given by the supremum or distance d infinity-- or always d upper infinity of f to g equal to-- no, I'll use lower infinity-- the supremum of m and M of the distance between f of x minus g of x. Oh, sorry. These should be m's.

Firstly, I should ask how do we know that this supremum exists and is finite. Anyone? I'll let y'all think through it.

Here we want to use the fact that f and g are bounded. So does that immediately imply that the supremum is bounded?

Yeah. The answer is yes. You can just apply the triangle inequality, right? The absolute value of f of m minus g of m is less than or equal to absolute value f plus absolute value g. And we can take the supremum of that, and that will have to be less than infinity.

So this supremum definitely exists, and it's finite, which is helpful. We want to be able to say metrics must be finite, unless you're looking at some weird version of a metric.

So we know that this is, in fact, a metric. The proof of this is just using the same proof of supremums that we did for continuous functions on a bounded interval. The only difference is instead of the extreme value theorem, we used boundedness, right? That's the main difference, because we don't know that it's compact. So we don't know that there exists an extreme value. But we do know that it's bounded. So that's all we needed from the extreme value theorem.

So that shows that this is, in fact, a metric. What's the next thing that we might want to say about this? Well, in fact, C infinity of M is Cauchy complete. The proof of this is just the same as it was on your homework.

You take a Cauchy sequence of functions. So the function such that the distance between two points gets small. And then what we know is we can take the limit points of them by continuity. And then you have to show that, in fact, the limit point is bounded.

But the proof of this is nearly exactly the same as your second problem set. So unless anyone wants me to work through the details right here, I would highly suggest doing it yourself. But I think it's the sort of thing that, because you've all done the problem set, you can believe me that this is true.

All right. So how are you going to use this fact? Well, the fact that this is Cauchy complete is super helpful for us. We want to somehow view our metric space M as a subset of C infinity of M. Now, how do we do that? So I'm going to prove this theorem now.

The way that we do that is we simply map points to-- we simply map points in our metric space to a function. So fix m prime M arbitrary and define the map from M and m.

We're going to map this to the function g of m which is defined pointewise at p as the distance of p to m minus the distance from p to m prime. So-- oh, sorry-- yeah. So fix sum m prime and M, which is arbitrary. And we're defining the map from our point in a metric space to a continuous function.

How do we know this is continuous? Well, firstly, you can show that a metric is continuous, which is what we did on our second day, right? We had that the distance between xn and y goes to the distance of x and y. And, in fact, distance of x and ym converges to d x, y.

So we have already briefly, in loose terms, shown continuity of a metric. But how do we show that this is bounded? Well, the fact that this is bounded has to do with the fact that both of these metrics are less than infinity, right? That's by definition of the metric.

So we're never going to get something that's nonfinite. So this tells us that we can take the supremum of it. So we now have a way to view our metric M as a subset of C infinity of M because you've mapped it to a continuous and bounded function.

Now, our goal is simply to take the closure of this subset. But to do so, we need to clarify what we mean by this, quotation mark, "subset," right? What I mean by this is notice that gm of p-- how do I want to say this?

Notice, then, that the distance from m1 to m2, this is, in fact, equal to the supremum of m and M of the distance between gm1 of p minus gm2 of p, because you can simply plug in-- oh, sorry this should be supremum over p-- because you can simply plug in m1 and m2 and take their difference. And then you just want the maximum distance between them.

And notice that this is precisely the same metric as is on our C infinity space. So this is equal to the distance to infinity between gm1 and gm2. What this tells us is that this map-- this map that goes from m to gm-- is an isometry, which means that it just preserves distances.

The distance between two points is the same as the distance between two functions in this different space. And, in fact, not only is it an isometry. It's going to be bijective, right? For each m, there exists a unique gm defined as this distance.

So this tells us that we really can view m as a subset of C infinity because we're just viewing it as a subset such that distances are preserved. Does everyone get why this fact is true? OK.

So move over to the next board. The proof from here is pretty straightforward. We have a subset of our metric space inside of a Cauchy complete metric space. Doo-doo-doo.

So what we have is that M is a subset of C infinity of M. And notice that this is complete, infinite, Cauchy complete. So it contains all of its limit points.

And secondly, we can just take the closure of M. So let M bar be equal to C infinity-- oh, sorry, not C infinity-- be the closure of M in C infinity of M.

So is this the completion that we want? Well, to show that it's the completion that we want is not too bad. You just have to check each of these properties. I'll say that again in case it was a little bit loud. We just have to check these four properties.

Well, one, we know that M is definitely a subset of its closure. That's by definition. And, in fact, it also satisfies four.

Secondly, we know the distances are preserved because the distances are preserved on the infinity of M when we take the closure. That's by the subspace metric.

The last thing you want to check is that, in fact, this is Cauchy complete. But that fact follows from the fact that it's a closed subset of a Cauchy complete space. Recall the closed subs-- or a closed subset of a Cauchy complete metric space is Cauchy complete.

How do I check this? Well, take any Cauchy sequence in your space. We know that it has to converge to something because it's in a Cauchy complete space. But that thing it converges to has to be in your closed subset because it's closed. So it contains all of its limit points.

So what this tells us is that every Cauchy sequence in our closed subset converges in our closed subset. So this implies that it's Cauchy complete. So then we're basically done, right? M bar is a subset of C infinity of M. But by this isometry, we can map it back to just the metric space. So the proof is then done.

So why is this so helpful? I've already given the example of the rationals. The rationals are a completion of the real numbers.

But that's one that we've already known, right? That that's one that has been known for quite some time in our real analysis studies. But there's quite a few more examples that we can consider. These examples show up, for instance, in 18.102, in very, very important ways.

So example-- consider normed spaces. These are vector spaces with a norm put on them. Recall this from our third lecture.

What if this isn't Cauchy complete? We want Cauchy completeness for a number of our theorems, and it's very important that we do. Well, we can just take the closure of them and end up with-- or the completion of them-- and end up with what are known as Banach spaces.

And Banach space theory is literally the majority of 18.102. In fact, we could do slightly more if you've heard of this from perhaps 18.06. You can complete inner product spaces, which I won't define here. But I'll simply note that the completion of them are known as Hilbert spaces.

And these are just names, right? These are just names that aren't necessarily too important to you right now. But I think that they're super cool because they show up in 18.102.

But perhaps an example that would be more interesting right now is how we end up with integrable functions, right? Recall Riemann integrable functions have a ton of holes-- "has a ton of holes."

So, for instance, an example of this would be to take the indicator function on rationals, where the indicator function at the point x is 1 where x is in the rationals and 0 otherwise, i.e., if it's irrational.

How do you integrate this function using Riemann integration? The answer is that you can't. It doesn't quite work. Somehow, this should both have an integral 0 because most terms are irrational, but it should also have an integral 1 because most terms are rational.

It's a huge issue, right? Riemann integration has a ton of flaws, this just being one of them. But, in fact, I can complete them. I can complete Riemann integrable functions.

So, specifically, we can consider compactly supported functions. So, specifically, consider C C0 functions on, let's say, a metric space-- what do I want to say? I'll say the metric space x. No, I'll just say on R.

So this is a set of compactly supported functions on the real numbers that are continuous. We've already talked about this before on our third lecture when we were talking about integration. Here what's so nice about this set is that the integral of a function f in the space over R-- the integral of f of x dx will be less than infinity because I know that the terms have to eventually go to 0.

But, in fact, this gives us a metric on our space. I can take absolute values of f of x. And then this is suddenly our L1 metric from day one, right? Our L1 metric was such that the distance-- I think I called it I1-- I1 of f, g is equal to the integral over R of f of x minus g of x dx, where it's called L1 because it's raised to the first power, then raised to the 1 over 1 power.

So this is our metric on the set of compactly supported functions. I can simply complete the subset of the compactly supported functions under this metric because, again, our definition of completion depends on what metric we're choosing. Right here, our metric has to restrict to the metric on our original metric space.

So if I change this metric, it's going to change what the completion looks like. In our case, what we would end up with-- so, in other words, the completion of the metric space C0 C of R with respect to the I1 metric, what this will give us is the set of LP functions-- or L1 functions, sorry-- on R, which are simply the set of functions that have integral 1. So this is functions f such that the integral of f is 1.

Now, here, our functions can look quite a bit weirder. They don't have to be Riemann integrable anymore. They can be Lesbegue integrable, which is what this L stands for. This integral is then known as the Lesbegue integral.

And this is used all the time. This is-- oh, sorry. This shouldn't be 1. This should be less than infinity.

This is used all the time, right? Because we want to be able to integrate functions that actually integrate well. And Riemann integrable functions are not it. They have a ton of holes, right?

In fact, we can generalize this quite a bit more. Remember on that first day that I defined Ip, which is the integral of the distance between them to the pth power, all of which raised to the 1 over p.

If I consider completing the metric of continuous functions under lp, then what I'll end up with are Lp functions, where all of a sudden this is raised to the pth power. So this is the space of Lp functions-- deeply, deeply important.

This also shows up in 18.102. But it's perhaps slightly easier to see why it's so important. So I'll note one more thing, I think, and then that will mostly be it, which is recall that this set was, in fact, a normed space, right? I can define this distance as a norm.

What this tells us is that the Lp spaces are Cauchy complete, right, because they're Banach spaces. They are completion of a normed space.

Now, this doesn't say that we've said all the things that there are to say about Lp functions. There's still questions like, what theorems about limits do we have? What theorems about approximations do we have? But this does give us a useful intuition behind what's happening.

All that's happening is we're looking at equivalence classes of Cauchy sequences. It makes a lot of sense to do this when it comes to the real numbers. But it might be weirder to think about with respect to functions or with respect to topologies and weird things like that.

So, yeah, I'll just leave y'all with that thought is equivalence classes of Cauchy sequences is a useful intuition behind all of these things. I should note that this is the Lp metric. I forgot to change that one. OK.

Is there anything else I want to say? I think that's it. Yeah, so unless you all have any questions, then we can end today slightly early.

But, yeah, this was very neat. Next time, I'm going to talk about more applications to 18.101, 102, 103, 152. There's a ton of applications of all of this material to this later class-- oh, 901, if you're planning to take 18.901. There's a ton of applications of what we've talked about in this class so far to these later classes. And I'm going to bring up how that actually works. So see y'all next time.