

40. Forms of semisimple Lie algebras over an arbitrary field

40.1. Automorphisms of semisimple Lie algebras. We showed in Corollary 17.10 that for a complex semisimple \mathfrak{g} , the group $\text{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra \mathfrak{g} . We also showed in Theorem 20.10 that its connected component of the identity $\text{Aut}(\mathfrak{g})^\circ$ acts transitively on the set of Cartan subalgebras in \mathfrak{g} . This group is called the **adjoint group** attached to \mathfrak{g} , and we will denote it by G_{ad} .

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset G_{\text{ad}}$ be the corresponding connected Lie subgroup. This subgroup can be viewed as the group of linear operators $\mathfrak{g} \rightarrow \mathfrak{g}$ which act by 1 on \mathfrak{h} and by $e^{\alpha(x)}$, $x \in \mathfrak{h}$, on each \mathfrak{g}_α . Thus the exponential map $\mathfrak{h} \rightarrow H$ defines an isomorphism $\mathfrak{h}/2\pi i P^\vee \cong H$. The group H is called the **maximal torus** of G_{ad} corresponding to \mathfrak{h} .

Proposition 40.1. *The normalizer $N(H)$ of H in G_{ad} coincides with the stabilizer of \mathfrak{h} and contains H as a normal subgroup, so that $N(H)/H$ is naturally isomorphic to the Weyl group W .*

Proof. First note that since $SL_2(\mathbb{C})$ is simply connected, for any simple root α_i we have a homomorphism $\eta_i : SL_2(\mathbb{C}) \rightarrow G_{\text{ad}}$ which identifies $\text{Lie}(SL_2(\mathbb{C}))$ with the \mathfrak{sl}_2 -subalgebra of \mathfrak{g} corresponding to this simple root. Let

$$(40.1) \quad S_i := \eta_i \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

Given $w \in W$, pick a decomposition $w = s_{i_1} \dots s_{i_n}$, and let $\tilde{w} := S_{i_1} \dots S_{i_n} \in G_{\text{ad}}$.²¹ Note that \tilde{w} acts on \mathfrak{h} by w . So if $w = w_1 w_2 \in W$ then $\tilde{w} = \tilde{w}_1 \tilde{w}_2 h$, where h preserves the root decomposition and acts trivially on \mathfrak{h} . Thus if $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$ then $h = \exp(\sum_j b_j \omega_j^\vee) \in H$. So the elements \tilde{w} and H generate a subgroup $N \subset N(H)$ of G_{ad} such that $N/H \cong W$.

It remains to show that $N(H) = N$. To this end, for $x \in N(H)$, let $\alpha'_i = x(\alpha_i)$. Then α'_i form a system of simple roots, so there exists $w \in W$ such that $w(\alpha'_i) = \alpha_{p(i)}$, where p is some permutation. Then $\tilde{w}x(\alpha_i) = \alpha_{p(i)}$. So $\tilde{w}x$ defines a Dynkin diagram automorphism of \mathfrak{g} . Since this automorphism is defined by an element of G_{ad} , it stabilizes all fundamental representations, so $p = \text{id}$, hence $\tilde{w}x \in H$, as claimed. \square

In particular, we see that H is a maximal commutative subgroup of G_{ad} , hence the terminology “maximal torus”.

²¹The element \tilde{w} in general depends on the decomposition of w as a product of simple reflections. One can show it does not if we take only reduced decompositions, but we will not need this.

Remark 40.2. Note that in general $N(H)$ is **not** isomorphic to $W \rtimes H$: it can be a non-split extension of W by H .

Another obvious subgroup of $\text{Aut}(\mathfrak{g})$ is the finite group $\text{Aut}(D)$ of automorphisms of the Dynkin diagram of \mathfrak{g} , which just permutes the generators e_i, f_i, h_i in the Serre presentation. Thus we have a natural homomorphism

$$\xi : \text{Aut}(D) \rtimes G_{\text{ad}} \rightarrow \text{Aut}(\mathfrak{g}),$$

which is the identity map on the connected components of 1. This homomorphism is clearly injective, since the center of G_{ad} is trivial and any nontrivial element of $\text{Aut}(D)$ nontrivially permutes fundamental representations of \mathfrak{g} .

Proposition 40.3. ξ is an isomorphism.

Proof. Our job is to show that ξ is surjective, i.e. for $a \in \text{Aut}(\mathfrak{g})$ show that $a \in \text{Im}\xi$. By Theorem 20.10, we may assume without loss of generality that a preserves a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (indeed, this can be arranged by multiplying by an element of G_{ad} , since G_{ad} acts transitively on Cartan subalgebras of \mathfrak{g}). Then by multiplying by an element of $\text{Aut}(D) \cdot N(H)$ we can make sure that a acts trivially on \mathfrak{h} and \mathfrak{g}_{α_i} . Then $a = 1$, which implies the proposition. \square

40.2. Forms of semisimple Lie algebras. We have classified semisimple Lie algebras over \mathbb{C} , but what about other fields (say of characteristic zero), notably \mathbb{R} (the case relevant to the theory of Lie groups)?

To address this question, note that the Serre presentation of a semisimple Lie algebra is defined over \mathbb{Q} , so it defines a Lie algebra of the same dimension over any such field, by imposing the same generators and relations. Such a Lie algebra is called **split**. So for example, over an algebraically closed field of characteristic zero, any semisimple Lie algebra is automatically split.

Now let \mathfrak{g} be a semisimple Lie algebra over a field K of characteristic zero which splits over a Galois extension L of K , i.e., $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ is split (corresponds to a Dynkin diagram via Serre's presentation). Can we classify such \mathfrak{g} ?

To this end, let $\Gamma = \text{Gal}(L/K)$ be the Galois group of L over K and observe that we can recover \mathfrak{g} as the subalgebra of invariants \mathfrak{g}_L^Γ . So \mathfrak{g} is determined by the action of Γ on the split semisimple Lie algebra \mathfrak{g}_L . Note that this action is **twisted-linear**, i.e., additive and $g(\lambda x) = g(\lambda)g(x)$ for $x \in \mathfrak{g}_L$, $\lambda \in L$, $g \in \Gamma$. The simplest example of such an action is the action $\rho_0(g)$ which preserves all the generators e_i, f_i, h_i and just acts on the scalars, which corresponds to the split form of \mathfrak{g} .

So any twisted-linear action ρ can be written as

$$\rho(g) = \eta(g)\rho_0(g)$$

for some map

$$\eta : \Gamma \rightarrow \text{Aut}(\mathfrak{g}_L).$$

In order that ρ be a homomorphism, we need

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

which is equivalent to

$$\eta(gh) = \eta(g) \cdot g(\eta(h)),$$

where for $a \in \text{Aut}(\mathfrak{g}_L)$, $g(a) := \rho_0(g)a\rho_0(g)^{-1}$. In other words, η is a **1-cocycle**. We will denote the Lie algebra attached to such cocycle η by \mathfrak{g}_η .

It remains to determine when \mathfrak{g}_{η_1} is isomorphic to \mathfrak{g}_{η_2} . This will happen exactly when the corresponding representations ρ_1 and ρ_2 are isomorphic, i.e., there is $a \in \text{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$, i.e.,

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

or

$$\eta_1(g) = a\eta_2(g)g(a)^{-1}.$$

Two 1-cocycles related in this way are called **cohomologous** (obviously, an equivalence relation), and the set of equivalence classes of cohomologous cocycles is called the **first Galois cohomology** of Γ with coefficients in $\text{Aut}(\mathfrak{g}_L)$ and denoted by $H^1(\Gamma, \text{Aut}(\mathfrak{g}_L))$. Note that this is cohomology with coefficients in a nonabelian group, so it is just a set and not a group.

So we obtain

Proposition 40.4. *Semisimple Lie algebras \mathfrak{g} over K which split over a Galois extension L of K are classified by the first Galois cohomology $H^1(\Gamma, \text{Aut}(\mathfrak{g}_L))$.*

Remark 40.5. There is nothing special about semisimplicity or about Lie algebras here – this works for any kind of linear algebraic structures, such as associative algebras, algebraic varieties, schemes, etc.

40.3. Real forms of a semisimple Lie algebra. Let us now make this classification more concrete in the case $K = \mathbb{R}$, $L = \mathbb{C}$, which is relevant to classification of real semisimple Lie groups. In this case, $\Gamma = \mathbb{Z}/2$ generated by complex conjugation $s \mapsto \bar{s}$ and, as we have shown, $\text{Aut}(\mathfrak{g}_L) = \text{Aut}(D) \rtimes G_{\text{ad}}$, where D is the Dynkin diagram of \mathfrak{g} and G_{ad} is the corresponding connected adjoint complex Lie group. Also since we always have $\eta(1) = 1$, the cocycle η is determined by

the element $s = \eta(-1) \in \text{Aut}(D) \times G_{\text{ad}}$. Moreover, s must satisfy the cocycle condition

$$s\bar{s} = 1$$

and the corresponding real Lie algebra, up to isomorphism, depends only on the cohomology class of s , which is the equivalence class modulo transformations $s \mapsto as\bar{a}^{-1}$. We thus obtain the following theorem.

Theorem 40.6. *Real semisimple Lie algebras whose complexification is \mathfrak{g} (i.e., **real forms of \mathfrak{g}**) are classified by $s \in \text{Aut}(D) \times G_{\text{ad}}$ such that $s\bar{s} = 1$ modulo equivalence $s \mapsto as\bar{a}^{-1}$, $a \in \text{Aut}(\mathfrak{g})$, where complex conjugation acts trivially on $\text{Aut}(D)$.*

We denote the real form of \mathfrak{g} corresponding to s by $\mathfrak{g}_{(s)}$. Namely, $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \bar{x} = s(x)\}$. For example, $\mathfrak{g}_{(1)}$ is the split form, consisting of real $x \in \mathfrak{g}$, i.e., such that $\bar{x} = x$.

Alternatively, one may define the **antilinear involution** $\sigma_s(x) = \overline{s(x)}$, and $\mathfrak{g}_{(s)}$ is the set of fixed points of σ_s in \mathfrak{g} .

In particular, such s defines an element $s_0 \in \text{Aut}(D)$ such that $s_0^2 = 1$. Note that the conjugacy class of s_0 is invariant under equivalences. The element s_0 permutes connected components of D , preserving some and matching others into pairs. Thus every semisimple real Lie algebra is a direct sum of simple ones, and each simple one either has a connected Dynkin diagram D (i.e., the complexified Lie algebra \mathfrak{g} is still simple) or consists of two identical components (i.e., the complexified Lie algebra is $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$ for some simple complex \mathfrak{a}). In the latter case $s = (g, \bar{g}^{-1})s_0$ where s_0 is the transposition and $g \in \text{Aut}(\mathfrak{a})$, so s is cohomologous to s_0 by taking $a = (g, 1)$. Thus in this case $\mathfrak{g}_{(s)} = \mathfrak{g}_{(s_0)} = \mathfrak{a}$, a complex simple Lie algebra regarded as a real Lie algebra.

It remains to consider the case when D is connected, i.e., \mathfrak{g} is simple.

Definition 40.7. (i) A real form $\mathfrak{g}_{(s)}$ of a complex simple Lie algebra \mathfrak{g} is said to be **inner** to $\mathfrak{g}_{(s')}$ if $s' = gs$ up to equivalence, where $g \in G_{\text{ad}}$ (i.e., s and s' differ by an inner automorphism). The **inner class** of $\mathfrak{g}_{(s)}$ is the collection of all real forms inner to $\mathfrak{g}_{(s)}$. In particular, an **inner form** is a form inner to the split form.

(ii) $\mathfrak{g}_{(s)}$ is called **quasi-split** if $s = s_0 \in \text{Aut}(D)$ (modulo equivalence).

So in particular any real form is inner to a unique quasi-split form, and a real form that is both inner and quasi-split is split.

Exercise 40.8. Let $\mathfrak{g}_{\mathbb{R}}$ be a real semisimple Lie algebra and $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ a Cartan subalgebra (the centralizer of a regular semisimple element

of $\mathfrak{g}_{\mathbb{R}}$). Let $\mathfrak{h} \subset \mathfrak{g}$ be their complexifications, and $H \subset G_{\text{ad}}$ the corresponding complex Lie groups. Let \mathbf{K} be the kernel of the natural map of Galois cohomology sets $H^1(\mathbb{Z}/2, N(H)) \rightarrow H^1(\mathbb{Z}/2, G_{\text{ad}})$ (i.e., the preimage of the unit element), where $\mathbb{Z}/2$ acts on G_{ad} by complex conjugation associated to the real form $\mathfrak{g}_{\mathbb{R}}$ of \mathfrak{g} .

(i) Show that conjugacy classes of Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$ are bijectively labeled by elements of \mathbf{K} , with the unit element corresponding to $\mathfrak{h}_{\mathbb{R}}$.

(ii) Show that \mathbf{K} is a finite set.²²

²²For classical Lie algebras the set \mathbf{K} will be computed explicitly in Exercise 44.18. The explicit answer is known for exceptional Lie algebras as well, but we will not discuss it here.

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