32. **Maximal root, exponents, Coxeter numbers, dual representations**

32.1. **Duals of irreducible representations.** Now let $\mathfrak{g}$ be any complex semisimple Lie algebra. How to compute the dual of the irreducible representation L_λ? It is clear that the highest weight of L_λ^* equals $-\mu$, where μ is the lowest weight of L_λ, so we should compute the latter. For this purpose, recall that the Weyl group W of $\mathfrak{g}$ contains a unique element w_0 which maps dominant weights to antidominant weights, i.e., maps positive roots to negative roots. This is the maximal element, which is the unique element whose length is $|R_+|$. For example, if $-1 \in W$ then clearly $w_0 = -1$. It is easy to see that the lowest weight of L_λ is $w_0\lambda$.

Thus we get

Proposition 32.1. $L_\lambda^* = L_{-w_0\lambda}$.

The map $-w_0$ permutes fundamental (co)weights and simple (co)roots, so it is induced by an automorphism of the Dynkin diagram of $\mathfrak{g}$. So if $\mathfrak{g}$ is simple and its Dynkin diagram has no nontrivial automorphisms, we have $w_0 = -1$, so $-w_0 = 1$ and thus $L_\lambda^* = L_\lambda$ for all λ. This happens for A_1, B_n, C_n, G_2, F_4, E_7 and E_8. In general, note that s_i and hence the whole Weyl group W acts trivially on P/Q, which implies that $-w_0$ acts on P/Q by inversion. Thus we see that for A_{n-1}, $n \geq 3$, when $P/Q = \mathbb{Z}/n$, the map $-w_0$ is the flip of the chain. Another way to see it is to note that $L_{\omega_1}^* = V^* = \wedge^{n-1}V = L_{\omega_{n-1}}$ (as $\dim V = n$). For E_6, $P/Q = \mathbb{Z}/3$, so $-w_0$ must exchange the two nonzero minuscule weights and thus must also be the flip.

Exercise 32.2. (i) Show that for D_{2n+1} we have $S_+^* = S_-$ while for D_{2n} we have $S_+^* = S_+$, $S_-^* = S_-$. (**Hint:** Show that in the first case $P/Q \cong \mathbb{Z}/4$ while in the second case $P/Q \cong (\mathbb{Z}/2)^2$.)

(ii) Show that the restriction of the spin representation S of $\mathfrak{so}_{2n+1}$ to $\mathfrak{so}_{2n}$ is $S_+ \oplus S_-$.

(iii) Show that there exist unique up to scaling nonzero **Clifford multiplication** homomorphisms

$$V \otimes S \to S,\ V \otimes S_+ \to S_-,\ V \otimes S_- \to S_+.$$

(iv) Compute the decomposition of the tensor products

$$S \otimes S^*,\ S_+ \otimes S_+^*,\ S_- \otimes S_-^*,\ S_+ \otimes S_-^*$$

into irreducible representations.

Hint. In the odd dimensional case, use that $\mathrm{Cl}(V) = 2S \otimes S^*$ as an $\mathfrak{so}(V)$-module, that $\mathrm{grCl}(V) = \wedge V$, and that representations of $\mathfrak{so}(V)$ are completely reducible.

The even case is similar:

$$\mathrm{Cl}(V) = S_+ \otimes S_+^* \oplus S_- \otimes S_-^* \oplus S_- \otimes S_+^* \oplus S_+ \otimes S_-^*.$$

If $\dim V = 2n$ and n is even, use that all representations of $\mathfrak{so}(V)$ are selfdual to conclude that the last two summands are isomorphic. (If n is odd, they will not be isomorphic).

Also in this case you need to pay attention to the middle exterior power - it should split into two parts. Namely, if $\dim V = 2n$ then on $\wedge^n V$ we have two invariant bilinear forms: one symmetric coming from the one on V, denoted $B(\xi, \eta)$, and the other given by wedge product $\wedge : \wedge^n V \times \wedge^n V \to \wedge^{2n} V = \mathbb{C}$, which is symmetric for even n and skew-symmetric for odd n. Since the wedge product form is nondegenerate, there is a unique linear operator $* : \wedge^n V \to \wedge^n V$ called the **Hodge *-operator** such that $B(\xi, \eta) = \xi \wedge *\eta$. You should show that $*^2 = 1$ in the even case and $*^2 = -1$ in the odd case (use an orthonormal basis of V). Thus we have an eigenspace decomposition $\wedge^n V = \wedge^n_+ V \oplus \wedge^n_- V$, into eigenspaces of $*$ with eigenvalues ± 1 in the even case (called **selfdual** and **anti-selfdual** forms respectively) and $\pm i$ in the odd case. You will see that these pieces are irreducible and isomorphic to each other in the odd case but not in the even case, and that one of them (which?) goes into $S_+ \otimes S_+^*$ and the other into $S_- \otimes S_-^*$.

32.2. **The maximal root.** Let $\mathfrak{g}$ be a complex simple Lie algebra and θ be the maximal root of $\mathfrak{g}$, i.e., the highest weight of the adjoint representation. For example, for $\mathfrak{g} = \mathfrak{sl}_n$ the adjoint representation is generated by the highest weight vector of $V \otimes V^*$, where $V = \mathbb{C}^n$ is the vector representation. Thus we have

$$\theta = \omega_1 + \omega_{n-1} = (2, 1, ..., 1, 0) = (1, 0, ..., 0, -1),$$

the sum of the highest weights of V and V^* (recall that weights for $\mathfrak{sl}_n$ are n-tuples of complex numbers modulo simultaneous translation by the same number). Thus, θ is not fundamental. Similarly, for $\mathfrak{g} = \mathfrak{sp}_{2n}$, we have $\mathfrak{g} = S^2 V$ where V is the vector representation, so $\theta = 2\omega_1$ is again not fundamental. Nevertheless, we have the following proposition.

Proposition 32.3. *For any simple Lie algebra $\mathfrak{g} \neq \mathfrak{sl}_n, \mathfrak{sp}_{2n}$, θ is a fundamental weight.*

Proof. If $\mathfrak{g} = \mathfrak{so}_N$, $N \geq 7$ (i.e. of type B or D but not A or C) then $\mathfrak{g} = \wedge^2 V = L_{\omega_2}$, so $\theta = \omega_2$.

If $\mathfrak{g} = G_2$, $\alpha_1 = \alpha$ is the long simple root and $\alpha_2 = \beta$ is the short one, then we easily see that $\theta = 2\alpha_1 + 3\alpha_2 = \omega_1$.

If $\mathfrak{g} = F_4$ then using the conventions of Subsection 23.3, we have $\theta = \mathbf{e}_1 + \mathbf{e}_2 = \omega_4$.

If $\mathfrak{g} = E_8$ then using the conventions of Subsection 23.4, we have $\theta = \mathbf{e}_1 + \mathbf{e}_2 = \omega_8$.

If $\mathfrak{g} = E_7$ then using the conventions of Subsection 23.5, we have $\theta = \mathbf{e}_1 - \mathbf{e}_2 = \omega_1$.

If $\mathfrak{g} = E_6$ then using the conventions of Subsection 23.6, we have

$$\theta = \tfrac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + \sum_{i=4}^{8} \mathbf{e}_i) = \omega_2.$$

$\square$

32.3. **Principal $\mathfrak{sl}_2$, exponents.** Let $\mathfrak{g}$ be a simple Lie algebra and let $e = \sum_i e_i$ and $h \in \mathfrak{h}$ be such that $\alpha_i(h) = 2$ for all i (i.e., $h = 2\rho^\vee$). We have $[h, e] = 2e$ and $h = \sum_i (2\rho^\vee, \omega_i) h_i$. So defining $f := \sum_i (2\rho^\vee, \omega_i) f_i$, we have $[h, f] = -2f$, $[e, f] = h$. So e, f, h span an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ called the **principal $\mathfrak{sl}_2$-subalgebra.**

Exercise 32.4. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Show that the restriction of the $n+1$-dimensional vector representation V of $\mathfrak{g}$ to the principal $\mathfrak{sl}_2$-subalgebra is the irreducible representation L_n.

Consider now $\mathfrak{g}$ as a module over its principal $\mathfrak{sl}_2$-subalgebra. How does it decompose? To see this, we can look at the weight decomposition of $\mathfrak{g}$ under h. We have $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and these summands correspond to negative, zero and positive weights, respectively. Moreover, all weights are even, and for $m > 0$, $\dim \mathfrak{g}[2m] = r_m$ is the number of positive roots of height m, i.e., representable as a sum of m simple roots, while $\mathfrak{g}[0] = \mathfrak{h}$ (as $\rho^\vee$ is a regular coweight), so $\dim \mathfrak{g}[0] = r$, the rank of $\mathfrak{g}$.

Definition 32.5. m is called an **exponent** of $\mathfrak{g}$ if $r_m > r_{m+1}$. The multiplicity of m is $r_m - r_{m+1}$.

Since r_m is zero for large m while $r_0 = r$, there are r exponents counting multiplicities. The exponents of $\mathfrak{g}$ are denoted m_i and are arranged in non-decreasing order: $m_1 \le m_2 \le \dots \le m_r$ (including multiplicities). Note that roots of height 2 are $\alpha_i + \alpha_j$ where i, j are connected by an edge. Thus we have $r_0 = r_1 = r$, $r_2 = r - 1$ (as the Dynkin diagram of $\mathfrak{g}$ is a tree), so $m_1 = 1$ and $m_2 > 1$. We also have $m_r = (\rho^\vee, \theta) := \mathrm{h}_{\mathfrak{g}} - 1$, where θ is the maximal root. The number $\mathrm{h}_{\mathfrak{g}}$ is called the **Coxeter number** of $\mathfrak{g}$. Finally, we have $\sum_{i=1}^r m_i = |R_+|$.

Proposition 32.6. *The restriction of $\mathfrak{g}$ to the principal $\mathfrak{sl}_2$-subalgebra decomposes as $\oplus_{i=1}^r L_{2m_i+1}$.*

Proof. This easily follows from the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4) and the definition of m_i. $\square$

Example 32.7. The exponents of $\mathfrak{sl}_n$ are $1, 2, ..., n-1$.

Exercise 32.8. (i) Show that the exponents of $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ are $1, 3, ..., 2n-1$, and the exponents of $\mathfrak{so}_{2n+2}$ are $1, 3, ..., 2n-1$ and n (so in the latter case, when n is odd, the exponent n has multiplicity 2).

(ii) Show that the exponents of G_2 are 1 and 5.

Exercise 32.9. Show that the exponents of F_4 are $1, 5, 7, 11$, the exponents of E_6 are $1, 4, 5, 7, 8, 11$, the exponents of E_7 are $1, 5, 7, 9, 11, 13, 17$, and the exponents of E_8 are $1, 7, 11, 13, 17, 19, 23, 29$.

Hint: For $m \geq 1$, use the data from Subsections 23.3,23.4,23.5,23.6 to count roots satisfying the equation $(\rho^\vee, \alpha) = m$, and find m where the number of such roots drops as m is increased.

Exercise 32.10. Use the Weyl character formula for the adjoint representation and the Weyl denominator formula to prove the following identity for a simple Lie algebra $\mathfrak{g}$:

$$\sum_{i=1}^{r} \frac{q^{2m_i+1} - q^{-2m_i-1}}{q - q^{-1}} = \prod_{\alpha \in R_+ : (\theta, \alpha^\vee) > 0} \frac{q^{(\theta+\rho, \alpha^\vee)} - q^{-(\theta+\rho, \alpha^\vee)}}{q^{(\rho, \alpha^\vee)} - q^{-(\rho, \alpha^\vee)}}.$$

(**Hint:** Compute the character of $\mathfrak{g}$ as a module over the principal $\mathfrak{sl}_2$-subalgebra in two different ways.)

32.4. **The Coxeter number and the dual Coxeter number.** We have defined the Coxeter number of a simple complex Lie algebra $\mathfrak{g}$ (or a reduced irreducible root system R) to be $\mathrm{h}_R = \mathrm{h}_{\mathfrak{g}} := (\theta, \rho^\vee) + 1 = m_r + 1$, where m_r is the largest exponent of $\mathfrak{g}$. One can also define the **dual Coxeter number** of $\mathfrak{g}$ (or R) as $\mathrm{h}_R^\vee = \mathrm{h}_{\mathfrak{g}}^\vee := (\widetilde{\theta}^\vee, \rho) + 1$, cf. footnote 15 (clearly, $\mathrm{h}_R^\vee = \mathrm{h}_R$ if R is simply laced). So the dual Coxeter number is the eigenvalue $\frac{1}{2}(\theta, \theta + 2\rho)$ of $\frac{1}{2}C$ on the adjoint representation $\mathfrak{g}$, where $C \in U(\mathfrak{g})$ is the quadratic Casimir element defined using the inner product in which $(\theta, \theta) = 2$ (or, equivalently, long roots have squared length 2). Indeed, if we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using this inner product then θ gets identified with $\widetilde{\theta}^\vee$.

Using the formulas from Subsections 23.7 and 32.2, we get

$$\mathrm{h}_{A_{n-1}} = n,$$

$$\mathrm{h}_{B_n} = 2n,\ \mathrm{h}_{B_n}^\vee = 2n - 1,$$

$$\mathrm{h}_{C_n} = 2n,\ \mathrm{h}_{C_n}^\vee = n + 1,$$

$$\mathrm{h}_{D_n} = 2n - 2,$$

$$\mathrm{h}_{G_2} = (2\alpha+3\beta, 5\alpha^\vee+3\beta^\vee)+1 = 6,\ h^\vee_{G_2} = \frac{1}{3}(2\alpha+3\beta, 3\alpha+5\beta)+1 = 4,$$

$$\mathrm{h}_{F_4} = (8,3,2,1)\cdot(1,1,0,0)+1 = 12,\ \mathrm{h}^\vee_{F_4} = (\tfrac{11}{2},\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2})\cdot(1,1,0,0)+1 = 9,$$

$$\mathrm{h}_{E_8} = (23,6,5,4,3,2,1,0)\cdot(1,1,0,0,0,0,0,0)+1 = 30,$$

$$\mathrm{h}_{E_7} = (\tfrac{17}{2},-\tfrac{17}{2},5,4,3,2,1,0)\cdot(1,-1,0,0,0,0,0,0)+1 = 18,$$

$$\mathrm{h}_{E_6} = (4,-4,-4,4,3,2,1,0)\cdot\tfrac{1}{2}(1,-1,-1,1,1,1,1,1)+1 = 12.$$

Note that we always have $\mathrm{h}_R = \mathrm{h}_{R^\vee}$, but if R is not simply laced then, as we see, the numbers h_R, $\mathrm{h}^\vee_{R^\vee}$, $\mathrm{h}^\vee_R$ are different, in general.

32.5. **Representations of complex, real and quaternionic type.**

Definition 32.11. An irreducible finite dimensional $\mathbb{C}$-representation V of a group G or Lie algebra $\mathfrak{g}$ is **complex type** when $V \not\cong V^*$, **real type** if there is a symmetric isomorphism $V \to V^*$ (i.e., an invariant symmetric inner product of V), and **quanternionic type** if there is a skew-symmetric isomorphism $V \to V^*$ (i.e., an invariant skew-symmetric inner product of V).

It is easy to see that any irreducible finite dimensional representation is of exactly one of these three types (check it!).

Exercise 32.12. Let V be an irreducible finite dimensional representation of a finite group G.

(i) Show that $\mathrm{End}_{\mathbb{R}G}V$ is $\mathbb{C}$ for complex type, $\mathrm{Mat}_2(\mathbb{R})$ for real type and the quaternion algebra $\mathbb{H}$ for quaternionic type. This explains the terminology.

(ii) Show that V is of real type if and only if in some basis of V the matrices of all elements of G have real entries.

You may find helpful to look at [E], Problem 5.1.2 (it contains a hint).

Example 32.13. Let L_n be the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight n (i.e., of dimension $n+1$). Then L_n is of real type for even n and quaternionic type for odd n. Indeed, $L_n = S^nV$, where $V = L_1 = \mathbb{C}^2$, so the invariant form on L_n is S^nB, where B is the invariant form on V, which is skew-symmetric.

Now let $\mathfrak{g}$ be any simple Lie algebra and $\lambda \in P_+$ be such that $\lambda = -w_0\lambda$, so that L_λ is selfdual. How to tell if it is of real or quaternionic type?

Proposition 32.14. *L_λ is of real type if $(2\rho^\vee, \lambda)$ is even and of quaternionic type if it is odd.*

Proof. The number $n := (2\rho^\vee, \lambda)$ is the eigenvalue of the element h of the principal $\mathfrak{sl}_2$-subalgebra on the highest weight vector v_λ. All the other eigenvalues are strictly less. Thus the restriction of L_λ to the principal $\mathfrak{sl}_2$-subalgebra is of the form $L_n \oplus \bigoplus_{m<n} k_m L_m$, i.e., L_n occurs with multiplicity 1. Hence the nondegenerate invariant form on L_λ restricts to a nondegenerate invariant form on L_n, so by Example 32.13 it is skew-symmetric if n is odd and symmetric if n is even. $\square$

Example 32.15. Consider $\mathfrak{g} = \mathfrak{so}_{2n}$. Then we have

$$\rho^\vee = \rho = \sum_i \omega_i = (n-1, n-2, ..., 1, 0).$$

So $(2\rho^\vee, \omega_{n-1}) = (2\rho^\vee, \omega_n) = \frac{n(n-1)}{2}$. This is odd if $n = 2, 3$ modulo 4 and even if $n = 0, 1$ modulo 4. Thus $S_\pm$ carry a symmetric form when $n = 0$ mod 4 and a skew-symmetric form if $n = 2$ mod 4.

Consider now $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Then $\rho^\vee = \sum_i \omega_i^\vee = (n, n-1, ..., 1)$. So $(2\rho^\vee, \omega_n) = \frac{n(n+1)}{2}$. So S carries a skew-symmetric form if $n = 1, 2$ mod 4 and a symmetric form if $n = 0, 3$ mod 4.

We obtain the following result.

Theorem 32.16. *(Bott periodicity for spin representations) The behavior of the spin representations of the orthogonal Lie algebra $\mathfrak{so}_m$ is determined by the remainder r of m modulo 8. Namely:*

For $r = 1, 7$, S is of real type.

For $r = 3, 5$, S is of quaternionic type.

For $r = 0$, S_+, S_- are of real type.

For $r = 2, 6$, $S_+^* = S_-$ (complex type).

For $r = 4$, S_+, S_- are of quaternionic type.