21. **Root systems**

21.1. **Abstract root systems.** Let $E \cong \mathbb{R}^r$ be a Euclidean space with a positive definite inner product.

Definition 21.1. An **abstract root system** is a finite set $R \subset E \setminus 0$ satisfying the following axioms:

(R1) R spans E;

(R2) For all $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer;

(R3) If $\alpha, \beta \in R$ then $s_\alpha(\beta) := \beta - n_{\alpha\beta}\alpha \in R$.

Elements of R are called **roots**. The number $r = \dim E$ is called the **rank** of R.

In particular, taking $\beta = \alpha$ in R3 yields that R is centrally symmetric, i.e., $R = -R$. Also note that s_α is the reflection with respect to the hyperplane $(\alpha, x) = 0$, so R3 just says that R is invariant under such reflections.

Note also that if $R \subset E$ is a root system, $\overline{E} \subset E$ a subspace, and $R' = R \cap \overline{E}$ then R' is also a root system inside $E' = \mathrm{Span}(R') \subset \overline{E}$.

For a root α the corresponding **coroot** $\alpha^\vee \in E^*$ is defined by the formula $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$. Thus $\alpha^\vee(\alpha) = 2$, $n_{\alpha\beta} = \alpha^\vee(\beta)$ and $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha$.

Definition 21.2. A root system R is **reduced** if for $\alpha, c\alpha \in R$, we have $c = \pm 1$.

Proposition 21.3. *If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra then the corresponding set of roots R is a reduced root system, and $\alpha^\vee = h_\alpha$.*

Proof. This follows immediately from Theorem 19.19. □

Example 21.4. 1. The root system of $\mathfrak{sl}_n$ is called A_{n-1}. In this case, as we have seen in Example 19.14, the roots are $\mathbf{e}_i - \mathbf{e}_j$, and $s_{\mathbf{e}_i - \mathbf{e}_j} = (ij)$, the transposition of the i-th and j-th coordinates.

2. The subset $\{1, 2, -1, -2\}$ of $\mathbb{R}$ is a root system which is not reduced.

Definition 21.5. Let $R_1 \subset E_1, R_2 \subset E_2$ be root systems. An **isomorphism of root systems** $\phi : R_1 \to R_2$ is an isomorphism $\phi : E_1 \to E_2$ which maps R_1 to R_2 and preserves the numbers $n_{\alpha\beta}$.

So an isomorphism does not have to preserve the inner product, e.g. it may rescale it.

21.2. **The Weyl group.**

Definition 21.6. The **Weyl group** of a root system R is the group of automorphisms of E generated by s_α.

Proposition 21.7. *W is a finite subgroup of $O(E)$ which preserves R.*

Proof. Since s_α are orthogonal reflections, $W \subset O(E)$. By R3, s_α preserves R. By R1 an element of W is determined by its action on R, hence W is finite. $\square$

Example 21.8. For the root system A_{n-1}, $W = S_n$, the symmetric group. Note that for $n \geq 3$, the automorphism $x \mapsto -x$ of R is not in W, so W is, in general, a proper subgroup of $\mathrm{Aut}(R)$.

21.3. **Root systems of rank 2.** If α, β are linearly independent roots in R and $E' \subset E$ is spanned by α, β then $R' = R \cap E'$ is a root system in E' of rank 2. So to classify reduced root systems, it is important to classify reduced root systems of rank 2 first.

Theorem 21.9. *Let R be a reduced root system and $\alpha, \beta \in R$ be two linearly independent roots with $|\alpha| \geq |\beta|$. Let ϕ be the angle between α and β. Then we have one of the following possibilities:*
(1) $\phi = \pi/2$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$;
(2a) $\phi = 2\pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$;
(2b) $\phi = \pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$;
(3a) $\phi = 3\pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -2$;
(3b) $\phi = \pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 2$;
(4a) $\phi = 5\pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -3$;
(4b) $\phi = \pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 3$.

Proof. We have $(\alpha, \beta) = 2|\alpha| \cdot |\beta| \cos\phi$, so $n_{\alpha\beta} = 2\frac{|\beta|}{|\alpha|}\cos\phi$. Thus $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\phi$. Hence this number can only take values $0, 1, 2, 3$ (as it is an integer by R2) and $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$ if $n_{\alpha\beta} \neq 0$. The rest is obtained by analysis of each case. $\square$

In fact, all these possibilities are realized. Namely, we have root systems $A_1 \times A_1$, A_2, $B_2 = C_2$ (the root system of the Lie algebras $\mathfrak{sp}_4$ and $\mathfrak{so}_5$, which are in fact isomorphic, consisting of the vertices and midpoints of edges of a square), and G_2, generated by α, β with $(\alpha, \alpha) = 6$, $(\beta, \beta) = 2$, $(\alpha, \beta) = -3$, and roots being $\pm\alpha, \pm\beta$, $\pm(\alpha+\beta)$, $\pm(\alpha + 2\beta)$, $\pm(\alpha + 3\beta)$, $\pm(2\alpha + 3\beta)$.

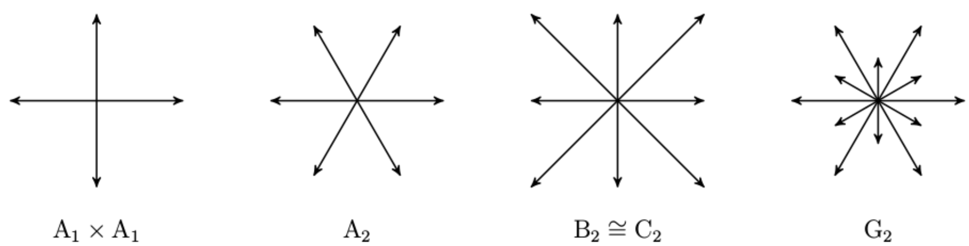
Theorem 21.10. *Any reduced rank 2 root system R is of the form $A_1 \times A_1$, A_2, B_2 or G_2.*

Proof. Pick independent roots $\alpha, \beta \in R$ such that the angle ϕ is as large as possible. Then $\phi \geq \pi/2$ (otherwise can replace α with $-\alpha$), so we are in one of the cases $1, 2a, 3a, 4a$. Now the statement follows by inspection of each case, giving $A_1 \times A_1$, A_2, B_2 and G_2 respectively. $\square$

Corollary 21.11. *If $\alpha, \beta \in R$ are independent roots with $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$.*

Proof. This is easy to see from the classification of rank 2 root systems. $\square$

The root systems of rank 2 are shown in the following picture.



21.4. **Positive and simple roots.** Let R be a reduced root system and $t \in E^*$ be such that $t(\alpha) \neq 0$ for any $\alpha \in R$. We say that a root is **positive** (with respect to t) if $t(\alpha) > 0$ and **negative** if $t(\alpha) < 0$. The set of positive roots is denoted by R_+ and of negative ones by R_-, so $R_+ = -R_-$ and $R = R_+ \cup R_-$ (disjoint union). This decomposition is called a **polarization** of R; it depends on the choice of t.

Example 21.12. Let R be of type A_{n-1}. Then for $t = (t_1, ..., t_n)$ we have $t(\alpha) \neq 0$ for all α iff $t_i \neq t_j$ for any i, j. E.g. suppose $t_1 > t_2 > ... > t_n$, then we have $\mathbf{e}_i - \mathbf{e}_j \in R_+$ iff $i < j$. We see that polarizations are in bijection with permutations in S_n, i.e., with elements of the Weyl group, which acts simply transitively on them. We will see that this is, in fact, the case for any reduced root system.

Definition 21.13. A root $\alpha \in R_+$ is **simple** if it is not a sum of two other positive roots.

Lemma 21.14. *Every positive root is a sum of simple roots.*

Proof. If α is not simple then $\alpha = \beta + \gamma$ where $\beta, \gamma \in R_+$. We have $t(\alpha) = t(\beta) + t(\gamma)$, so $t(\beta), t(\gamma) < t(\alpha)$. If β or γ is not simple, we can continue this process, and it will terminate since t has finitely many values on R. $\square$

Lemma 21.15. *If $\alpha, \beta \in R_+$ are simple roots then $(\alpha, \beta) \leq 0$.*

Proof. Assume $(\alpha,\beta) > 0$. Then $(-\alpha,\beta) < 0$ so by Lemma 21.11 $\gamma := \beta - \alpha$ is a root. If γ is positive then $\beta = \alpha + \gamma$ is not simple. If γ is negative then $-\gamma$ is positive so $\alpha = \beta + (-\gamma)$ is not simple. $\square$

Theorem 21.16. *The set $\Pi \subset R_+$ of simple roots is a basis of E.*

Proof. We will use the following linear algebra lemma:

Lemma 21.17. *Let v_i be vectors in a Euclidean space E such that $(v_i, v_j) \le 0$ when $i \neq j$ and $t(v_i) > 0$ for some $t \in E^*$. Then v_i are linearly independent.*

Proof. Suppose we have a nontrivial relation

$$\sum_{i\in I} c_i v_i = \sum_{i\in J} c_i v_i$$

where I, J are disjoint and $c_i > 0$ (clearly, every nontrivial relation can be written in this form). Evaluating t on this relation, we deduce that both sides are nonzero. Now let us compute the square of the left hand side:

$$0 < |\sum_{i\in I} c_i v_i|^2 = (\sum_{i\in I} c_i v_i, \sum_{j\in J} c_j v_j) \le 0.$$

This is a contradiction. $\square$

Now the result follows from Lemma 21.15 and Lemma 21.17. $\square$

Thus the set Π of simple roots has r elements: $\Pi = (\alpha_1, ..., \alpha_r)$.

Example 21.18. Let us describe simple roots for classical root systems. Suppose the polarization is given by $t = (t_1, ..., t_n)$ with decreasing coordinates. Then:

1. For type A_{n-1}, i.e., $\mathfrak{g} = \mathfrak{sl}_n$, the simple roots are $\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}$, $1 \le i \le n-1$.

2. For type C_n, i.e., $\mathfrak{g} = \mathfrak{sp}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = 2\mathbf{e}_n.$$

3. For type B_n, i.e., $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we have the same story as for C_n except $\alpha_n = \mathbf{e}_n$ rather than $2\mathbf{e}_n$. Thus the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = \mathbf{e}_n.$$

4. For type D_n, i.e., $\mathfrak{g} = \mathfrak{so}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1}, \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n.$$

We thus obtain

Corollary 21.19. *Any root $\alpha \in R$ can be uniquely written as $\alpha = \sum_{i=1}^r n_i\alpha_i$, where $n_i \in \mathbb{Z}$. If α is positive then $n_i \geq 0$ for all i and if α is negative then $n_i \leq 0$ for all i.*

For a positive root α, its **height** $h(\alpha)$ is the number $\sum n_i$. So simple roots are the roots of height 1, and the height of $\mathbf{e}_i - \mathbf{e}_j$ in $R = A_{n-1}$ is $j - i$.

21.5. **Dual root system.** For a root system R, the set $R^\vee \subset E^*$ of $\alpha^\vee$ for all $\alpha \in R$ is also a root system, such that $(R^\vee)^\vee = R$. It is called the **dual root system** to R. For example, B_n is dual to C_n, while A_{n-1}, D_n and G_2 are self-dual.

Moreover, it is easy to see that any polarization of R gives rise to a polarization of $R^\vee$ (using the image $t^\vee$ of t under the isomorphism $E \to E^*$ induced by the inner product), and the corresponding system $\Pi^\vee$ of simple roots consists of $\alpha_i^\vee$ for $\alpha_i \in \Pi$.

21.6. **Root and weight lattices.** Recall that a **lattice** in a real vector space E is a subgroup $Q \subset E$ generated by a basis of E. Of course, every lattice is conjugate to $\mathbb{Z}^n \subset \mathbb{R}^n$ by an element of $GL_n(\mathbb{R})$. Also recall that for a lattice $Q \subset E$ the **dual lattice** $Q^* \subset E^*$ is the set of $f \in E^*$ such that $f(v) \in \mathbb{Z}$ for all $v \in Q$. If Q is generated by a basis $\mathbf{e}_i$ of E then Q^* is generated by the dual basis $\mathbf{e}_i^*$.

In particular, for a root system R we can define the **root lattice** $Q \subset E$, which is generated by the simple roots α_i with respect to some polarization of R. Since Q is also generated by all roots in R, it is independent on the choice of the polarization. Similarly, we can define the **coroot lattice** $Q^\vee \subset E^*$ generated by $\alpha^\vee, \alpha \in R$, which is just the root lattice of $R^\vee$.

Also we define the **weight lattice** $P \subset E$ to be the dual lattice to $Q^\vee$: $P = (Q^\vee)^*$, and the **coweight lattice** $P^\vee \subset E^*$ to be the dual lattice to Q: $P^\vee = Q^*$, so $P^\vee$ is the weight lattice of $R^\vee$. Thus

$$P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in R\},\ P^\vee = \{\lambda \in E^* : (\lambda, \alpha) \in \mathbb{Z}\, \forall \alpha \in R\}.$$

Since for $\alpha, \beta \in R$ we have $(\alpha^\vee, \beta) = n_{\alpha\beta} \in \mathbb{Z}$, we have $Q \subset P$, $Q^\vee \subset P^\vee$.

Given a system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, we define **fundamental coweights** $\omega_i^\vee$ to be the dual basis to α_i and **fundamental weights** ω_i to be the dual basis to $\alpha_i^\vee$: $(\omega_i, \alpha_j^\vee) = (\omega_i^\vee, \alpha_j) = \delta_{ij}$. Thus P is generated by ω_i and $P^\vee$ by $\omega_i^\vee$.

Example 21.20. Let R be of type A_1. Then $(\alpha, \alpha^\vee) = 2$ for the unique positive root α, so $\omega = \frac{1}{2}\alpha$, thus $P/Q = \mathbb{Z}/2$. More generally, if R is of type A_{n-1} and we identify $Q \cong Q^\vee, P \cong P^\vee$, then P becomes the set of

$\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ such that $\sum_i \lambda_i = 0$ and $\lambda_i - \lambda_j \in \mathbb{Z}$. So we have a homomorphism $\phi : P \to \mathbb{R}/\mathbb{Z}$ given by $\phi(\lambda) = \lambda_i \bmod \mathbb{Z}$ (for any i). Since $\sum_i \lambda_i = 0$, we have $\phi : P \to \mathbb{Z}/n$, and $\mathrm{Ker}\phi = Q$ (integer vectors with sum zero). Also it is easy to see that ϕ is surjective (we may take $\lambda_i = \frac{k}{n}$ for $i \neq n$ and $\lambda_n = \frac{k}{n} - k$, then $\phi(\lambda) = \frac{k}{n}$). Thus $P/Q \cong \mathbb{Z}/n$.