10. **Proofs of the fundamental theorems of Lie theory**

10.1. **Distributions and the Frobenius theorem.**

The proofs of the fundamental theorems of Lie theory are based on the notion of an integrable distribution in differential geometry, and the Frobenius theorem about such distributions.

Definition 10.1. A k-dimensional **distribution** on a manifold X is a rank k subbundle $D \subset TX$.

This means that in every tangent space T_xX we fix a k-dimensional subspace D_x which varies regularly with x. In other words, on some neighborhood $U \subset X$ of every $x \in X$, D is spanned by vector fields $\mathbf{v}_1, ..., \mathbf{v}_k$ linearly independent at every point of U.

Definition 10.2. A distribution D is **integrable** if every point $x \in X$ has a neighborhood U and local coordinates $x_1, ..., x_n$ on U such that D is defined at every point of U by the equations $dx_{k+1} = ... = dx_n = 0$, i.e., it is spanned by vector fields $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, ..., k$.

This is equivalent to saying that every point x of X is contained in an **integral submanifold** for D, i.e., an immersed submanifold $S = S_x \subset X$ such that for any $y \in S$ the tangent space $T_yS \subset T_yX$ coincides with D_y. Namely, S_x is the set of all points of $y \in X$ that can be connected to x by a smooth curve $\gamma : [0, 1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ and $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [0, 1]$ (show it!).

For this reason an integrable distribution is also called a **foliation** and the integral submanifolds S_x are called the **sheets of the foliation**. The manifold X falls into a disjoint union of such sheets. But note that the sheets need not be closed (i.e., think of the irrational torus winding!)

Example 10.3. A 1-dimensional distribution is the same thing as a **direction field.** It is always integrable, as follows from the existence theorem for ODE, and its integral submanifolds are called **integral curves**. They are geometric realizations of solutions of the corresponding ODE.

However, for $k \geq 2$ a distribution is not always integrable.

Theorem 10.4. *(The Frobenius theorem) A distribution D is integrable if and only if for every two vector fields $\mathbf{v}, \mathbf{w}$ contained in D, their commutator $[\mathbf{v}, \mathbf{w}]$ is also contained in D.*

Example 10.5. Let $\mathbf{v} = \partial_x$, $\mathbf{w} = x\partial_y + \partial_z$ in $\mathbb{R}^3$, and D be the 2-dimensional distribution spanned by $\mathbf{v}, \mathbf{w}$. Then $[\mathbf{v}, \mathbf{w}] = \partial_y \notin D$. So D is not integrable.

Proof. If D is integrable, a vector field is contained in D iff it is tangent to integral submanifolds of D. But the commutator of two vector fields tangent to a submanifold is itself tangent to this submanifold. This establishes the "only if" part.

It remains to prove the "if " part. The proof is by induction in the rank k of D. The base case $k = 0$ is trivial, so it suffices to establish the inductive step. The question is local, so we may work in a neighborhood U of $P \in X$. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathrm{Vect}(U)$ is a basis of D in U (on every tangent space). By local existence and uniqueness of solutions of ODE, in some local coordinates $x_1, ..., x_n = z$, the vector field $\mathbf{v}_k$ equals ∂_z. By subtracting from $\mathbf{v}_i, i < k$ a suitable multiple of $\mathbf{v}_k$ we can make sure that $\mathbf{v}_i$ has no ∂_z-component. Then

$$\mathbf{v}_i = \sum_{j=1}^{n-1} a_{ij}(x_1, ..., x_{n-1}, z)\partial_{x_j}.$$

Thus, since by assumption $[\partial_z, \mathbf{v}_i] = [\mathbf{v}_k, \mathbf{v}_i]$ is a linear combination of $\mathbf{v}_m$ with functional coefficients, we have

$$[\partial_z, \mathbf{v}_i] = \sum_{m=1}^{k-1} b_{im}(x_1, ..., x_{n-1}, z)\mathbf{v}_m$$

($\mathbf{v}_k$ does not occur since there is no ∂_z component on the left hand side). Hence

$$\partial_z a_{ij}(x_1, ..., x_{n-1}, z) = \sum_{m=1}^{k-1} b_{im}(x_1, ..., x_{n-1}, z)a_{mj}(x_1, ..., x_{n-1}, z).$$

So, setting $A = (a_{mj}(x_1, ..., x_{n-1}, z))$ (a $(k-1) \times (n-1)$-matrix) and $B = (b_{im}(x_1, ..., x_{n-1}, z))$ (a $(k-1) \times (k-1)$ matrix), we have

$$\partial_z A = BA.$$

Let A_0 be the solution of this linear ODE in $(k-1) \times (k-1)$ matrices with $A_0(x_1, ..., x_{n-1}, 0) = 1$. Then $A = A_0 C$, where $C = C(x_1, ..., x_{n-1})$ is a $(k-1) \times (n-1)$-matrix which does not depend on z. So we have a new basis of D given by $\mathbf{w}_k = \partial_z$ and

$$\mathbf{w}_i = \sum_j c_{ij}(x_1, ..., x_{n-1})\partial_{x_j},\ 1 \le i \le k-1.$$

Thus there is a neighborhood U of P which can be represented as $U = (-a, a) \times U'$, where $\dim U' = n-1$, so that $D = \mathbb{R} \oplus D'$, where D' is a $k-1$-dimensional distribution on U' spanned by $\mathbf{w}_i$, $1 \le i \le k-1$. It is clear that for any two vector fields $\mathbf{v}, \mathbf{w}$ on U' contained in D', so is

$[\mathbf{v},\mathbf{w}]$. Hence D' is integrable by the induction assumption. Therefore, so is D, justifying the inductive step.

$\square$

10.2. **Proofs of the fundamental theorems of Lie theory.**

10.2.1. *Proof of Theorem 9.11.* Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. We need to show that there is a unique (not necessarily closed) connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$. The proof of existence of H is based on the Frobenius theorem.

Define the distribution D on G by left-translating $\mathfrak{h} \subset \mathfrak{g} = T_1G$, i.e., $D_g = L_g\mathfrak{h}$. So any vector field contained in D is of the form

$$\mathbf{v} = \sum f_i \mathbf{L}_{a_i},$$

where a_i is a basis of $\mathfrak{h}$ and f_i are regular functions. Now if

$$\mathbf{w} = \sum g_j \mathbf{L}_{a_j}$$

is another such field then

$$[\mathbf{v},\mathbf{w}] = \sum_{i,j} (f_i \mathbf{L}_{a_i}(g_j)\mathbf{L}_{a_j} - g_j \mathbf{L}_{a_j}(f_i)\mathbf{L}_{a_i} + f_i g_j [\mathbf{L}_{a_i}, \mathbf{L}_{a_j}]).$$

But $[a_i, a_j] = \sum_k c_{ij}^k a_k$, so

$$[\mathbf{L}_{a_i}, \mathbf{L}_{a_j}] = \sum_k c_{ij}^k \mathbf{L}_{a_k}.$$

Thus if $\mathbf{v},\mathbf{w}$ are contained in D then so is $[\mathbf{v},\mathbf{w}]$. Hence by the Frobenius theorem, D is integrable.

Now consider the integral (embedded) submanifold H of D going through $1 \in G$. We claim that H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Indeed, it suffices to show that H is a subgroup of G. But this is clear since H is the collection of elements of G of the form

$$g = \exp(a_1)...\exp(a_m),$$

where $a_i \in \mathfrak{h}$.

Moreover, H is unique since it has to be generated by the image of the exponential map $\exp : \mathfrak{h} \to G$.

10.2.2. *Proof of Theorem 9.12.* We need to show that the natural map $\mathrm{Hom}(G,K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$ is a bijection if G is simply connected.

We know this map is injective so we only need to establish surjectivity. For any morphism $\psi : \mathrm{Lie}G \to \mathrm{Lie}K$, consider the morphism

$$\theta = (\mathrm{id}, \psi) : \mathrm{Lie}G \to \mathrm{Lie}(G \times K) = \mathrm{Lie}G \oplus \mathrm{Lie}K$$

The previous proposition implies that there is a connected Lie subgroup $H \subset G \times K$ whose Lie algebra is $\mathrm{Im}\theta$. We have projection homomorphisms $p_1 : H \to G$, $p_2 : H \to K$, and $(p_1)_* = \mathrm{id}$, so p_1 is a covering. Since G is simply connected, p_1 is an isomorphism, so we can define $\phi := p_2 \circ p_1^{-1} : G \to K$, and it is easy to see that $\psi = \phi_*$.

10.2.3. *Proof of Theorem 9.13.* Finally, let us discuss a proof of Theorem 9.13, stating that any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the Lie algebra of a Lie group. We will deduce it from the following purely algebraic *Ado's theorem*.

Theorem 10.6. *Any finite dimensional Lie algebra over $\mathbb{K}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.*

Ado's theorem in fact holds over any ground field, but it is rather nontrivial and we won't prove it now. A proof can be found, for example, in [J]. But Ado's theorem immediately implies Theorem 9.13. Indeed, using Theorem 9.11, Ado's theorem implies the following even stronger statement:

Theorem 10.7. *Any finite dimensional $\mathbb{K}$-Lie algebra is the Lie algebra of a Lie subgroup of $GL_n(\mathbb{K})$ for some n.*

This implies

Corollary 10.8. *Any simply connected Lie group is the universal covering of a linear Lie group, i.e., of a Lie subgroup of $GL_n(\mathbb{K})$.*

However, it is not true that any Lie group is isomorphic to a Lie subgroup of $GL_n(\mathbb{K})$, see Exercise 11.20.

One can also prove Theorem 9.13 directly and then deduce Ado's theorem as a corollary. We will do this in Sections 49 and 50. We note that Theorem 9.13 will not be used in proofs of other results until that point.

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