41. **Classification of real forms of semisimple Lie algebras**

41.1. **The compact real form.** An important example of a real form of simple complex Lie algebra $\mathfrak{g}$ is the **compact real form**. It is determined by the automorphism τ (called the **Cartan involution**) defined by the formula

$$\tau(h_j) = -h_j,\ \tau(e_j) = -f_j,\ \tau(f_j) = -e_j.$$

Let us denote this real form $\mathfrak{g}_{(\tau)}$ by $\mathfrak{g}^c$.

Proposition 41.1. *The Killing form of $\mathfrak{g}^c$ is negative definite.*

Proof. We have an orthogonal decomposition

$$\mathfrak{g}^c = (\mathfrak{h} \cap \mathfrak{g}^c) \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c.$$

Moreover, the Killing form is clearly negative definite on $\mathfrak{h} \cap \mathfrak{g}^c$, since the inner product on the coroot lattice is positive definite, and $\{i\alpha_j^\vee\}$ is a basis of $\mathfrak{h} \cap \mathfrak{g}^c$. So it suffices to show that the Killing form is negative definite on $(\mathfrak{g}_\alpha \cap \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^c$ for any $\alpha \in R_+$.

First consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathfrak{g}^c$ is spanned by the Pauli matrices ih, $e - f$, $i(e + f)$, so $\mathfrak{g}^c = \mathfrak{su}(2)$. It follows that the trace form of any finite dimensional representation of $\mathfrak{g}^c$ is negative definite.

Thus for a general $\mathfrak{g}$, the elements S_i given by (40.1) preserve $\mathfrak{g}^c$; this follows since the matrix $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to $SU(2)$, and $\mathrm{Lie}(SU(2)_i) \subset \mathfrak{g}^c$. It follows that for any $w \in W$ the element $\widetilde{w}$ preserves $\mathfrak{g}^c$. Thus the restriction of the Killing form of $\mathfrak{g}^c$ to $\mathfrak{g}^c \cap (\mathfrak{sl}_2)_\alpha$ is negative definite for any root α (since it is so for simple roots, as follows from the case of $\mathfrak{sl}_2$). This implies the statement. $\square$

Now consider the group $\mathrm{Aut}(\mathfrak{g}^c)$. Since the Killing form on $\mathfrak{g}^c$ is negative definite, it is a closed subgroup in the orthogonal group $O(\mathfrak{g}^c)$, hence is compact. Moreover, it is a Lie group with Lie algebra $\mathfrak{g}^c$. Thus we obtain

Corollary 41.2. *Let $G^c_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}^c)^\circ$. Then G^c_{ad} is a connected compact Lie group with Lie algebra $\mathfrak{g}^c$.*

In particular, this gives a new proof that representations of a finite dimensional semisimple Lie algebra are completely reducible (by using Weyl's unitary trick, see Subsection 35.1).

Exercise 41.3. (i) Show that if $\mathfrak{g} = \mathfrak{sl}_n$ then $G^c_{\mathrm{ad}} = PSU(n) = SU(n)/\mu_n$, where μ_n is the group of roots of unity of order n.

(ii) Show that if $\mathfrak{g} = \mathfrak{so}_n$ then $G^c_{\rm ad} = SO(n)$ for odd n and $SO(n)/\pm 1$ for even n.

(iii) Show that if $\mathfrak{g} = \mathfrak{sp}_{2n}$ then $G^c_{\rm ad} = U(n,\mathbb{H})/\pm 1$, where $U(n,\mathbb{H})$ is the quaternionic unitary group $Sp_{2n}(\mathbb{C}) \cap U(2n)$ (see Exercise 6.15).

Exercise 41.4. (i) Compute the signature of the Killing form of the split form $\mathfrak{g}^{\rm spl}$ of a complex simple Lie algebra $\mathfrak{g}$ in terms of its dimension and rank, and show that the compact form is never split.

(ii) Show that the compact form is inner to the quasi-split form defined by the flip of the Dynkin diagram corresponding to taking the dual representation (i.e., induced by $-w_0$), but is never quasi-split itself (show that the quasi-split form contains nonzero nilpotent elements). For which simple Lie algebras is the compact form inner?

41.2. **Other examples of real forms.** So let us list real forms of simple Lie algebras that we know so far.

1. Type A_{n-1}. We have the split form $\mathfrak{sl}_n(\mathbb{R})$, the compact form $\mathfrak{su}(n)$, and also for $n > 2$ the quasi-split form associated to the automorphism $s(A) = -JA^TJ^{-1}$, where $J_{ij} = (-1)^i\delta_{i,n+1-j}$ (this automorphism sends e_i, f_i, h_i to $e_{n+1-i}, f_{n+1-i}, h_{n+1-i}$). So the corresponding real Lie algebra is the Lie algebra of traceless matrices preserving the hermitian or skew-hermitian form defined by the matrix J, which has signature (p,p) if $n = 2p$ and $(p+1,p)$ or $(p,p+1)$ if $n = 2p+1$. Thus in the first case we have $\mathfrak{su}(p,p)$ and in the second case we have $\mathfrak{su}(p+1,p)$. Note that for $n = 2$ we have $\mathfrak{su}(1,1) = \mathfrak{sl}_2(\mathbb{R})$, so in this special case this form is not new. We also observe that for $n \geq 4$ there are other forms, e.g. $\mathfrak{su}(n-p,p)$ with $1 \leq p \leq \frac{n}{2} - 1$.

2. Type B_n. We have the split form $\mathfrak{so}(n+1,n)$, the compact form $\mathfrak{so}(2n+1)$. The Dynkin diagram has no nontrivial automorphisms, so there are no non-split quasi-split forms. In particular, since $A_1 = B_1$, we have $\mathfrak{so}(3) = \mathfrak{su}(2)$ and $\mathfrak{so}(2,1) = \mathfrak{su}(1,1)$.

3. Type C_n. We have the split form $\mathfrak{sp}_{2n}(\mathbb{R})$ and compact form $\mathfrak{u}(n,\mathbb{H})$. The Dynkin diagram has no nontrivial automorphisms, so there are no non-split quasi-split forms. The equality $B_2 = C_2$ implies that $\mathfrak{so}(3,2) = \mathfrak{sp}_4(\mathbb{R})$ and $\mathfrak{so}(5) = \mathfrak{u}(2,\mathbb{H})$.

4. Type D_n. We have the split form $\mathfrak{so}(n,n)$, the compact form $\mathfrak{so}(2n)$. Moreover, in this case we have a unique nontrivial involution of the Dynkin diagram. More precisely, this is true for $n \neq 4$, while for $n = 4$ we have $\mathrm{Aut}(D) = S_3$, but there is still a unique nontrivial involution up to conjugation. So we also have a non-split quasi-split form. To compute it, recall that the split form is defined by the equation $A = -JA^TJ^{-1}$ where $J_{ij} = \delta_{i,2n+1-j}$. The quasi-split form is obtained by replacing J by $J' = gJ$, where g permutes e_n

and e_{n+1} (this is the automorphism that switches α_{n-1} and α_n while keeping other simple roots fixed). The signature of the form defined by J' is $(n+1,n-1)$, so we get that the non-split quasi-split form is $\mathfrak{so}(n+1,n-1)$. In particular, since $D_2 = A_1 + A_1$, for $n=2$ we get

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2),\ \mathfrak{so}(2,2) = \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1),\ \mathfrak{so}(3,1) = \mathfrak{sl}_2(\mathbb{C})$$

(the Lie algebra of the Lorentz group of special relativity). Also, since $D_3 = A_3$, for $n=3$ we get $\mathfrak{so}(6) = \mathfrak{su}(4)$, $\mathfrak{so}(3,3) = \mathfrak{sl}_4(\mathbb{R})$, and $\mathfrak{so}(4,2) = \mathfrak{su}(2,2)$.

5. Type G_2. We have the split and compact forms $G_2(\mathbb{R}), G_2^c$.
6. Type F_4. We have the split and compact forms $F_4(\mathbb{R}), F_4^c$.
7. Type E_6. We have the split and compact forms $E_6(\mathbb{R}), E_6^c$ and the quasi-split form E_6^{qs} attached to the non-trivial automorphism.
8. Type E_7. We have the split and compact forms $E_7(\mathbb{R}), E_7^c$.
9. Type E_8. We have the split and compact forms $E_8(\mathbb{R}), E_8^c$.

41.3. **Classification of real forms.** However, we are not done with the classification of real forms yet, as we still need to find all real forms and show there are no others. To this end, consider a complex simple Lie algebra $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$. We have the compact antilinear involution $\omega = \sigma_\tau$ of $\mathfrak{g}$ whose set of fixed points is $\mathfrak{g}^c$. Another real structure on $\mathfrak{g}$ is then defined by the antilinear involution $\sigma = \omega \circ g$, where $g \in \mathrm{Aut}(\mathfrak{g})$ is such that $\omega(g)g = 1$. But it is easy to see that

$$\omega(g) = (g^\dagger)^{-1},$$

where $x^\dagger$ is the adjoint to $x \in \mathrm{End}(\mathfrak{g})$ under the negative definite Hermitian form $(X,Y) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}\omega(Y))$ (the Hermitian extension of the Killing form on $\mathfrak{g}^c$ to $\mathfrak{g}$). It follows that the operator g is self-adjoint. Thus it is diagonalizable with real eigenvalues, and we have a decomposition

$$\mathfrak{g} = \oplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma),$$

where $\mathfrak{g}(\gamma)$ is the γ-eigenspace of g, such that $[\mathfrak{g}(\beta), \mathfrak{g}(\gamma)] = \mathfrak{g}(\beta\gamma)$. Now consider the operator $|g|^t$ for any $t \in \mathbb{R}$. It acts on $\mathfrak{g}(\gamma)$ by $|\gamma|^t$, so $|g|^t = \exp(t \log |g|) \in G_{\mathrm{ad}}$ is a 1-parameter subgroup. Now define $\theta := g|g|^{-1}$. We have $\theta \circ \omega = \omega \circ \theta$ and $\theta^2 = 1$. Also g and θ define the same real structure since $\theta = |g|^{-1/2} g \omega(|g|^{1/2})$. This shows that without loss of generality we may assume that $g = \theta$ with $\theta \circ \omega = \omega \circ \theta$ (i.e., $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$) and $\theta^2 = 1$.[23]

[23]The advantage of passing from g to θ is that the equation $\theta^2 = 1$ is much easier to solve than $g\omega(g) = 1$, as it just means that we have a decomposition of $\mathfrak{g}$ into the $+1$- and -1-eigenspaces of θ.

Moreover, another such element θ' defines the same real form if and only if $\theta' = x\theta\omega(x)^{-1}$ for some $x \in \mathrm{Aut}(\mathfrak{g})$. So we get

$$x\theta\omega(x)^{-1} = \omega(x)\theta x^{-1},$$

so setting $z := \omega(x)^{-1}x$, we get $\omega(z) = z^{-1}$, $\theta z = z^{-1}\theta$. Note that $z = x^\dagger x$ is positive definite. So setting $y = xz^{-1/2}$, we have

$$\omega(y) = \omega(x)z^{1/2} = xz^{-1/2} = y$$

i.e., $y \in \mathrm{Aut}(\mathfrak{g}^c)$ and

$$\theta' = x\theta\omega(x)^{-1} = x\theta z x^{-1} = xz^{-1/2}\theta z^{1/2}x^{-1} = y\theta y^{-1}.$$

Thus we obtain

Theorem 41.5. *Real forms of $\mathfrak{g}$ are in bijection with conjugacy classes of involutions $\theta \in \mathrm{Aut}(\mathfrak{g}^c)$, via $\theta \mapsto \omega_\theta := \theta \circ \omega = \omega \circ \theta$.*

Theorem 41.5 provides a different classification of real forms from the one given in Subsection 40.3, obtained by "counting" from the compact form rather than the split form (as we did in Subsection 40.3). We denote the real form of $\mathfrak{g}$ assigned in Theorem 41.5 to an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\mathfrak{g}_\theta$. For example, $\mathfrak{g}_1 = \mathfrak{g}^c = \mathfrak{g}_{(\tau)}$.

Thus we have a canonical (up to automorphisms of $\mathfrak{g}^c$) decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, into the eigenspaces of θ with eigenvalues 1 and -1, such that $\mathfrak{k}$ is a Lie subalgebra, $\mathfrak{p}$ is a module over $\mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. We also have the corresponding decomposition for the underlying real Lie algebra $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$. Moreover, the corresponding real form $\mathfrak{g}_\theta$ is just $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, where $\mathfrak{p}_\theta := i\mathfrak{p}^c$.

Exercise 41.6. Show that $\mathfrak{k}$ is a reductive Lie algebra. Does it have to be semisimple?

Proposition 41.7. *There exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ invariant under θ, such that $\mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra in $\mathfrak{k}$.*

Proof. Take a generic $t \in \mathfrak{k}^c$; as $\mathfrak{k}$ is reductive, it is regular semisimple. Let $\mathfrak{h}_+^c$ be the centralizer of t in $\mathfrak{k}^c$. Then $\mathfrak{h}_+ := \mathfrak{h}_+^c \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{k}$ is a Cartan subalgebra. Let $\mathfrak{h}_-^c$ be a maximal subspace of $\mathfrak{p}^c$ for the property that $\mathfrak{h}^c := \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ is a commutative Lie subalgebra of $\mathfrak{g}^c$.

We claim that $\mathfrak{h} := \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra in $\mathfrak{g}$. Indeed, it obviously consists of semisimple elements (as all elements in $\mathfrak{g}^c$ are semisimple, being anti-hermitian operators on $\mathfrak{g}^c$). Now, if $z \in \mathfrak{g}$ commutes with $\mathfrak{h}$ then $z = z_+ + z_-$, $z_+ \in \mathfrak{k}$ and $z_- \in \mathfrak{p}$, and both z_+, z_- commute with $\mathfrak{h}$. Thus $z_+ \in \mathfrak{h}_+$ and $z_- = x + iy$, where $x, y \in \mathfrak{p}^c$ and both commute with $\mathfrak{h}$. Hence $x, y \in \mathfrak{h}_-^c$ by the definition of $\mathfrak{h}_-^c$. Thus $z \in \mathfrak{h}$, as claimed. It is clear that $\mathfrak{h}$ is θ-stable, so the proposition is proved. $\square$

Thus we have a decomposition $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, and θ acts by 1 on $\mathfrak{h}_+$ and by -1 on $\mathfrak{h}_-$.

Lemma 41.8. *The space $\mathfrak{h}_-$ does not contain any coroots of $\mathfrak{g}$.*

Proof. Suppose that $\alpha^\vee \in \mathfrak{h}_-$ is a coroot. Thus $\theta(\alpha^\vee) = -\alpha^\vee$, so $\theta(e_\alpha) = e_{-\alpha}$ and $\theta(e_{-\alpha}) = e_\alpha$ for some nonzero $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$. Let $x = e_\alpha + e_{-\alpha}$. We have $\theta(x) = x$, so $x \in \mathfrak{k}$. On the other hand, $x \notin \mathfrak{h}_+$ (as x is orthogonal to $\mathfrak{h}_+$ and nonzero) and $[\mathfrak{h}_+, x] = 0$ since α vanishes on $\mathfrak{h}_+$. This is a contradiction, since $\mathfrak{h}_+$ is a maximal commutative subalgebra of $\mathfrak{k}$. $\square$

By Lemma 41.8, a generic element $t \in \mathfrak{h}_+$ is regular in $\mathfrak{g}$. So let us pick one for which $\mathrm{Re}(t, \alpha^\vee)$ is nonzero for any coroot $\alpha^\vee$ of $\mathfrak{g}$, and use it to define a polarization of R: set $R_+ := \{\alpha \in R : \mathrm{Re}(t, \alpha^\vee) > 0\}$. Then $\theta(R_+) = R_+$. So $\theta(\alpha_i) = \alpha_{\theta(i)}$, where $\theta(i)$ is the action of θ on the Dynkin diagram D of $\mathfrak{g}$. Thus if $\theta(i) = i$ then $\theta(e_i) = \pm e_i$, $\theta(h_i) = h_i$, $\theta(f_i) = \pm f_i$ while if $\theta(i) \neq i$, we can normalize $e_i, e_{\theta(i)}, f_i, f_{\theta(i)}$ so that $\theta(e_i) = e_{\theta(i)}$, $\theta(f_i) = f_{\theta(i)}$, $\theta(h_i) = h_{\theta(i)}$. Thus θ can be encoded in a marked Dynkin diagram of $\mathfrak{g}$: we connect vertices i and $\theta(i)$ if $\theta(i) \neq i$ and paint a θ-stable vertex i white if $\theta(e_i) = e_i$ (i.e., $e_i \in \mathfrak{k}$, a **compact root**), and black if $\theta(e_i) = -e_i$ (i.e., $e_i \in \mathfrak{p}$, a **non-compact root**). Such a decorated Dynkin diagram is called a **Vogan diagram**. So we see that every Vogan diagram gives rise to a real form, and every real form is defined by some Vogan diagram.

Exercise 41.9. (i) Show that the signature of the Killing form of a real form $\mathfrak{g}_\theta$ of a complex semisimple Lie algebra $\mathfrak{g}$ corresponding to involution θ equals $(\dim \mathfrak{p}, \dim \mathfrak{k})$. In particular, the Killing form of $\mathfrak{g}_\theta$ is negative definite if and only if $\theta = 1$, i.e., $\mathfrak{g}_\theta = \mathfrak{g}^c$ is the compact form.

(ii) Deduce that for the split form $\dim \mathfrak{k} = |R_+|$, the number of positive roots of $\mathfrak{g}$.

(iii) Show that for a real form of $\mathfrak{g}$ in the compact inner class, we have $\mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$.

41.4. **Real forms of classical Lie algebras.** We are not finished yet with the classification of real forms since different Vogan diagrams can define the same real form (they could arise from different choices of R_+ coming from different choices of the element t). However, we are now ready to classify real forms of classical Lie algebras.

1. Type A_{n-1}, compact inner class. In this case θ is an inner automorphism, conjugation by an element of order ≤ 2 in $PSU(n)$. Obviously, such an element can be lifted to $g \in U(n)$ such that $g^2 = 1$, so $\theta(x) = gxg^{-1}$. Thus $g = \mathrm{Id}_p \oplus (-\mathrm{Id}_q)$ where $p + q = n$ and we may assume that $p \geq q$. It is easy to see that this defines the real form $\mathfrak{g}_\theta = \mathfrak{su}(p, q)$, and $\mathfrak{k} = \mathfrak{gl}_p \oplus \mathfrak{sl}_q$. These are all pairwise non-isomorphic since the corresponding automorphisms θ are not conjugate to each other. So we get $[\frac{n}{2}] + 1$ real forms. Note that for $n = 2$ this exhausts all real forms, so we have only two – $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1) = \mathfrak{sl}_2(\mathbb{R})$ with $\mathfrak{k} = \mathfrak{gl}_1$.

2. Type A_{n-1}, $n > 2$, the split inner class. If n is odd, there is no choice as all the vertices of the Vogan diagram are connected into pairs, so we only get the split form $\mathfrak{g}_\theta = \mathfrak{sl}_n(\mathbb{R})$. However, if $n = 2k$ is even, there is one unmatched vertex in the middle of the Vogan diagram, which can be either white or black. It is easy to check that in the first case (white vertex) $\mathfrak{k} = \mathfrak{sp}_{2k}$ and in the second one (black vertex) $\mathfrak{k} = \mathfrak{so}_{2k}$. So the first case is $\mathfrak{g}_\theta = \mathfrak{sl}(k, \mathbb{H})$, the Lie algebra of quaternionic matrices of size k whose trace has zero real part (See Subsection 6.3), while the second case is the split form $\mathfrak{g}_\theta = \mathfrak{sl}_n(\mathbb{R})$.

3. Type B_n. Then θ is an inner automorphism, given by an element of order ≤ 2 in $SO(2n+1)$. So $\theta = \mathrm{Id}_{2p+1} \oplus (-\mathrm{Id}_{2q})$ where $p + q = n$. Thus all the real forms are $\mathfrak{so}(2p+1, 2q)$ (all distinct), $\mathfrak{k} = \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2q}$.

4. Type C_n. Then θ is an inner automorphism, given by an element $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ such that $g^2 = 1$ or $g^2 = -1$. In the first case the 1-eigenspace of g has dimension $2p$ and the -1-eigenspace has dimension $2q$ (since they are symplectic), where $p + q = n$, and we may assume $p \geq q$ (replacing g by $-g$ if needed). So the real form we get is $\mathfrak{g}_\theta = \mathfrak{u}(p, q, \mathbb{H})$, the quaternionic pseudo-unitary Lie algebra for a quaternionic Hermitian form (see Subsection 6.3). In this case $\mathfrak{k} = \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q}$. On the other hand, if $g^2 = -1$ then $\mathbb{C}^{2n} = V(i) \oplus V(-i)$ (eigenspaces of g, which in this case are Lagrangian subspaces), so $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. The corresponding real form is the split form $\mathfrak{g}_\theta = \mathfrak{sp}_{2n}(\mathbb{R})$.

5. Type D_n, compact inner class. We again have an inner automorphism θ given by $g \in SO(2n)$ such that $g^2 = \pm 1$. If $g^2 = 1$ then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$, the direct sum of eigenspaces, and since $\det(g) = 1$, the eigenspaces are even-dimensional, of dimensions $2p$ and $2q$ where $p+q = n$, and, as in the case of type C_n, we may assume $p \geq q$. So the corresponding real form is $\mathfrak{g}_\theta = \mathfrak{so}(2p, 2q)$ with $\mathfrak{k} = \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2q}$. On the other hand, if $g^2 = -1$ then we have $\mathbb{C}^{2n} = V(i) \oplus V(-i)$, and these are Lagrangian subspaces of dimension n. So $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$. The corresponding real form is the quaternionic orthogonal Lie algebra (symmetries of a quaternionic skew-Hermitian form), $\mathfrak{g}_\theta = \mathfrak{so}^*(2n)$ (see Subsection 6.3).

6. Type D_n, the other inner class. In this case θ is given by an element g of $O(2n)$ such that $\det(g) = -1$ and $g^2 = \pm 1$. Note that if $g^2 = -1$ then , as shown above, $\det(g) = 1$, so in the case at hand we always have $g^2 = 1$. Then $\mathbb{C}^{2n} = V(1) \oplus V(-1)$, but now the dimensions of these spaces are odd, $2p + 1$ and $2q - 1$ where $p + q = n$, and we may assume that $p + 1 \geq q$. So the real form is $\mathfrak{g}_\theta = \mathfrak{so}(2p+1, 2q-1)$, with $\mathfrak{k} = \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2q-1}$. Note that for $n = 3$, $D_3 = A_3$, so we have $\mathfrak{so}(5, 1) = \mathfrak{sl}(2, \mathbb{H})$. Note also that this agrees with what we found before: the split form $\mathfrak{so}(n, n)$ is in the compact inner class for even n and in the other one for odd n, and the quasi-split form $\mathfrak{so}(n+1, n-1)$ the other way around.

Exercise 41.10. Compute the subalgebras $\mathfrak{k}$ for all the real forms of classical simple Lie algebras.

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