9. **Fundamental theorems of Lie theory**

9.1. **Proofs of Theorem 3.13, Proposition 4.12, Proposition 4.7.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and X be a manifold with an action $a : G \times X \to X$. Then for any $z \in \mathfrak{g}$ we have a vector field $a_*(z)$ on X given by

$$(a_*(z)f)(x) = \tfrac{d}{dt}|_{t=0} f(\exp(-tz)x),$$

where $t \in \mathbb{R}$, $f \in O(U)$ for some open set $U \subset X$ and $x \in U$.

Proposition 9.1. *The map a_* is linear and we have*

$$a_*([z, w]) = [a_*(z), a_*(w)].$$

In other words, the map $a_* : \mathfrak{g} \to \mathrm{Vect}(X)$ is a homomorphism of Lie algebras.

Exercise 9.2. Prove Proposition 9.1.

This motivates the following definition.

Definition 9.3. An action of a Lie algebra $\mathfrak{g}$ on a manifold X is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$.

Thus an action of a Lie group G on X induces an action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}G$ on X.

Now let $x \in X$. Then we have a linear map $a_{*x} : \mathfrak{g} \to T_xX$ given by $a_{*x}(z) := a_*(z)(x)$.

Theorem 9.4. *(i) The stabilizer G_x is a closed subgroup of G with Lie algebra*

$$\mathfrak{g}_x := \mathrm{Ker}(a_{*x}).$$

(ii) The map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion. So the orbit Gx is an immersed submanifold of X, and

$$T_x(Gx) \cong \mathrm{Im}(a_{*x}) \cong \mathfrak{g}/\mathfrak{g}_x.$$

Part (i) of Theorem 9.4 is the promised weaker version of Theorem 3.13 sufficient for our purposes. Also, part (ii) implies Proposition 4.12.

Proof. (i) It is clear that G_x is closed in G, but we need to show it is a Lie subgroup and compute its Lie algebra.[8] It suffices to show that for some neighborhood U of 1 in G, $U \cap G_x$ is a (closed) submanifold of U such that $T_1(U \cap G_x) = \mathfrak{g}_x$.

Note that $\mathfrak{g}_x \subset \mathfrak{g}$ is a Lie subalgebra, since the commutator of vector fields vanishing at x also vanishes at x (by the formula for commutator

[8]Although we claimed in Theorem 3.13 that a closed subgroup of a Lie group is always a Lie subgroup, we did not prove it, so we need to prove it in this case.

in local coordinates). Also, for any $z \in \mathfrak{g}_x$, $\exp(tz)x$ is a solution of the ODE $\gamma'(t) = a_{*\gamma(t)}(z)$ with initial condition $\gamma(0) = x$, and $\gamma(t) = x$ is such a solution, so by uniqueness of ODE solutions $\exp(tz)x = x$, thus $\exp(tz) \in G_x$.

Now choose a complement $\mathfrak{u}$ of $\mathfrak{g}_x$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{u}$. Then $a_{*x} : \mathfrak{u} \to T_xX$ is injective. By the implicit function theorem, the map $\mathfrak{u} \to X$ given by $u \mapsto \exp(u)x$ is injective for small u, so $\exp(u) \in G_x$ for small $u \in \mathfrak{u}$ if and only if $u = 0$.

But in a small neighborhood U of 1 in G, any element g can be uniquely written as $g = \exp(u)\exp(z)$, where $u \in \mathfrak{u}$ and $z \in \mathfrak{g}_x$. So we see that $g \in G_x$ iff $u = 0$, i.e., $\log(g) \in \mathfrak{g}_x$. This shows that $U \cap G_x$ coincides with $U \cap \exp(\mathfrak{g}_x)$, as desired.

(ii) The same proof shows that we have an isomorphism $T_1(G/G_x) \cong \mathfrak{g}/\mathfrak{g}_x = \mathfrak{u}$, so the injectivity of $a_{*x} : \mathfrak{u} \to T_xX$ implies that the map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion, as claimed. $\square$

Corollary 9.5. *(Proposition 4.7) Let $\phi : G \to K$ be a morphism of Lie groups and $\phi_* : \mathrm{Lie}G \to \mathrm{Lie}K$ be the corresponding morphism of Lie algebras. Then $H := \mathrm{Ker}(\phi)$ is a closed normal Lie subgroup with Lie algebra $\mathfrak{h} := \mathrm{Ker}(\phi_*)$, and the map $\overline{\phi} : G/H \to K$ is an immersion. Moreover, if $\mathrm{Im}\overline{\phi}$ is a submanifold of K then it is a closed Lie subgroup, and we have an isomorphism of Lie groups $\overline{\phi} : G/H \cong \mathrm{Im}\overline{\phi}$.*

Proof. Apply Theorem 9.4 to the action of G on $X = K$ via $g \circ k = \phi(g)k$, and take $x = 1$. $\square$

Corollary 9.6. *Let V be a finite dimensional representation of a Lie group G, and $v \in V$. Then the stabilizer G_v is a closed Lie subgroup of G with Lie algebra $\mathfrak{g}_v := \{z \in \mathfrak{g} : zv = 0\}$.*

Example 9.7. Let A be a finite dimensional algebra (not necessarily associative, e.g. a Lie algebra). Then the group $G = \mathrm{Aut}(A) \subset GL(A)$ is a closed Lie subgroup with Lie algebra $\mathrm{Der}(A) \subset \mathrm{End}(A)$ of derivations of A, i.e., linear maps $d : A \to A$ such that

$$d(ab) = d(a) \cdot b + a \cdot d(b).$$

Indeed, consider the action of $GL(A)$ on $\mathrm{Hom}(A \otimes A, A)$. Then $G = G_\mu$ where $\mu : A \otimes A \to A$ is the multiplication map. Also, if g_t is a smooth family of automorphisms of A such that $g_0 = \mathrm{id}$ (i.e., $g_t(ab) = g_t(a)g_t(b)$) and $d = \frac{d}{dt}|_{t=0}g_t$ then $d(ab) = d(a) \cdot b + a \cdot d(b)$, and conversely, if d is a derivation then $g_t := \exp(td)$ is an automorphism.

9.2. **The center of G and $\mathfrak{g}$.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $Z = Z(G)$ the center of G, i.e. the set of $z \in G$ such that $zg = gz$

for all $g \in G$. Also let $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ be the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$; it is called the **center** of $\mathfrak{g}$.

Proposition 9.8. *If G is connected then Z is a closed (normal, commutative) Lie subgroup of G with Lie algebra $\mathfrak{z}$.*

Proof. Since G is connected, an element $g \in G$ belongs to Z iff it commutes with $\exp(tu)$ for all $u \in \mathfrak{g}$, i.e., iff $\mathrm{Ad}_g(u) = u$. Thus $Z = \mathrm{Ker}(\mathrm{Ad})$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation. Thus by Proposition 4.7, $Z \subset G$ is a closed Lie subgroup with Lie algebra $\mathrm{Ker}(\mathrm{ad})$, as claimed. □

Remark 9.9. In general (when G is not necessarily connected), it is easy to show that G/G° acts on $\mathfrak{z}$, and Z is a closed Lie subgroup of G with Lie algebra $\mathfrak{z}^{G/G^\circ}$ (the subspace of invariant vectors).

Definition 9.10. For a connected Lie group G, the group $G/Z(G)$ is called the **adjoint group** of G.

It is clear that $G/Z(G)$ is naturally isomorphic to the image of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, which motivates the terminology.

9.3. **The statements of the fundamental theorems of Lie theory.**

Theorem 9.11. *(First fundamental theorem of Lie theory) For a Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, given by $\mathfrak{h} = \mathrm{Lie}H$.*

Theorem 9.12. *(Second fundamental theorem of Lie theory) If G and K are Lie groups with G simply connected then the map*

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$$

given by $\phi \mapsto \phi_*$ is a bijection.

Theorem 9.13. *(Third fundamental theorem of Lie theory) Any finite dimensional Lie algebra is the Lie algebra of a Lie group.*

These theorems hold for real as well as complex Lie groups. Thus we have

Corollary 9.14. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the assignment $G \mapsto \mathrm{Lie}G$ is an equivalence between the category of simply connected $\mathbb{K}$-Lie groups and the category of finite dimensional $\mathbb{K}$-Lie algebras. Moreover, any connected Lie group K has the form G/Γ where G 'is simply connected and $\Gamma \subset G$ is a discrete central subgroup.*

Proof. The second fundamental theorem says that the functor $G \mapsto \mathrm{Lie}G$ is fully faithful, and the third fundamental theorem says that it is essentially surjective. Thus it is an equivalence of categories. The last statement follows from Proposition 3.5 (G is the universal covering of K). $\square$

We will discuss proofs of the fundamental theorems of Lie theory in Subsection 10.2. The third theorem is the hardest one, and we will give its complete proof only in Section 49.

9.4. **Complexification of real Lie groups and real forms of complex Lie groups.** Let $\mathfrak{k}$ be a real Lie algebra. Then $\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra. We say that $\mathfrak{g} := \mathfrak{k}_{\mathbb{C}}$ is the **complexification** of $\mathfrak{k}$, and $\mathfrak{k}$ is a **real form** of $\mathfrak{g}$. Thus a real form of $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ such that the natural map $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$ is an isomorphism.

In this case we have an antilinear involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ given by $\sigma(a + ib) = a - ib$ for $a, b \in \mathfrak{k}$, and $\mathfrak{k} := \mathfrak{g}^{\sigma}$ is the set of fixed points of σ. Conversely, it is easy to see that if σ is an antilinear involution of a complex Lie algebra $\mathfrak{g}$ (i.e., an automorphism as a real Lie algebra such that $\sigma^2 = 1$ and $\sigma(\lambda a) = \overline{\lambda}\sigma(a)$ for $a \in \mathfrak{g}, \lambda \in \mathbb{C}$), then $\mathfrak{k} := \mathfrak{g}^{\sigma} \subset \mathfrak{g}$ is a real form of $\mathfrak{g}$. Thus real forms of a complex Lie algebra are in natural bijection with its antilinear involutions.

Note that two non-isomorphic real Lie algebras can have isomorphic complexifications; in other words, the same complex Lie algebra can have non-isomorphic real forms. For example,

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{C})$$

while for $n > 1$,

$$\mathfrak{u}(n) \ncong \mathfrak{gl}_n(\mathbb{R}),$$

since in the first algebra any element x with nilpotent $\mathrm{ad}x$ must be zero, while in the second one it does not have to.

Let us now discuss real forms of complex Lie groups. By analogy with the case of Lie algebras, we make the following definition.

Definition 9.15. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and $\sigma : G \to G$ be an involutive automorphism of G as a real Lie group such that the induced map $\sigma : \mathfrak{g} \to \mathfrak{g}$ is antilinear (i.e., σ is antiholomorphic). Then the fixed point subgroup $K := G^{\sigma}$ is called a **real form** of G and G is called a **complexification** of K.[9]

[9]Note that this definition is not quite equivalent to Definition 3.51 in [K] of the same notion, which is less conventional. For example, according to the definition of [K], every complex elliptic curve has a real form, which does not agree with the definition from algebraic geometry (cf. Example 9.16).

Note that a real Lie group K may not admit a complexification. For example, Exercise 11.20 shows that this happens if $K^\circ \cong \widetilde{SL_2(\mathbb{R})}$, the universal cover of $SL_2(\mathbb{R})$. On the other hand, Example 9.16 shows that K may admit several (in fact, infinitely many) non-isomorphic complexifications.

For example, both $U(n)$ and $GL_n(\mathbb{R})$ are real forms of $GL_n(\mathbb{C})$, with $\sigma(g) = \overline{g}$ and $\sigma(g) = (\overline{g}^T)^{-1}$ respectively. Note that $GL_n(\mathbb{R})$ is not connected, so a real form of a connected Lie group may be disconnected.

We see that every real form (i.e., antilinear involution) of $\mathfrak{g}$ defines at most one such form for G. However, it could be none since the involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ may not lift to G. This is demonstrated by the following example.

Example 9.16. Let $\Lambda \subset \mathbb{C}$ be a lattice generated by 1 and $\tau \in \mathbb{C}$ with $\mathrm{Im}\tau > 0$, $-\frac{1}{2} < \mathrm{Re}\tau \le \frac{1}{2}$, and let $E := \mathbb{C}/\Lambda$ be the corresponding complex elliptic curve (a 1-dimensional complex Lie group). We have $\mathrm{Lie}E = \mathbb{C}$, so the only real form of $\mathrm{Lie}E$ is defined by the antilinear involution $\sigma(z) = \overline{z}$. The condition for this involution to lift to E is that $\sigma(\Lambda) = \Lambda$, or, equivalently, $\overline{\tau} = a\tau + b$ for some $a, b \in \mathbb{Z}$ coprime. Taking imaginary parts, we get that $a = -1$, so E has a real form if and only if $\overline{\tau} + \tau \in \mathbb{Z}$. This coincides with the definition of a real elliptic curve in algebraic geometry saying that E can be defined by a Weierstrass equation $y^2 = P(x)$ where P is a cubic polynomial with real coefficients (check it!). There are two types of such elliptic curves: $\tau \in i\mathbb{R}$ (P has one real root) and $\tau \in \frac{1}{2} + i\mathbb{R}$ (P has three real roots). In the first case the corresponding real group E^σ is $\mathbb{Z}/2 \times \mathbb{R}/\mathbb{Z}$ (the two components are the images of $\mathbb{R}$ and $\mathbb{R} + \frac{1}{2}\tau$), while in the second case it is $\mathbb{R}/\mathbb{Z}$ (the image of $\mathbb{R}$).

However, if G is a simply connected complex Le group, then every real form of $\mathfrak{g}$ necessarily defines one for G. Indeed, in this case by the second fundamental theorem of Lie theory, the antilinear involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ lifts to an antiholomorphic involution $G \to G$.

Exercise 9.17. (i) Classify complex Lie algebras of dimension at most 3, up to isomorphism.

(ii) Classify real Lie algebras of dimension at most 3.

(iii) Classify connected complex and real Lie groups of dimension at most 3.

18.745 Lie Groups and Lie Algebras I
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