16. Semisimple and reductive Lie algebras, the Cartan criteria

16.1. Semisimple and reductive Lie algebras, the radical. Let \mathfrak{g} be a finite dimensional Lie algebra over a field \mathbf{k} .

Proposition 16.1. The sum of all solvable ideals of \mathfrak{g} is a solvable ideal.

Definition 16.2. This ideal is called **the radical** of \mathfrak{g} and denoted rad(\mathfrak{g}).

Proof. Let I, J be solvable ideals of \mathfrak{g} . Then $I + J \subset \mathfrak{g}$ is an ideal, and $(I + J)/I = J/(I \cap J)$ is solvable, so I + J is solvable. Thus the sum of finitely many solvable ideals is solvable. Hence the sum of all solvable ideals in \mathfrak{g} is a solvable ideal, as desired.

Definition 16.3. (i) \mathfrak{g} is called **semisimple** if $rad(\mathfrak{g}) = 0$, i.e., \mathfrak{g} does not contain nonzero solvable ideals.

(ii) A non-abelian \mathfrak{g} is called **simple** if it contains no ideals other than $0, \mathfrak{g}$. In other words, a non-abelian \mathfrak{g} is simple if its adjoint representation is irreducible (=simple).

Thus if \mathfrak{g} is both solvable and semisimple then $\mathfrak{g} = 0$.

Proposition 16.4. (i) We have $rad(\mathfrak{g} \oplus \mathfrak{h}) = rad(\mathfrak{g}) \oplus rad(\mathfrak{h})$. In particular, the direct sum of semisimple Lie algebras is semisimple.

(ii) A simple Lie algebra is semisimple. Thus a direct sum of simple Lie algebras is semisimple.

Proof. (i) The images of $rad(\mathfrak{g} \oplus \mathfrak{h})$ in \mathfrak{g} and in \mathfrak{h} are solvable, hence contained in $rad(\mathfrak{g})$, respectively $rad(\mathfrak{h})$. Thus

 $\operatorname{rad}(\mathfrak{g} \oplus \mathfrak{h}) \subset \operatorname{rad}(\mathfrak{g}) \oplus \operatorname{rad}(\mathfrak{h}).$

But $rad(\mathfrak{g}) \oplus rad(\mathfrak{h})$ is a solvable ideal in $\mathfrak{g} \oplus \mathfrak{h}$, so

 $\operatorname{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \operatorname{rad}(\mathfrak{g}) \oplus \operatorname{rad}(\mathfrak{h}).$

(ii) The only nonzero ideal in \mathfrak{g} is \mathfrak{g} , and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ since \mathfrak{g} is not abelian. Hence \mathfrak{g} is not solvable. Thus \mathfrak{g} is semisimple.

Example 16.5. The Lie algebra $\mathfrak{sl}_2(\mathbf{k})$ is simple if char(\mathbf{k}) $\neq 2$. Likewise, $\mathfrak{so}_3(\mathbf{k})$ is simple.

Theorem 16.6. (weak Levi decomposition) The Lie algebra $\mathfrak{g}_{ss} = \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. Thus any \mathfrak{g} can be included in an exact sequence

$$0 \to \operatorname{rad}(\mathfrak{g}) \xrightarrow[87]{} \mathfrak{g} \to \mathfrak{g}_{ss} \to 0,$$

where $\operatorname{rad}(\mathfrak{g})$ is a solvable ideal and \mathfrak{g}_{ss} is semisimple. Moreover, if $\mathfrak{h} \subset \mathfrak{g}$ is a solvable ideal such that $\mathfrak{g}/\mathfrak{h}$ is semisimple then $\mathfrak{h} = \operatorname{rad}(\mathfrak{g})$.

Proof. Let $I \subset \mathfrak{g}_{ss}$ be a solvable ideal, and let \widetilde{I} be its preimage in \mathfrak{g} . Then \widetilde{I} is a solvable ideal in \mathfrak{g} . Thus $\widetilde{I} = \operatorname{rad}(\mathfrak{g})$ and I = 0.

In fact, in characteristic zero there is a stronger statement, which says that the extension in Theorem 16.6 splits. Namely, given a Lie algebra \mathfrak{h} and another Lie algebra \mathfrak{a} acting on \mathfrak{h} by derivations, we may form the **semidirect product** Lie algebra $\mathfrak{a} \ltimes \mathfrak{h}$ which is $\mathfrak{a} \oplus \mathfrak{h}$ as a vector space with commutator defined by

$$[(a_1, h_1), (a_2, h_2)] = ([a_1, a_2], a_1 \circ h_2 - a_2 \circ h_1 + [h_1, h_2]).$$

Note that a special case of this construction has already appeared in Example 15.23.

Theorem 16.7. (Levi decomposition) If $\operatorname{char}(\mathbf{k}) = 0$ then we have $\mathfrak{g} \cong \operatorname{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$ as vector spaces, where $\mathfrak{g}_{ss} \subset \mathfrak{g}$ is a semisimple subalgebra (but not necessarily an ideal); i.e., \mathfrak{g} is isomorphic to the semidirect product $\mathfrak{g}_{ss} \ltimes \operatorname{rad}(\mathfrak{g})$. In other words, the projection $p : \mathfrak{g} \to \mathfrak{g}_{ss}$ admits an (in general, non-unique) splitting $q : \mathfrak{g}_{ss} \to \mathfrak{g}$, i.e., a Lie algebra map such that $p \circ q = \operatorname{Id}$.

Theorem 16.7 will be proved in Subsection 48.2.

Example 16.8. Let G be the group of motions of the Euclidean space \mathbb{R}^3 (generated by rotations and translations). Then $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, so $\mathfrak{g} = \text{Lie}G = \mathfrak{so}_3(\mathbb{R}) \ltimes \mathbb{R}^3$, hence $\text{rad}(\mathfrak{g}) = \mathbb{R}^3$ (abelian Lie algebra) and $\mathfrak{g}_{ss} = \mathfrak{so}_3(\mathbb{R})$.

Proposition 16.9. Let $char(\mathbf{k}) = 0$, \mathbf{k} algebraically closed, and V be an irreducible representation of \mathfrak{g} . Then $rad(\mathfrak{g})$ acts on V by scalars, and $[\mathfrak{g}, rad(\mathfrak{g})]$ by zero.

Proof. By Lie's theorem, there is a nonzero $v \in V$ and $\lambda \in \operatorname{rad}(\mathfrak{g})^*$ such that $av = \lambda(a)v$ for $a \in \operatorname{rad}(\mathfrak{g})$. Let $x \in \mathfrak{g}$ and $\mathfrak{g}_x \subset \mathfrak{g}$ be the Lie subalgebra spanned by $\operatorname{rad}(\mathfrak{g})$ and x. Let W be the span of $x^n v$ for $n \geq 0$. By Lemma 15.18(i), W is a \mathfrak{g}_x -subrepresentation of V on which $a \in \operatorname{rad}(\mathfrak{g})$ has the only eigenvalue $\lambda(a)$. Thus by Lemma 15.18(iii), for $a \in \operatorname{rad}(\mathfrak{g})$ we have $\lambda([x, a]) = 0$, so the λ -eigenspace V_λ of $\operatorname{rad}(\mathfrak{g})$ in V is a \mathfrak{g} -subrepresentation of V, which implies that $V_\lambda = V$ since Vis irreducible. \Box

Definition 16.10. \mathfrak{g} is called **reductive** if $rad(\mathfrak{g})$ coincides with the center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} .

In other words, \mathfrak{g} is reductive if $[\mathfrak{g}, \operatorname{rad}(\mathfrak{g})] = 0$.

The Levi decomposition theorem implies that a reductive Lie algebra in characteristic zero is a direct sum of a semisimple Lie algebra and an abelian Lie algebra (its center). We will also prove this in Corollary 18.8.

16.2. Invariant inner products. Let B be a bilinear form on a Lie algebra \mathfrak{g} . Recall that B is invariant if B([x,y],z) = B(x,[y,z]) for any $x, y, z \in \mathfrak{g}$.

Example 16.11. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation of \mathfrak{g} then the form

$$B_V(x, y) := \operatorname{Tr}(\rho(x)\rho(y))$$

is an invariant symmetric bilinear form on $\mathfrak{g}.$ Indeed, the symmetry is obvious and

$$B_V([x,y],z) = B_V(x,[y,z]) = \text{Tr}|_V(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)).$$

Proposition 16.12. If B is a symmetric invariant bilinear form on \mathfrak{g} and $I \subset \mathfrak{g}$ is an ideal then the orthogonal complement $I^{\perp} \subset \mathfrak{g}$ is also an ideal. In particular, $\mathfrak{g}^{\perp} = \operatorname{Ker}(B)$ is an ideal in \mathfrak{g} .

Exercise 16.13. Prove Proposition 16.12.

Proposition 16.14. If B_V is nondegenerate for some V then \mathfrak{g} is reductive.

Proof. Let $V_1, ..., V_n$ be the simple composition factors of V; i.e., V has a filtration by subrepresentations such that $F_iV/F_{i-1}V = V_i$, $F_0V = 0$ and $F_nV = V$. Then $B_V(x, y) = \sum_i B_{V_i}(x, y)$. Now, if $x \in [\mathfrak{g}, \operatorname{rad}(\mathfrak{g})]$ then $x|_{V_i} = 0$, so $B_{V_i}(x, y) = 0$ for all $y \in \mathfrak{g}$, hence $B_V(x, y) = 0$. \Box

Example 16.15. It is clear that if $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{k})$ and $V = \mathbf{k}^n$ then the form B_V is nondegenerate, as $B_V(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}$. Thus \mathfrak{g} is reductive. Also if n is not divisible by the characteristic of \mathbf{k} then $\mathfrak{sl}_n(\mathbf{k})$ is semisimple, since it is orthogonal to scalars under B_V (hence reductive), and has trivial center. In fact, it is easy to show that in this case $\mathfrak{sl}_n(\mathbf{k})$ is a simple Lie algebra (another way to see that it is semisimple).

In fact, we have the following proposition.

Proposition 16.16. All classical Lie algebras over $\mathbb{K} = \mathbb{R}$ and \mathbb{C} are reductive.

Proof. Let \mathfrak{g} be a classical Lie algebra and V its standard matrix representation. It is easy to check that the form B_V on \mathfrak{g} is nondegenerate, which implies that \mathfrak{g} is reductive.

For example, the Lie algebras $\mathfrak{so}_n(\mathbb{K})$, $\mathfrak{sp}_{2n}(\mathbb{K})$, $\mathfrak{su}(p,q)$ have trivial center and therefore are semisimple.

16.3. The Killing form and the Cartan criteria.

Definition 16.17. The Killing form of a Lie algebra \mathfrak{g} is the form $B_{\mathfrak{g}}(x, y) = \text{Tr}(\text{ad}x \cdot \text{ad}y).$

The Killing form is denoted by $K_{\mathfrak{g}}(x, y)$ or shortly by K(x, y).

Theorem 16.18. (*Cartan criterion of solvability*) A Lie algebra \mathfrak{g} over a field \mathbf{k} of characteristic zero is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \operatorname{Ker}(K)$.

Theorem 16.19. (Cartan criterion of semisimplicity) A Lie algebra \mathfrak{g} over a field \mathbf{k} of characteristic zero is semisimple if and only if its Killing form is nondegenerate.

Theorems 16.18 and 16.19 will be proved in the next section.

Corollary 16.20. On a simple Lie algebra, the Killing form is the unique up to scaling invariant bilinear form.

Proof. Let \mathfrak{g} be a simple Lie algebra. Then the Killing form is a nonzero (in fact, nondegenerate) invariant bilinear form on \mathfrak{g} . Also any invariant bilinear form B on \mathfrak{g} can be viewed as a homomorphism of representations $B : \mathfrak{g} \to \mathfrak{g}^*$. Thus by Schur's lemma it is unique up to scaling.

16.4. Jordan decomposition. To prove the Cartan criteria, we will use the Jordan decomposition of a square matrix. Let us recall it.

Proposition 16.21. A square matrix $A \in \mathfrak{gl}_N(\mathbf{k})$ over a field \mathbf{k} of characteristic zero can be uniquely written as $A_s + A_n$, where $A_s \in \mathfrak{gl}_N(\mathbf{k})$ is semisimple (i.e. diagonalizes over the algebraic closure of \mathbf{k}) and $A_n \in \mathfrak{gl}_N(\mathbf{k})$ is nilpotent in such a way that $A_sA_n = A_nA_s$. Moreover, $A_s = P(A)$ for some $P \in \mathbf{k}[x]$.

Proof. By the Chinese remainder theorem, there exists a polynomial $P \in \overline{\mathbf{k}}[x]$ such that for every eigenvalue λ of A we have $P(x) = \lambda$ modulo $(x - \lambda)^N$, i.e.,

$$P(x) - \lambda = (x - \lambda)^N Q_\lambda(x)$$

for some polynomial Q_{λ} . Then on the generalized eigenspace $V(\lambda)$ for A, we have

$$P(A) - \lambda = (A - \lambda)^{N} Q_{\lambda}(A) = 0,$$

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so $A_s := P(A)$ is semisimple and $A_n = A - P(A)$ is nilpotent, with $A_n A_s = A_s A_n$. If $A = A'_s + A'_n$ is another such decomposition then A'_s, A'_n commute with A, hence with A_s and A_n . Also we have

$$A_s - A'_s = A'_n - A_n$$

Thus this matrix is both semisimple and nilpotent, so it is zero. Finally, since A_s, A_n are unique, they are invariant under the Galois group of $\overline{\mathbf{k}}$ over \mathbf{k} and therefore have entries in \mathbf{k} .

Remark 16.22. 1. If **k** is algebraically closed, then A admits a basis in which it is upper triangular, and A_s is the diagonal part while A_n is the off-diagonal part of A.

2. Proposition 16.21 holds with the same proof in characteristic p if the field **k** is perfect, i.e., the Frobenius map $x \to x^p$ is surjective on **k**. However, if **k** is not perfect, the proof fails: the fact that A_s , A_n are Galois invariant does not imply that their entries are in **k**. Also the statement fails: if $\mathbf{k} = \mathbb{F}_p(t)$ and $Ae_i = e_{i+1}$ for i = 1, ..., p-1 while $Ae_p = te_1$ then A has only one eigenvalue $t^{1/p}$, so $A_s = t^{1/p} \cdot \text{Id}$, i.e., does not have entries in **k**. 18.745 Lie Groups and Lie Algebras I Fall 2020

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