34. **Integration on manifolds**

34.1. **Integration of top differential forms on oriented manifolds.** An important operation with top degree differential forms is **integration**. Namely, if ω is a differential n-form on an open set $U \subset \mathbb{R}^n$ (with the usual orientation), $\omega = f(x_1, ..., x_n)dx_1 \wedge ... \wedge dx_n$, then we can set

$$\int_U \omega := \int_U f(x_1, ..., x_n)dx_1...dx_n.$$

(provided this integral is absolutely convergent). This, however, is not completely canonical: if we change coordinates (so that U maps diffeomorphically to U'), the change of variable formula in a multiple integral tells us that

$$\int_U f(x_1, ..., x_n)dx_1\wedge...\wedge dx_n = \int_{U'} f(x_1(\mathbf{y}), ..., x_n(\mathbf{y}))\left|\det\left(\frac{\partial x_i}{\partial y_j}\right)\right| dy_1\wedge...\wedge dy_n,$$

while the transformation law for ω is the same but without the absolute value. This shows that our definition is invariant only under orientation preserving transformations of coordinates, i.e., ones whose Jacobian $\det\left(\frac{\partial x_i}{\partial y_j}\right)$ is positive. Consequently, we will only be able to define integration of top differential forms on **oriented manifolds**, i.e., ones equipped with an atlas of charts in which transition maps have a positive Jacobian; such an atlas defines an **orientation** on M. To fix an orientation, we just need to say which local coordinate systems (or bases of tangent spaces) are right-handed, and do so in a consistent way. But this cannot always be done globally (the classic counterexamples are Möbius strip and Klein bottle).

Now let us proceed to define integration of a continuous top form ω over an oriented manifold M. For this pick an atlas of local charts $\{U_i, i \in I\}$ on M and pick a partition of unity $\{f_s\}$ subordinate to this cover, which is possible by Proposition 33.4. First assume that ω is nonnegative, i.e., $\omega(v_1, ..., v_n) \geq 0$ for a right-handed basis v_i of any tangent space of M. Then define

$$\int_M \omega := \sum_s \int_{U_{i(s)}} f_{i(s)}\omega \qquad (34.1)$$

where in each U_i we use a right-handed coordinate system to compute the corresponding integral. This makes sense (as a nonnegative real number or $+\infty$), and is also independent of the choice of a partition of unity. Indeed, it is easy to see that for two atlases $\{U_i\}$, $\{V_j\}$ and two partitions of unity $\{f_s\}, \{g_t\}$ the answer is the same, by comparing both to the answer for the atlas $\{U_i \cap V_j\}$ and partition of unity $\{f_s g_t\}$. In fact, this makes sense for any measurable ω (i.e., given by a measurable function in every local chart) if we use Lebesgue integration.

Now, if ω is not necessarily nonnegative, we may define the nonnegative form $|\omega|$ which is ω at points where ω is nonnegative and $-\omega$ otherwise. Then, if

$$\int_M |\omega| < \infty,$$

we can define $\int_M \omega$ by the same formula (34.1) which will now be a not necessarily positive but absolutely convergent series (a finite sum in the compact case).

Importantly, the same definition works for manifolds M with boundary ∂M (an $n-1$-manifold); the only difference is that at boundary points the manifold locally looks like $\mathbb{R}^n_+$ (the space of vectors with nonnegative last coordinate) rather than $\mathbb{R}^n$. Note that the boundary of an oriented manifold carries a canonical orientation as well (a basis of $T_p\partial M$ is right-handed if adding to it a vector looking inside M produces a right-handed basis of T_pM).

Remark 34.1. If the manifold M is non-orientable, we cannot integrate top differential forms on M. However, we can integrate **densities** on M, which are sections of the line bundle $|\wedge^n T^*M|$, the absolute value of the orientation bundle. This bundle is defined by transition functions $|g_{ij}(x)|$, where $g_{ij}(x)$ are the transition functions of $\wedge^n T^*M$. Thus its sections, called densities on M, transform under changes of coordinates according to the rule

$$f(x_1, ..., x_n)|dx_1\wedge...\wedge dx_n| = f(x_1(\mathbf{y}), ..., x_n(\mathbf{y}))|\det\left(\frac{\partial x_i}{\partial y_j}\right)|\cdot|dy_1\wedge...\wedge dy_n|,$$

i.e., exactly the one needed for the integral to be defined canonically. This procedure actually makes sense for any manifold, and in the oriented case reduces to integration of top forms described above.

Using partitions of unity, it is not hard to show that the bundle $|\wedge^n T^*M|$ is trivial (check it!). A positive smooth section of this bundle (i.e., positive in every chart) therefore exists and is nothing but a **positive smooth measure** on M, and any two such measures differ by multiplication by a positive smooth function. Moreover, given such a measure μ and a measurable function f on M such that $\int_M |f|d\mu < \infty$ (i.e., $f \in L^1(M,\mu)$), we can define $\int_M f d\mu$ as usual.

34.2. **Nonvanishing forms.** Let us say that a top degree continuous differential form ω on M is **non-vanishing** if for any $x \in M$, $\omega_x \in \wedge^n T_x^*M$ is nonzero. In this case, ω defines an orientation on M by declaring a basis $v_1, ..., v_n$ of T_xM right-handed if $\omega(v_1, ..., v_n) > 0$

(in particular, there are no non-vanishing top forms on non-orientable manifolds). Thus we can integrate top differential forms on M, and in particular ω defines a positive measure $\mu = \mu_\omega$ on M, namely

$$\mu(U) = \int_U \omega$$

for an open set $U \subset M$ (this integral may be $+\infty$, but is finite if U is a small enough neighborhood of any point $x \in M$). Thus we can integrate functions on M with respect to this measure:

$$\int_M f d\mu = \int_M f\omega.$$

This, of course, only makes sense if f is measurable and $\int_M |f| d\mu < \infty$, i.e., if $f \in L^1(M, \mu)$. Note also that if $\lambda \in \mathbb{R}^\times$ then $\mu_{\lambda\omega} = |\lambda|\mu_\omega$.

Example 34.2. If M is an open set in $\mathbb{R}^n$ with the usual orientation and $\omega = dx_1 \wedge ... \wedge dx_n$ then $\int_M \omega = \int_M dx_1...dx_n$ is just the volume of M. For this reason top differential forms are often called **volume forms**, especially when they are non-vanishing and thus define an orientation and a measure on M, and in the latter case $\int_M \omega$, if finite, is called the **volume of** M with respect to ω.

Proposition 34.3. *If M is compact and ω is non-vanishing then M has finite volume under the measure $\mu = \mu_\omega$, and every bounded measurable (in particular, any continuous) function on M is in $L^1(M, \mu)$.*

Proof. For each $x \in M$ choose a neighborhood U_x of x such that $\mu(U_x) < \infty$. The collection of sets U_x forms an open cover of M, so it has a finite subcover $U_1, ..., U_N$, and $\mu(M) \le \mu(U_1) + ... + \mu(U_N) < \infty$. Then $\int_M |f| d\mu \le \mu(M) \sup |f| < \infty$ for bounded measurable f. □

34.3. **Stokes formula.** A central result about integration of differential forms is

Theorem 34.4. *(Stokes formula) If M is an n-dimensional oriented manifold with boundary and ω a differential $n-1$-form on M of class C^1 then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, if M is closed (has no boundary) then $\int_M d\omega = 0$, and if ω is closed ($d\omega = 0$) then $\int_{\partial M} \omega = 0$.

When M is an interval in $\mathbb{R}$, this reduces to the fundamental theorem of calculus. If M is a region in $\mathbb{R}^2$, this reduces to Green's formula. If M is a surface in $\mathbb{R}^3$, this reduces to the classical Stokes formula from vector calculus. Finally, if M is a region in $\mathbb{R}^3$ then this reduces to the Gauss formula (Divergence theorem).

The proof of the Stokes formula is not difficult. Namely, by writing ω as $\sum_s f_s\omega$ for some partition of unity, it suffices to prove the formula for M being a box in $\mathbb{R}^n$, which easily follows from the fundamental theorem of calculus.

34.4. **Integration on Lie groups.** Now let G be a real Lie group of dimension n. In this case given any $\xi \in \wedge^n\mathfrak{g}^*$, we can extend it to a left-invariant skew-symmetric tensor field (i.e., top differential form) ω_ξ on G. Also, if $\xi \neq 0$ then $\omega = \omega_\xi$ is non-vanishing and thus defines an orientation and a left-invariant positive measure μ_ω on G. Note that ξ is unique up to scaling by a real number $\lambda \in \mathbb{R}^\times$. So, since $\mu_{\lambda\omega} = |\lambda|\mu_\omega$, we see that μ_ω is defined uniquely up to scaling by positive numbers. This measure is called the **left-invariant Haar measure** and we'll denote it just by μ_L (assuming that the normalization has been chosen somehow).

In a similar way we can define the **right invariant Haar measure** μ_R on G. One may ask if these measures coincide (or, rather, are proportional, since they are defined only up to normalization). This question is answered by the following proposition.

Given a 1-dimensional real representation V of a group G, let $|V|$ be the representation of G on the same space with $\rho_{|V|}(g) = |\rho_V(g)|$, where $\rho : G \to \mathrm{Aut}(V) = \mathbb{R}^\times$.

Proposition 34.5. *$\mu_L = \mu_R$ if and only if $|\wedge^n \mathfrak{g}^*|$ (or, equivalently, $|\wedge^n \mathfrak{g}|$) is a trivial representation of G.*

Proof. It is clear that $\mu_L = \mu_R$ if and only if the left-invariant top volume form ω on G is also right invariant up to sign. This is equivalent to saying that ω is conjugation invariant up to sign, i.e., that $\omega_1 \in \wedge^n\mathfrak{g}^*$ is invariant up to sign under the action of G. This implies the statement. □

If $\mu_L = \mu_R$ then G is called **unimodular**. In this case we have a **bi-invariant Haar measure** $\mu = \mu_L = \mu_R$ on G (under some normalization).

In particular, we see that if G has no nontrivial continuous characters $G \to \mathbb{R}^\times$ then it is unimodular.

Example 34.6. If G is a discrete countable group then G is unimodular and μ is the counting measure: $\mu(U) = |U|$ (number of elements in U).

Exercise 34.7. (i) Let us say that a finite dimensional real Lie algebra $\mathfrak{g}$ of dimension n is unimodular if $\wedge^n\mathfrak{g}$ is a trivial representation of $\mathfrak{g}$.

Show that a connected Lie group G is unimodular if and only if so is Lie$G$.

(ii) Show that a perfect Lie algebra (such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$) is unimodular. In particular, a semisimple Lie algebra is unimodular.

(iii) Show that a nilpotent (in particular, abelian) Lie algebra is unimodular.

(iv) Show that if $\mathfrak{g}_1, \mathfrak{g}_2$ are unimodular then so is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Deduce that a reductive Lie algebra is unimodular.

(v) Show that the Lie algebra of upper triangular matrices of size n is **not** unimodular for $n > 1$. Give an example of a Lie algebra $\mathfrak{g}$ and ideal I such that I and $\mathfrak{g}/I$ are unimodular but $\mathfrak{g}$ is not.

(vi) Give an example of a non-unimodular Lie group G such that its connected component of the identity G° is unimodular (try groups of the form $\mathbb{Z} \ltimes \mathbb{R}$).

For a unimodular Lie group G, we will sometimes denote the integral of a function f with respect to the Haar measure by

$$\int_G f(g)dg.$$

Proposition 34.8. *A compact Lie group is unimodular.*

Proof. The representation of G on $|\wedge^n \mathfrak{g}^*|$ defines a continuous homomorphism $\rho : G \to \mathbb{R}^+$. Since G is compact, the image $\rho(G)$ of ρ is a compact subgroup of $\mathbb{R}^+$. But the only such subgroup is the trivial group. This implies the statement. □

Thus, on a compact Lie group we have a (bi-invariant) Haar measure μ. Moreover, in this case $\int_G d\mu = \text{Volume}(G) < \infty$, so we have a canonical normalization of μ by the condition that it is a probability measure:

$$\int_G d\mu = 1.$$

E.g., for finite groups this normalization is the averaging measure, which is $|G|^{-1}$ times the counting measure. This is the normalization we will use if G is compact.

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