11. Representations of Lie groups and Lie algebras

11.1. **Representations.** We have previously defined (finite dimensional) representations of Lie groups and (iso)morphisms between them. We can do the same for Lie algebras:

Definition 11.1. A **representation of a Lie algebra** $\mathfrak{g}$ over a field $\mathbf{k}$ (or a $\mathfrak{g}$-module) is a vector space V over $\mathbf{k}$ equipped with a homomorphism of Lie algebras $\rho = \rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. A **(homo)morphism of representations** $A : V \to W$ (also called an **intertwining operator**) is a linear map which commutes with the $\mathfrak{g}$-action: $A\rho_V(b) = \rho_W(b)A$ for $b \in \mathfrak{g}$. Such A is an isomorphism if it is an isomorphism of vector spaces.

The first and second fundamental theorems of Lie theory imply:

Corollary 11.2. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}G$.*

(i) Any finite dimensional representation $\rho : G \to GL(V)$ gives rise to a Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and any morphism of G-representations is also a morphism of $\mathfrak{g}$-representations.

(ii) If G is connected then any morphism of $\mathfrak{g}$-representations is a morphism of G-representations.

(iii) If G is simply connected then the assignment $\rho \mapsto \rho_*$ is an equivalence of categories $\mathrm{Rep}\, G \to \mathrm{Rep}\, \mathfrak{g}$ between the corresponding categories of finite dimensional representations. In particular, any finite dimensional representation of the Lie algebra $\mathfrak{g}$ can be uniquely exponentiated to the group G.

Example 11.3. 1. The trivial representation: $\rho(g) = 1, g \in G$, $\rho_*(x) = 0$, $x \in \mathfrak{g}$.

2. The adjoint representation: $\rho(g) = \mathrm{Ad}_g, \rho_*(x) = \mathrm{ad}x$.

Exercise 11.4. Let $\mathfrak{g}$ be a complex Lie algebra. Show that $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$. Deduce that if G is a simply connected complex Lie group then $\mathrm{Rep}_{\mathbb{R}}\, G \cong \mathrm{Rep}(\mathfrak{g} \oplus \mathfrak{g})$, where $\mathrm{Rep}_{\mathbb{R}}\, G$ is the category of finite dimensional representations of G regarded as a real Lie group.

As usual, a **subrepresentation** of a representation V is a subspace $W \subset V$ invariant under the G-action (resp. $\mathfrak{g}$-action). In this case the quotient space V/W has a natural structure of a representation, called the **quotient representation**. The notion of **direct sum** of representations is defined in an obvious way:

$$\rho_{V \oplus W}(x) = \rho_V(x) \oplus \rho_W(x).$$

Also we have the notion of **dual representation**:

$$\rho_{V^*}(g) = \rho_V(g^{-1})^*, g \in G;\ \rho_{V^*}(x) = -\rho_V(x)^*, x \in \mathfrak{g},$$

and **tensor product**:

$$\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g),\ \rho_{V\otimes W}(x) = \rho_V(x) \otimes 1_W + 1_V \otimes \rho_W(x).$$

Thus we have the notion of **symmetric and exterior powers** $S^mV, \wedge^m V$ of a representation V, which can be defined either as quotients or (over a field of characteristic zero) as subrepresentations of $V^{\otimes n}$. Also for representations V, W, $\mathrm{Hom}(V, W)$ is a representation via

$$g \circ A = \rho_W(g)A\rho_V(g^{-1}),\ x \circ A = \rho_W(x)A - A\rho_V(x),$$

so if V is finite dimensional then $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Finally, for every representation V we have the notion of invariants:

$$V^G = \{v \in V : gv = v\ \forall g \in G\},\ V^{\mathfrak{g}} = \{v \in V : xv = 0\ \forall x \in \mathfrak{g}\}.$$

Thus $V^G \subset V^{\mathfrak{g}}$ and $V^G = V^{\mathfrak{g}}$ for connected G (in general, $V^G = (V^{\mathfrak{g}})^{G/G^\circ}$). Also $\mathrm{Hom}(V, W)^G \cong \mathrm{Hom}_G(V, W)$ and $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$, the spaces of intertwining operators. Note that in all cases the formula for Lie algebras is determined by the formula for groups by the requirement that these definitions should be consistent with the assignment $\rho \mapsto \rho_*$.

Definition 11.5. A representation $V \neq 0$ of G or $\mathfrak{g}$ is **irreducible** if any subrepresentation $W \subset V$ is either 0 or V and is **indecomposable** if for any decomposition $V \cong V_1 \oplus V_2$, we have $V_1 = 0$ or $V_2 = 0$.

It is clear that any finite dimensional representation is isomorphic to a direct sum of indecomposable representations (in fact, uniquely so up to order of summands by the *Krull-Schmidt theorem*). However, not any V is a direct sum of irreducible representations, e.g.

$$\rho : \mathbb{C} \to GL_2(\mathbb{C}),\ \rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Definition 11.6. A representation V is called **completely reducible** if it is isomorphic to a direct sum of irreducible representations.

Some of the main problems of representation theory are:

1) Classify irreducible representations;

2) If V is a completely reducible representation, find its decomposition into irreducibles.

3) For which G are all representations completely reducible?

Example 11.7. Let V be a finite dimensional $\mathbb{C}$-representation of $\mathfrak{g}$ or G and $A : V \to V$ be a homomorphism of representations (e.g., defined by a central element). Then we have a decomposition of representations $V = \oplus_\lambda V(\lambda)$, where $V(\lambda)$ is the generalized eigenspace of A with eigenvalue λ.

Example 11.8. Let V be the vector representation of $GL(V)$. Then V is irreducible, and more generally so are $S^mV, \wedge^n V$ (show it!). Thus $V \otimes V$ is completely reducible: $V \otimes V \cong S^2V \oplus \wedge^2 V$.

11.2. **Schur's lemma.**

Lemma 11.9. *(Schur's lemma) Let V, W be irreducible finite dimensional complex representations of G or $\mathfrak{g}$. Then $\mathrm{Hom}_{G,\mathfrak{g}}(V, W) = 0$ if V, W are not isomorphic, and every endomorphism of the representation V is a scalar.*

Proof. Let $A : V \to W$ be a nonzero morphism of representations. Then $\mathrm{Im}(A) \subset W$ is a nonzero subrepresentation, hence $\mathrm{Im}(A) = W$. Also $\mathrm{Ker}(A) \subset V$ is a proper subrepresentation, so $\mathrm{Ker}(A) = 0$. Thus A is an isomorphism, i.e., we may assume that $W = V$. In this case, let λ be an eigenvalue of A. Then $A - \lambda \cdot \mathrm{Id} : V \to V$ is a morphism of representations but not an isomorphism, hence it must be zero, so $A = \lambda \cdot \mathrm{Id}$. □

Note that the second statement of Schur's lemma (unlike the first one) does not hold over $\mathbb{R}$. For example, consider the rotation group $SO(2)$ (or any of its finite subgroups of order > 2) acting on $V = \mathbb{R}^2$ by rotations. Then $\mathrm{End}(V) = \mathbb{C} \neq \mathbb{R}$. Similarly, if V is the representation of $SU(2)$ on $\mathbb{H}$ defined by right multiplication by unit quaternions then V is an irreducible real representation but $\mathrm{End}(V) = \mathbb{H} \neq \mathbb{R}$. For this reason, in representation theory of Lie groups and Lie algebras one usually considers complex representations. Thus from now on all representations we consider will be assumed complex unless specified otherwise.[10]

Corollary 11.10. *The center of $G, \mathfrak{g}$ acts on an irreducible representation by a scalar. In particular, if G or $\mathfrak{g}$ is abelian then every irreducible representation of G is 1-dimensional.*

Example 11.11. Irreducible representations of $\mathbb{R}$ are χ_s given by $\chi_s(a) = \exp(sa)$, $s \in \mathbb{C}$. Irreducible representations of $\mathbb{R}^\times = \mathbb{R}_{>0} \times \mathbb{Z}/2$ are $\chi_{s,+}(a) = |a|^s$, $\chi_{s,-}(a) = |a|^s \mathrm{sign}(a)$. Irreducible representations of S^1 are $\chi_n(z) = z^n$, $n \in \mathbb{Z}$. Irreducible representations of the real group $\mathbb{C}^\times = \mathbb{R}_{>0} \times S^1$ are $\chi_{s,n}(z) = |z|^s (z/|z|)^n$, $s \in \mathbb{C}$, $n \in \mathbb{Z}$.

Corollary 11.12. *Let V_i be irreducible and $V = \oplus_i n_i V_i, W = \oplus_i m_i V_i$ be completely reducible complex representations of G or $\mathfrak{g}$. Then we have a natural linear isomorphism*

$$\mathrm{Hom}_{G,\mathfrak{g}}(V, W) \cong \oplus_i \mathrm{Mat}_{m_i,n_i}(\mathbb{C}).$$

[10]An exception is the adjoint representation of a real Lie group and associated tensor representations, which are real.

Moreover, if $V = W$ then this is an isomorphism of algebras.

11.3. **Unitary representations.** A finite dimensional representation V of G is said to be **unitary** if it is equipped with a positive definite Hermitian inner product $B(,)$ invariant under G, i.e., $B(gv, gw) = B(v, w)$ for $v, w \in V$, $g \in G$.

Proposition 11.13. *Any unitary representation can be written as an orthogonal direct sum of irreducible unitary representations. In particular, it is completely reducible.*

Proof. If $W \subset V$ is a subrepresentation of a unitary representation V then let $W^\perp$ be its orthogonal complement under B. Then $W^\perp$ is also a subrepresentation since B is invariant, and $V = W \oplus W^\perp$ since B is positive definite.

Now we can prove that V is an orthogonal direct sum of irreducible unitary representations by induction in $\dim V$. The base $\dim V = 1$ is clear so let us make the inductive step. Pick an irreducible $W \subset V$. Then $V = W \oplus W^\perp$, and $W^\perp$ is a unitary representation of dimension smaller than $\dim V$, so is an orthogonal direct sum of irreducible unitary representations by the induction assumption. $\square$

Proposition 11.14. *Any finite dimensional representation V of a finite group G is unitary. Moreover, if V is irreducible, the unitary structure is unique up to a positive factor.*

Proof. Let B be any positive definite inner product on V. Let

$$\widehat{B}(v, w) := \sum_{g \in G} B(gv, gw).$$

Then $\widehat{B}$ is positive definite and invariant, so V is unitary.

If V is irreducible and B_1, B_2 are two unitary structures on V then $B_1(v, w) = B_2(Av, w)$ for some homomorphism $A : V \to V$. Thus by Schur's lemma $A = \lambda \cdot \mathrm{Id}$, and $\lambda > 0$ since B_1, B_2 are positive definite. $\square$

Corollary 11.15. *Every finite dimensional complex representation of a finite group G is completely reducible.*

11.4. **Representations of $\mathfrak{sl}_2$.** The Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with commutator

$$[e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f.$$

Since 2-by-2 matrices act on variables x, y, they also act on the space $V = \mathbb{C}[x, y]$ of polynomials in x, y. Namely, this action is given by the formulas

$$e = x\partial_y,\ f = y\partial_x,\ h = x\partial_x - y\partial_y.$$

This infinite-dimensional representation has the form $V = \oplus_{n\ge 0} V_n$, where V_n is the space of polynomials of degree n. The space V_n is invariant under e, f, h, so it is an $n+1$-dimensional representation of $\mathfrak{sl}_2$. It has basis $v_{pq} = x^p y^q$, such that

$$hv_{pq} = (p - q)v_{pq},\ ev_{pq} = qv_{p+1,q-1},\ fv_{pq} = pv_{p-1,q+1}.$$

Thus V_0 is the trivial representation, and V_1 is the tautological representation by 2-by-2 matrices. Also it is easy to see that V_2 is the adjoint representation.

Theorem 11.16. *(i) V_n is irreducible.*

(ii) If $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}_2$ then $e|_V$ and $f|_V$ are nilpotent, so $U := \mathrm{Ker}(e) \neq 0$. Moreover, h preserves U and acts diagonalizably on it, with nonnegative integer eigenvalues.

(iii) Any irreducible finite dimensional representation V of $\mathfrak{sl}_2$ is isomorphic to V_n for some n.

(iv) Any finite dimensional representation V of $\mathfrak{sl}_2$ is completely reducible.

Proof. (i) Let $W \subset V_n$ be a nonzero subrepresentation. Since it is h-invariant, it must be spanned by vectors $v_{p,n-p}$ for p from a nonempty subset $S \subset [0, n]$. Since W is e-invariant and f-invariant, if $m \in S$ then so are $m + 1, m - 1$ (if they are in $[0, n]$). Thus $S = [0, n]$ and $W = V_n$.

(ii) Let V be a finite dimensional representation of $\mathfrak{sl}_2$. We can write V as a direct sum of generalized eigenspaces of h: $V = \oplus_\lambda V(\lambda)$. Since $he = e(h + 2)$, $hf = f(h - 2)$, we have $e : V(\lambda) \to V(\lambda + 2)$, $f : V(\lambda) \to V(\lambda - 2)$. Thus $e|_V$, $f|_V$ are nilpotent, so $U \neq 0$.

If $v \in U$ then $e(hv) = (h-2)ev = 0$, so $hv \in U$, i.e., U is h-invariant.

Given $v \in U$, consider the vector $v_m := e^m f^m v$. We have

(11.1)
$$ef^m v = fef^{m-1}v + hf^{m-1}v = fef^{m-1}v + f^{m-1}(h - 2(m - 1))v = ...$$
$$= f^{m-1}m(h - m + 1)v.$$

Thus

$$v_m = e^{m-1}f^{m-1}m(h - m + 1)v = m(h - m + 1)v_{m-1}.$$

Hence

$$v_m = m!h(h - 1)...(h - m + 1)v.$$

But for large enough m, $v_m = 0$, since f is nilpotent, so

$$h(h-1)...(h-m+1)v = 0.$$

Thus h acts diagonalizably on U with nonnegative integer eigenvalues.

(iii) Let $v \in U$ be an eigenvector of h, i.e., $hv = \lambda v$. Let $w_m = f^m v$. Then

$$fw_m = w_{m+1}, hw_m = (\lambda - 2m)w_m.$$

Also, it follows from (11.1) that

$$ew_m = m(\lambda - m + 1)w_{m-1}.$$

Thus if $w_m \neq 0$ and $\lambda \neq m$ then $w_{m+1} \neq 0$. Also the nonzero vectors w_m are linearly independent since they have different eigenvalues of h. Thus $\lambda = n$ must be a nonnegative integer (as also follows from (ii)), and $w_{n+1} = 0$. So V, being irreducible, has a basis w_m, $m = 0, ..., n$. Now it is easy to see that $V \cong V_n$, via the assignment

$$w_m \mapsto n(n-1)...(n-m+1)x^m y^{n-m}.$$

(iv) Consider the **Casimir operator**

$$C = 2fe + \frac{h^2}{2} + h.$$

It is easy to check that $[C, e] = [C, f] = [C, h] = 0$, so $C : V \to V$ is a homomorphism. Thus $C|_{V_n} = \frac{n(n+2)}{2}$ (it is a scalar by Schur's lemma, and acts with such eigenvalue on $v_{n0} \in V_n$); note that these are different for different n. For a general representation, we have $V = \oplus_c V_c$, the direct sum of generalized eigenspaces of C.

Assume V is indecomposable. Then by Example 11.7 C has a single eigenvalue c on V. Fix a **Jordan-Hölder filtration** on V, i.e. a filtration

$$0 = F_0V \subset F_1V \subset ... \subset F_mV = V$$

such that $Y_i := F_iV/F_{i-1}V$ are irreducible for all i. By (iii), for each i we have $Y_i \cong V_n$ for some n, so $c = \frac{n(n+2)}{2}$ and thus this n is the same for all i. Thus $V(k)$ has dimension m, with h acting on it by $k \cdot \mathrm{Id}$ for $k = n, n-2, ..., -n$ and $V(k) = 0$ otherwise, by (ii); in particular, $\dim V = m(n+1)$. Let $u_1, ..., u_m$ be a basis of $V(n)$. As in (iii), we define subrepresentations $W_i \subset V$ generated by u_i. It is easy to see that $W_i \cong V_n$ and the natural morphism $W_1 \oplus ... \oplus W_m \to V$ is injective. Hence it is an isomorphism by dimension count, i.e., V is completely reducible. □

Corollary 11.17. *(The Jacobson-Morozov lemma for $GL(V)$) Let V be a finite dimensional complex vector space and $N : V \to V$ be a* *nilpotent operator. Then there is a unique up to isomorphism action of $\mathfrak{sl}_2$ on V for which e acts by N.*

Proof. This follows from Theorem 11.16 and the Jordan normal form theorem for operators on V. □

For a representation V define its **character** by

$$\chi_V(z) = \mathrm{Tr}_V(z^h) = \sum_m \dim V(m) z^m.$$

Thus

$$\chi_{V_n}(z) = z^n + z^{n-2} + ... + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}.$$

It is easy to see that

$$\chi_{V \oplus W} = \chi_V + \chi_W, \chi_{V \otimes W} = \chi_V \chi_W.$$

Since the functions χ_{V_n} are linearly independent, we see that a finite dimensional representation of $\mathfrak{sl}_2$ is determined by its character.

Theorem 11.18. *(The Clebsch-Gordan rule) We have*

$$V_m \otimes V_n \cong \oplus_{i=0}^{\min(m,n)} V_{|m-n|+2i}.$$

Proof. It suffices to note that we have the corresponding character identity:

$$\chi_{V_m} \chi_{V_n} = \sum_{i=0}^{\min(m,n)} \chi_{V_{|m-n|+2i}}.$$

□

Exercise 11.19. Show that V_n has an invariant nondegenerate inner product (i.e., such that $(av, w) + (v, aw) = 0$ for $a \in \mathfrak{sl}_2$, $v, w \in V_n$) which is symmetric for even n and skew-symmetric for odd n. In particular, $V_n^* \cong V_n$.

Exercise 11.20. Let G be the universal cover of $SL_2(\mathbb{R})$. Show that G is not isomorphic to a Lie subgroup of $GL_n(\mathbb{R})$ for any n and that moreover, the only quotients of G that are such subgroups are $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$.

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