33. **Differential forms, partitions of unity**

Now we want to develop an integration theory on Lie groups. First we need to recall the basics about integration on manifolds.

33.1. **Locally compact spaces.** A Hausdorff topological space X is called **locally compact** if every point has a neighborhood whose closure is compact. For example, $\mathbb{R}^n$ and thus every manifold is locally compact.

Lemma 33.1. *If X is a locally compact topological space with a countable base then it can be represented as a nested union of compact subsets: $X = \cup_{n\in\mathbb{N}} K_n$, $K_i \subset K_{i+1}$, such that every point $x \in X$ has a neighborhood U_x contained in some K_n.*

Proof. For each $x \in X$ fix a neighborhood U_x of x such that $\overline{U}_x$ is compact. By Lemma 1.4 the open cover $\{U_x\}$ of X has a countable subcover $\{W_i, i \in \mathbb{N}\}$. Then the sets $K_n = \cup_{i=0}^n \overline{W_i}$ form a desired nested sequence of compact subsets of X. $\square$

An open cover of a topological space X is said to be **locally finite** if every point of X has a neighborhood intersecting only finitely many members of this cover.

Lemma 33.2. *Let X be a locally compact topological space with a countable base. Then every base of X has a countable, locally finite subcover.*

Proof. Use Lemma 33.1 to write X as a nested union of compact sets K_n such that every point is contained in some K_n together with its neighborhood. We construct the required subcover inductively as follows. Choose finitely many sets $U_1, ..., U_{N_0}$ of the base covering K_0, and remove all other members of the base which meet K_0. The remaining collection of open sets is no longer a base but still an open cover of X. So add finitely many new sets $U_{N_0+1}, ..., U_{N_1}$ from this cover (all necessarily disjoint from K_0) to our list so that it now covers K_1, and remove all other members that meet K_1, and so on. The remaining sequence $U_1, U_2, ...$ has only finitely many members which meets every K_n, so every point of X has a neighborhood meeting only finitely many U_i. $\square$

33.2. **Reminder on differential forms.** Let M be a real smooth n-dimensional manifold. Recall that a differential k-form on M is a smooth section of the vector bundle $\wedge^i T^*M$, i.e., a skew-symmetric $(n, 0)$-tensor field (see Subsection 5.3). Thus, for example, a 1-form is a section of T^*M. If $x_1, ..., x_n$ are local coordinates on M near some point $p \in M$ then the differentials $dx_1, ..., dx_n$ form a basis in fibers of T^*M near this point, so a general 1-form in these coordinates has the form

$$\omega = \sum_{i=1}^{n} f_i(x_1, ..., x_n) dx_i.$$

If we change the coordinates $x_1, ..., x_n$ to $y_1, ..., y_n$ then x_i are smooth functions of $y_1, ..., y_n$ and in the new coordinates ω looks like

$$\omega = \sum_{i,j=1}^{n} f_i(x_1, ..., x_n) \frac{\partial x_i}{\partial y_j} dy_j.$$

Similarly, a differential k-form in the coordinates x_i looks like

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1,...,i_k}(x_1, ..., x_n) dx_{i_1} \wedge ... \wedge dx_{i_k}$$

where $f_{i_1,...,i_k}$ are smooth functions, and in the coordinates y_j it looks like

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} \sum_{1 \le j_1 < ... < j_k \le n} f_{i_1,...,i_k}(x_1, ..., x_n) \det \left(\frac{\partial x_{i_r}}{\partial y_{j_s}} \right) dy_{j_1} \wedge ... \wedge dy_{j_k}.$$

The space of differential k-forms on M is denoted $\Omega^k(M)$. For instance, $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = 0$ for $k > n$. Consider now the extremal case $k = n$. The bundle $\wedge^n T^*M$ is a line bundle (a vector bundle of rank 1), so locally any differential n-form in coordinates x_i has the form

$$\omega = f(x_1, ..., x_n) dx_1 \wedge ... \wedge dx_n,$$

which in coordinates y_j takes the form

$$\omega = f(x_1, ..., x_n) \det \left(\frac{\partial x_i}{\partial y_j} \right) dy_1 \wedge ... \wedge dy_n.$$

We have a canonical differentiation operator $d : \Omega^0(M) \to \Omega^1(M)$ given in local coordinates by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

It is easy to check that this operator does not depend on the choice of coordinates (this becomes obvious if you define it without coordinates, $df(v) = \partial_v f$ for $v \in T_pM$). Also $\Omega^\bullet(M) := \oplus_{k=0}^{n} \Omega^k(M)$ is a graded algebra under wedge product, and d naturally extends to a degree 1 derivation $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ defined in coordinates by

$$d(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

Namely, this is independent on choices and gives rise to a derivation in the "graded" sense:

$$d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db.$$

A form ω is **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$ for some η. It is easy to check that $d^2 = 0$, so any exact form is closed. However, not every closed form is exact: on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ the form dx is closed but the function x is defined only up to adding integers, so dx is not exact. The space $\Omega^k_{\text{closed}}(M)/\Omega^k_{\text{exact}}(M)$ is called the k-th **de Rham cohomology** of M, denoted $H^k(M)$.

If $f : M \to N$ is a differentiable mapping then for a differential form $\omega \in \Omega^k(N)$ we can define the pullback $f^*\omega \in \Omega^k(N)$, given by $(f^*\omega)(v_1, , , .v_k) = \omega(f_*v_1, ..., f_*v_k)$ for $v_1, ..., v_k \in T_pM$. This operation commutes with wedge product and the differential, and $(f \circ g)^* = g^* \circ f^*$.

33.3. **Partitions of unity.** Let M be a manifold and $\{U_i, i \in I\}$ be an open cover of M.

Definition 33.3. A smooth **partition of unity** subordinate to $\{U_i, i \in I\}$ is a collection $\{f_s, s \in S\}$ of smooth nonnegative functions on M such that

(i) for all s the support of f_s is contained in U_i for some $i = i(s)$;

(ii) Any $y \in M$ has a neighborhood in which all but finitely many f_s are zero;

(iii) $\sum_s f_s = 1$.

Note that the sum in (iii) makes sense because of condition (ii).

Note also that given any partition of unity $\{f_s\}$ subordinate to $\{U_i\}$, we can define

$$F_i := \sum_{s:i(s)=i} f_s,$$

and this is a new partition of unity subordinate to the same cover now labeled by the set I, with the support of F_i contained in U_i.

Finally, note that in every partition of unity on M, the set of s such that f_s is not identically zero is countable, and moreover finite if M is compact. This follows from the fact that by Lemma 1.4, any open cover of a manifold M has a countable subcover, and moreover a finite one if M is compact (applied to the neighborhoods from condition (ii)).

Proposition 33.4. *Any open cover $\{U_i, i \in I\}$ of a manifold M admits a partition of unity subordinate to this cover.*

Proof. Define a function $h : [0, \infty] \to \mathbb{R}$ given by $h(t) = 0$ for $t \geq 1$ and $h(t) = \exp(\frac{1}{t-1})$ for $t < 1$. It is easy to check that h is smooth.

Thus we can define the smooth **hat function** $H(x) := h(|x|^2)$ on $\mathbb{R}^n$, supported on the closed unit ball $\overline{B(0,1)}$.

If $\phi : \overline{B(0,1)} \to M$ is a C^∞-map which is a diffeomorphism onto the image, we will say that the image of ϕ is a **closed ball** in M. Thus given a closed ball $\overline{B}$ on M (equipped with a diffeomorphism $\phi : \overline{B(0,1)} \to \overline{B}$), we have a hat function $H_B(y) := H(\phi^{-1}(y))$ on $\overline{B}$, which we extend by zero to a smooth function on M whose support is $\overline{B}$ and which is strictly positive in its interior $B \subset \overline{B}$.

Now let $\{\overline{B}_s, s \in J\}$ be the collection of all closed balls in M such that their interiors B_s are contained in some U_i. Then $\{B_s, s \in J\}$ is clearly a base for M. Thus by Lemma 33.2, this base has a countable, locally finite subcover $\{B_s, s \in S\}$. Picking diffeomorphisms $\phi_s : \overline{B(0,1)} \to \overline{B}_s, s \in S$, we can define the smooth function $F(y) := \sum_{s \in S} H_{B_s}(y)$, which is strictly positive on M since B_s cover M (this makes sense by the local finiteness). Now define the smooth functions $f_s(y) := \frac{H_{B_s}(y)}{F(y)}$. This collection is a partition of unity subordinate to the cover $\{U_i\}$, as desired. $\square$

18.755 Lie Groups and Lie Algebras II
Spring 2024