17. Proofs of the Cartan criteria, properties of semisimple Lie algebras

17.1. Proof of the Cartan solvability criterion. It is clear that \mathfrak{g} is solvable if and only if so is $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$, so we may assume that \mathbf{k} is algebraically closed.

For the "only if" part, note that by Lie's theorem, \mathfrak{g} has a basis in which the operators $\operatorname{ad} x, x \in \mathfrak{g}$, are upper triangular. Then $[\mathfrak{g}, \mathfrak{g}]$ acts in this basis by strictly upper triangular matrices, so K(x, y) = 0 for $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.

To prove the "if" part, let us prove the following lemma.

Lemma 17.1. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that for any $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ we have $\operatorname{Tr}(xy) = 0$. Then \mathfrak{g} is solvable.

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Let $\lambda_i, i = 1, ..., m$, be the distinct eigenvalues of x. Let $E \subset \mathbf{k}$ be a Q-span of λ_i . Let $b : E \to \mathbb{Q}$ be a linear functional. There exists an interpolation polynomial $Q \in \mathbf{k}[t]$ such that $Q(\lambda_i - \lambda_j) = b(\lambda_i - \lambda_j) = b(\lambda_i) - b(\lambda_j)$ for all i, j.

By Proposition 16.21, we can write x as $x = x_s + x_n$. Then the operator adx_s is diagonalizable with eigenvalues $\lambda_i - \lambda_j$. So

$$Q(\mathrm{ad}x_s) = \mathrm{ad}b$$

where $b: V \to V$ is the operator acting by $b(\lambda_j)$ on the generalized λ_j -eigenspace of x.

Also we have

$$\mathrm{ad}x = \mathrm{ad}x_s + \mathrm{ad}x_n$$

a sum of commuting semisimple and nilpotent operators. Thus

$$\mathrm{ad}x_s = (\mathrm{ad}x)_s = P(\mathrm{ad}x),$$

and P(0) = 0 since 0 is an eigenvalue of adx. Thus

$$adb = R(adx),$$

where R(t) = Q(P(t)) and R(0) = 0.

Let $x = \sum_{j} [y_j, z_j], y_j, z_j \in \mathfrak{g}$, and d_j be the dimension of the generalized λ_j -eigenspace of x. Then

$$\sum_{j} d_j b(\lambda_j) \lambda_j = \operatorname{Tr}(bx) =$$

$$\operatorname{Tr}(\sum_{j} b[y_j, z_j]) = \operatorname{Tr}(\sum_{j} [b, y_j] z_j) = \operatorname{Tr}(\sum_{j} R(\operatorname{ad} x)(y_j) z_j).$$
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Since R(0) = 0, we have $R(adx)(y_j) \in [\mathfrak{g}, \mathfrak{g}]$, so by assumption we get

$$\sum_{j} d_j b(\lambda_j) \lambda_j = 0.$$

Applying b, we get $\sum_j d_j b(\lambda_j)^2 = 0$. Thus $b(\lambda_j) = 0$ for all j. Hence b = 0, so E = 0.

Thus, the only eigenvalue of x is 0, i.e., x is nilpotent. But then by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Thus \mathfrak{g} is solvable. Thus proves the lemma.

Now the "if" part of the Cartan solvability criterion follows easily by applying Lemma 17.1 to $V = \mathfrak{g}$ and replacing \mathfrak{g} by the quotient $\mathfrak{g}/\mathfrak{g}(\mathfrak{g})$.

17.2. Proof of the Cartan semisimplicity criterion. Assume that \mathfrak{g} is semisimple, and let $I = \operatorname{Ker}(K_{\mathfrak{g}})$, an ideal in \mathfrak{g} . Then $K_I = (K_{\mathfrak{g}})|_I = 0$. Thus by Cartan's solvability criterion I is solvable. Hence I = 0.

Conversely, suppose $K_{\mathfrak{g}}$ is nondegenerate. Then \mathfrak{g} is reductive. Moreover, the center of \mathfrak{g} is contained in the kernel of $K_{\mathfrak{g}}$, so it must be trivial. Thus \mathfrak{g} is semisimple.

17.3. Properties of semisimple Lie algebras.

Proposition 17.2. Let $char(\mathbf{k}) = 0$ and \mathfrak{g} be a finite dimensional Lie algebra over \mathbf{k} . Then \mathfrak{g} is semisimple iff $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ is semisimple.

Proof. Immediately follows from Cartan's criterion of semisimplicity. Here is another proof (of the nontrivial direction): if \mathfrak{g} is semisimple and I is a nonzero solvable ideal in $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ then it has a finite Galois orbit I_1, \ldots, I_n and $I_1 + \ldots + I_n$ is a Galois invariant solvable ideal, so it comes from a solvable ideal in \mathfrak{g} .

Remark 17.3. This theorem fails if we replace the word "semisimple" by "simple": e.g., if \mathfrak{g} is a simple complex Lie algebra regarded as a real Lie algebra then $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$ is semisimple but not simple.

Theorem 17.4. Let \mathfrak{g} be a semisimple Lie algebra and $I \subset \mathfrak{g}$ an ideal. Then there is an ideal $J \subset \mathfrak{g}$ such that $\mathfrak{g} = I \oplus J$.

Proof. Let I^{\perp} be the orthogonal complement of I with respect to the Killing form, an ideal in \mathfrak{g} . Consider the intersection $I \cap I^{\perp}$. It is an ideal in \mathfrak{g} with the zero Killing form. Thus, by the Cartan solvability criterion, it is solvable. By definition of a semisimple Lie algebra, this means that $I \cap I^{\perp} = 0$, so we may take $J = I^{\perp}$.

We will see below (in Proposition 17.7) that J is in fact unique and must equal I^{\perp} .

Corollary 17.5. A Lie algebra \mathfrak{g} is semisimple iff it is a direct sum of simple Lie algebras.

Proof. We have already shown that a direct sum of simple Lie algebras is semisimple. The opposite direction easily follows by induction from Theorem 17.4. $\hfill \Box$

Corollary 17.6. If \mathfrak{g} is a semisimple Lie algebra, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Proof. For a simple Lie algebra it is clear because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in \mathfrak{g} which cannot be zero (otherwise, \mathfrak{g} would be abelian). So the result follows from Corollary 17.5.

Proposition 17.7. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be a semisimple Lie algebra, with \mathfrak{g}_i being simple. Then any ideal I in \mathfrak{g} is of the form $I = \bigoplus_{i \in S} \mathfrak{g}_i$ for some subset $S \subset \{1, ..., k\}$.

Proof. The proof goes by induction in k. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projection. Consider $p_k(I) \subset \mathfrak{g}_k$. Since \mathfrak{g}_k is simple, either $p_k(I) = 0$, in which case $I \subset \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{k-1}$ and we can use the induction assumption, or $p_k(I) = \mathfrak{g}_k$. Then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, p_k(I)] = \mathfrak{g}_k$. Since I is an ideal, $I \supset \mathfrak{g}_k$, so $I = I' \oplus \mathfrak{g}_k$ for some subspace $I' \subset \mathfrak{g}_1 \oplus \oplus \mathfrak{g}_{k-1}$. It is immediate that then I' is an ideal in $\mathfrak{g}_1 \oplus \oplus \mathfrak{g}_{k-1}$ and the result again follows from the induction assumption.

Corollary 17.8. Any ideal in a semisimple Lie algebra is semisimple. Also, any quotient of a semisimple Lie algebra is semisimple.

Let Derg be the Lie algebra of derivations of a Lie algebra \mathfrak{g} . We have a homomorphism $\text{ad} : \mathfrak{g} \to \text{Derg}$ whose kernel is the center $\mathfrak{z}(\mathfrak{g})$. Thus if \mathfrak{g} has trivial center (e.g., is semisimple) then the map ad is injective and identifies \mathfrak{g} with a Lie subalgebra of Derg. Moreover, for $d \in \text{Derg}$ and $x \in \mathfrak{g}$, we have

$$(d, adx](y) = d[x, y] - [x, dy] = [dx, y] = ad(dx)(y).$$

Thus $\mathfrak{g} \subset \operatorname{Der}\mathfrak{g}$ is an ideal.

Proposition 17.9. If \mathfrak{g} is semisimple then $\mathfrak{g} = \text{Derg}$.

Proof. Consider the invariant symmetric bilinear form

$$K(a,b) = \mathrm{Tr}|_{\mathfrak{g}}(ab)$$

on Derg. This is an extension of the Killing form of \mathfrak{g} to Derg, so its restriction to \mathfrak{g} is nondegenerate. Let $I = \mathfrak{g}^{\perp}$ be the orthogonal complement of \mathfrak{g} in Derg under K. It follows that I is an ideal, $I \cap \mathfrak{g} = 0$, and $I \oplus \mathfrak{g} = \text{Derg}$. Since both I and \mathfrak{g} are ideals, we have $[\mathfrak{g}, I] = 0$. Thus for $d \in I$ and $x \in \mathfrak{g}$, $[d, \mathrm{ad} x] = \mathrm{ad}(dx) = 0$, so dx belongs to the center of \mathfrak{g} . Thus dx = 0, i.e., d = 0. It follows that I = 0, as claimed.

Corollary 17.10. Let \mathfrak{g} be a real or complex semisimple Lie algebra, and $G = \operatorname{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Then G is a Lie group with $\operatorname{Lie} G = \mathfrak{g}$. Thus G acts on \mathfrak{g} by the adjoint action.

Proof. It is easy to show that for any finite dimensional real or complex Lie algebra \mathfrak{g} , Aut(\mathfrak{g}) is a Lie group with Lie algebra Der(\mathfrak{g}), so the statement follows from Proposition 17.9.

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