35. **Representations of compact Lie groups**

35.1. **Unitary representations.** Now we can extend to compact groups the result that representations of finite groups are unitary. Namely, let V be a finite dimensional (continuous) complex representation of a compact Lie group G.

Proposition 35.1. *V admits a G-invariant unitary structure.*

Proof. Fix a positive Hermitian form B on V and define a new Hermitian form on V by

$$B_{\mathrm{av}}(v, w) = \int_G B(\rho_V(g)v, \rho_V(g)w)dg.$$

This form is well defined since G is compact and is G-invariant by construction (since the measure dg is invariant). Also $B_{\mathrm{av}}(v, v) > 0$ for $v \neq 0$ since $B(w, w) > 0$ for any $w \neq 0$. □

Corollary 35.2. *Every finite dimensional representation V of a compact Lie group G is completely reducible.*

Proof. Let $W \subset V$ be a subrepresentation and B be an invariant positive Hermitian form on V. Let $W^\perp \subset V$ be the orthogonal complement of W under B. Then $V = W \oplus W^\perp$, which implies the statement. □

In particular, this applies to the special unitary group $SU(n)$. Recall that $SU(n)/SU(n-1) = S^{2n-1}$, which implies that $SU(n)$ is simply connected. Thus (smooth) representations of $SU(n)$ is the same thing as representations of the Lie algebra $\mathfrak{su}(n)$ or its complexification $\mathfrak{sl}_n$. Thus we get a new, analytic proof that finite dimensional representations of $\mathfrak{sl}_n$ are completely reducible (this is called **Weyl's unitary trick**). In fact, we will see that complete reducibility of finite dimensional representations of all semisimple Lie algebras can be proved in this way.

35.2. **Matrix coefficients.** Let V be a finite dimensional continuous complex representation of a Lie group G. A **matrix coefficient** of V is a function $G \to \mathbb{C}$ of the form $(f, \rho_V(g)v)$ for some $v \in V$ and $f \in V^*$. Obviously, such a function is continuous.

Proposition 35.3. *Matrix coefficients are smooth.*

Proof. Let us say that $v \in V$ is smooth if the function $f(\rho_V(g)v)$ is smooth for any $f \in V^*$; it is clear that such vectors form a subspace V_{sm} of V. Our job is to show that, in fact, $V_{\mathrm{sm}} = V$. To this end let us first construct some smooth vectors. For this let $\phi : G \to \mathbb{C}$ be a smooth function with compact support, and let

$$w = w(\phi, v) := \int_G \phi(g)\rho_V(g)v dg,$$

where dg is a left-invariant Haar measure on G and $v \in V$. We claim that w is a smooth vector. Indeed,

$$f(\rho_V(h)w) = f\left(\rho_V(h)\int_G \phi(g)\rho_V(g)vdg\right) =$$

$$\int_G f(\phi(g)\rho_V(hg)v)dg = \int_G f(\phi(h^{-1}g)\rho_V(g)v)dg,$$

and this is manifestly smooth in h (we can differentiate indefinitely under the integral sign).

Define a **delta-like sequence** (or a **Dirac sequence**) around a point $x_0 \in M$ on a manifold M with a smooth measure dx to be a sequence of continuous functions ϕ_n on M such that for every neighborhood U of x_0 the supports of almost all ϕ_n are contained in U, and $\int_M \phi_n(x)dx = 1$. The "hat" function construction implies that delta-like sequences exist and can be chosen non-negative and smooth. Namely, we can pick a sequence of non-negative smooth functions satisfying the first condition and then normalize it to satisfy the second one.

Now let ϕ_n be a smooth delta-like sequence around 1 on G with left-invariant Haar measure. Let $w_n := w(\phi_n, v)$. It is obvious that $w_n \to v$ as $n \to \infty$. Thus $V_{\rm sm}$ is dense in V. Since V is finite dimensional, it follows that $V_{\rm sm} = V$, as claimed. $\square$

Now let V be an irreducible representation of a compact Lie group G. As shown above, it has an invariant positive Hermitian inner product, which we'll denote by $(,)$. Moreover, this product is unique up to scaling. Pick an orthonormal basis $v_1, ..., v_n$ of V under this inner product, and let $v_1^*, ..., v_n^*$ be the dual basis of V^*. Now consider the matrix coefficients of V in this basis:

$$\psi_{V,ij}(g) := v_j^*(\rho_V(g)v_i) = (\rho_V(g)v_i, v_j).$$

Note that these functions are independent on the normalization of $(,)$.

Suppose now that we also have another such representation W with orthonormal basis w_i.

Theorem 35.4. *(Orthogonality of matrix coefficients) We have*

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,kl}(g)}dg = 0$$

if V is not isomorphic to W. Also

$$\int_G \psi_{V,ij}(g)\overline{\psi_{V,kl}(g)}dg = \frac{\delta_{ik}\delta_{jl}}{\dim V}.$$

Proof. We have

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,kl}(g)}dg = \int_G ((\rho_V(g)\otimes\rho_{\overline{W}}(g))(v_i\otimes w_k), v_j\otimes w_l)dg =$$
$$(P(v_i\otimes w_k), v_j\otimes w_l)$$

where

$$P := \int_G \rho_V(g)\otimes\rho_{\overline{W}}(g)dg = \int_G \rho_{V\otimes\overline{W}}(g)dg.$$

Since W is unitary, $\overline{W}\cong W^*$, so we have

$$P = \int_G \rho_{V\otimes W^*}(g)dg : V\otimes W^* \to V\otimes W^*.$$

By construction, $\mathrm{Im}(P)\subset (V\otimes W^*)^G$, which is zero if $V\ncong W$. Thus we have proved the proposition in this case.

It remains to consider the case $V = W$. In this case $V\otimes W^* = V\otimes V^* = V\otimes\overline{V}$, and the only invariant in this space up to scaling is $\mathbf{u} := \sum_k v_k\otimes v_k$. Also P is conjugation invariant under G, so by decomposing $V\otimes V^*$ into irreducibles we see that it is the orthogonal projector to $\mathbb{C}\mathbf{u}$:

$$P\mathbf{x} = \frac{(\mathbf{x},\mathbf{u})}{(\mathbf{u},\mathbf{u})}\mathbf{u} = \frac{(\mathbf{x},\mathbf{u})\mathbf{u}}{\dim V}.$$

In particular,

$$(P(v_i\otimes w_k), v_j\otimes w_l) = \frac{\delta_{ik}\delta_{jl}}{\dim V},$$

as claimed. □

35.3. **The Peter-Weyl theorem.** Thus we see that the functions $\psi_{V,ij}$ for various V,i,j form an orthogonal system in the Hilbert space $L^2(G) = L^2(G,dg)$ of measurable functions $f : G\to\mathbb{C}$ such that

$$||f||^2 = \int_G |f(g)|^2 dg < \infty.$$

A fundamental result about compact Lie groups is that this system is, in fact, complete:

Theorem 35.5. *(Peter-Weyl theorem) The functions $\psi_{V,ij}$ form an orthogonal basis of $L^2(G)$.*

Theorem 35.5 will be proved in Section 36.

35.4. **An alternative formulation of the Peter-Weyl theorem.** Given a finite dimensional irreducible representation V of G, consider the space $\mathrm{Hom}_G(V, L^2(G))$ of G-homomorphisms for the action of G on $L^2(G)$ by left translations. We have an obvious inclusion

$$\iota_V : V^* \hookrightarrow \mathrm{Hom}_G(V, L^2(G))$$

via the matrix coefficient map $f \mapsto [v \mapsto (\rho_{V^*}(-)f)(v)]$. Clearly, this is a map of G-modules, where now G acts on $L^2(G)$ by right translations. We claim that ι_V is surjective, i.e., an isomorphism. For this, note that an element $\phi \in \mathrm{Hom}_G(V, L^2(G))$ can be viewed a left G-equivariant L^2-function $\phi : G \to V^*$, i.e. such that for almost all $g \in G$ (with respect to the Haar measure) we have

$$\phi(x) = \rho_{V^*}(xg^{-1})\phi(g) \tag{35.1}$$

for almost all $x \in G$. But then by changing ϕ on a set of measure zero if needed, we may replace it by a continuous function (the right hand side of (35.1)). Then, setting $g = 1$, we have $\phi(x) = \rho_{V^*}(x)\phi(1)$, as claimed.

Thus we have a natural inclusion

$$\xi : \oplus_{V \in \mathrm{Irrep}(G)} V \otimes V^* \cong \oplus_{V \in \mathrm{Irrep}(G)} V \otimes \mathrm{Hom}_G(V, L^2(G)) \hookrightarrow L^2(G),$$

which is actually an embedding of $G \times G$-modules, and we will denote the image of ξ by $L^2_{\mathrm{alg}}(G)$ (the "algebraic part" of $L^2(G)$). Note that if $\psi \in L^2(G)$ generates a finite dimensional representation V under the action of G by left translations then ψ belongs to the image of a homomorphism $V \to L^2(G)$, hence to $L^2_{\mathrm{alg}}(G)$. Thus $L^2_{\mathrm{alg}}(G)$ is just the subspace of $\psi \in L^2(G)$ which generate a finite dimensional representation under left translations by G. We also see that it may be equivalently characterized as the subspace of $\psi \in L^2(G)$ which generate a finite dimensional representation under right translations by G.

Theorem 35.6. *(Peter-Weyl theorem, alternative formulation) The space $L^2_{\mathrm{alg}}(G)$ is dense in $L^2(G)$. In other words, the map ξ gives rise to an isomorphism*

$$\widehat{\oplus}_{V \in \mathrm{Irrep}(G)} V \otimes V^* \to L^2(G)$$

where the first copy of G acts on V and the second one on V^* and the hat denotes the Hilbert space completion of the direct sum.

Note that this is again an instance of the double centralizer property! Namely, it expresses representation-theoretically the fact that the centralizer of the group of left translations on G is the group of right translations on G, and vice versa.

For example, let $G = S^1$. Then the irreducible representations of G are the characters $\psi_n(\theta) = e^{in\theta}$. So the Peter-Weyl theorem in this case says that $\{e^{in\theta}\}$ is an orthonormal basis of $L^2(S^1)$ with norm

$$||f||^2 := \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta,$$

which is the starting point for Fourier analysis. So the Peter-Weyl theorem is similarly a starting point for **nonabelian Fourier (or harmonic) analysis**.

Exercise 35.7. Let G be a compact Lie group and $H \subset G$ a closed subgroup. Then we have a compact homogeneous space G/H and the Haar measure on G defines a probability measure on G/H. So we can define the infinite dimensional unitary representation $L^2(G/H)$ of G.

(i) Show that have a decomposition

$$L^2(G/H) = \widehat{\oplus}_{V\in \mathrm{Irrep}G} N_H(V)V,$$

where $N_H(V) = \dim V^H$, the dimension of the space of H-invariants of V.

(ii) Let $G = SO(3)$, so the irreducible representations are L_{2m} for $m \ge 0$. Thus

$$L^2(G/H) = \widehat{\oplus}_{m\ge 0} N_H(m) L_{2m}.$$

Compute this decomposition (i.e., the numbers $N_H(m)$) for $H = \mathbb{Z}/n\mathbb{Z}$ acting by rotations around an axis by angles $2\pi k/n$ (rotations of a regular n-gon).

(iii) Do the same for the dihedral group $H = \mathbf{D}_n$ of symmetries of the regular n-gon (where reflections in the plane are realized as rotations around a line in this plane).

(iv) Do the same for the groups $H = SO(2)$ and $H = O(2)$ of rotations and symmetries of the circle.

(v) Do the same for H being the group of symmetries of a platonic solid (tetrahedron, cube, icosahedron).

It may be more convenient to give $N_V(m)$ in the form of the generating function $\sum_m N_V(m)t^m$.

Exercise 35.8. Let $G = GL_n(\mathbb{C})$. A **regular algebraic function** on G is a polynomial of X_{ij} and $\det(X)^{-1}$ for $X \in G$. Denote by $\mathcal{O}(G)$ the algebra of regular algebraic functions on G.

(i) Show that $G \times G$ acts on $\mathcal{O}(G)$ by left and right multiplication.

(ii) (Algebraic Peter-Weyl theorem) Show that as a $G \times G$-module, we have

$$\mathcal{O}(G) = \oplus_{V\in \mathrm{Irrep}(G)} V \otimes V^*.$$

Hint. Compute $\mathrm{Hom}_G(V, \mathcal{O}(G))$ where G acts on $\mathcal{O}(G)$ by right translations. For this, interpret elements of this space as equivariant functions $G \to V^*$ and show that such functions are automatically regular algebraic.

(iii) Generalize (i) and (ii) to orthogonal and symplectic groups.

35.5. **Orthogonality and completeness of characters.**

Corollary 35.9. *Let $\chi_V(g) = \mathrm{Tr}(\rho_V(g))$ be the character of V. Then $\{\chi_V(g), V \in \mathrm{Irrep} G\}$ is an orthonormal basis of $L^2(G)^G$, the space of conjugation-invariant functions in $L^2(G)$ (i.e., such that $f(gxg^{-1}) = f(x)$).*

Proof. We have $\chi_V(g) = \sum_i \psi_{V,ii}(g)$, so by orthogonality of matrix coefficients χ_V are orthonormal in $L^2(G)^G$. So it remains to show that they are complete. For this observe that $L^2_{\mathrm{alg}}(G)^G = \xi(\oplus_V (V \otimes V^*)^G) = \oplus_V \mathbb{C}\chi_V$. Thus our job is to show that $L^2_{\mathrm{alg}}(G)^G$ is dense in $L^2(G)^G$. To this end, for $\psi \in L^2(G)^G$ fix a sequence $\psi_n \in L^2_{\mathrm{alg}}(G)$ such that $\psi_n \to \psi$ as $n \to \infty$. Such a sequence exists by the Peter-Weyl theorem. Let

$$\psi_n^{\mathrm{av}}(x) = \int_G \psi_n(gxg^{-1})dg.$$

It is easy to see that $\psi_n^{\mathrm{av}} \in L^2_{\mathrm{alg}}(G)$. Also $||\psi_n^{\mathrm{av}} - \psi|| \le ||\psi_n - \psi|| \to 0$, $n \to \infty$, as claimed. $\square$

18.755 Lie Groups and Lie Algebras II
Spring 2024