8. **Lie algebras**

8.1. **The Jacobi identity.** The matrix commutator $[x, y] = xy - yx$ obviously satisfies the identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

called the **Jacobi identity**. Thus it is satisfied for any Lie subgroup of $GL_n(\mathbb{K})$.

Proposition 8.1. *The Jacobi identity holds for any Lie group G.*

Proof. Let $\mathfrak{g} = T_1G$. The Jacobi identity is equivalent to $\mathrm{ad}x$ being a derivation of the commutator:

$$\mathrm{ad}x([y, z]) = [\mathrm{ad}x(y), z] + [y, \mathrm{ad}x(z)],\ \ x, y, z \in \mathfrak{g}.$$

To show that it is indeed a derivation, let $g(t) = \exp(tx)$, then

$$\mathrm{Ad}_{g(t)}([y, z]) = [\mathrm{Ad}_{g(t)}(y), \mathrm{Ad}_{g(t)}(z)].$$

The desired identity is then obtained by differentiating this equality by t at $t = 0$ and using the Leibniz rule and Proposition 7.11(iv). $\square$

Corollary 8.2. *We have* $\mathrm{ad}[x, y] = [\mathrm{ad}x, \mathrm{ad}y]$.

Proof. This is also equivalent to the Jacobi identity. $\square$

Proposition 8.3. *For $x \in \mathfrak{g}$ one has* $\exp(\mathrm{ad}x) = \mathrm{Ad}_{\exp(x)} \in GL(\mathfrak{g})$.

Proof. We will show that $\exp(t\mathrm{ad}x) = \mathrm{Ad}_{\exp(tx)}$ for $t \in \mathbb{R}$. Let $\gamma_1(t) = \exp(t\mathrm{ad}x)$ and $\gamma_2(t) = \mathrm{Ad}_{\exp(tx)}$. Then γ_1, γ_2 both satisfy the differential equation $\gamma'(t) = \gamma(t)\mathrm{ad}x$ and equal 1 at $t = 0$. Thus $\gamma_1 = \gamma_2$. $\square$

8.2. **Lie algebras.**

Definition 8.4. A **Lie algebra** over a field $\mathbf{k}$ is a vector space $\mathfrak{g}$ over $\mathbf{k}$ equipped with bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **commutator** or **(Lie) bracket** which satisfies the following identities:
(i) $[x, x] = 0$ for all $x \in \mathfrak{g}$;
(ii) the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

A **(homo)morphism of Lie algebras** is a linear map between Lie algebras that preserves the commutator.

Remark 8.5. If $\mathbf{k}$ has characteristic $\neq 2$ then the condition $[x, x] = 0$ is equivalent to skew-symmetry $[x, y] = -[y, x]$, but in characteristic 2 it is stronger.

Example 8.6. Any subspace of $\mathfrak{gl}_n(\mathbf{k})$ closed under $[x, y] := xy - yx$ is a Lie algebra.

Example 8.7. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a morphism of Lie algebras.

Thus we have

Theorem 8.8. *If G is a $\mathbb{K}$-Lie group (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) then $\mathfrak{g} := T_1G$ has a natural structure of a Lie algebra over $\mathbb{K}$. Moreover, if $\phi : G \to K$ is a morphism of Lie groups then $\phi_* : T_1G \to T_1K$ is a morphism of Lie algebras.*

We will denote the Lie algebra $\mathfrak{g} = T_1G$ by Lie$G$ or Lie(G) and call it the **Lie algebra of** G. We see that the assignment $G \mapsto \mathrm{Lie}G$ is a functor from the category of Lie groups to the category of Lie algebras. Thus we have a map $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$, which is injective if G is connected.

Motivated by Proposition 7.11(v), a Lie algebra $\mathfrak{g}$ is said to be **commutative** or **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

8.3. **Lie subalgebras and ideals.** A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ closed under the commutator. It is called a **Lie ideal** if moreover $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Proposition 8.9. *Let $H \subset G$ be a Lie subgroup. Then:*

(i) Lie$H \subset$ Lie$G$ *is a Lie subalgebra;*

(ii) If H is normal then Lie$H$ *is a Lie ideal in* Lie$G$*;*

(iii) If G, H are connected and Lie$H \subset$ LieG *is a Lie ideal then H is normal in G.*

Proof. (i) If $x, y \in \mathfrak{h}$ then $\exp(tx), \exp(sy) \in H$, so by Proposition 7.11(iv)

$$[x, y] = \lim_{t,s\to 0} \frac{\log(\exp(tx)\exp(sy)\exp(-tx)\exp(-sy))}{ts} \in \mathfrak{h}.$$

(ii) We have $ghg^{-1} \in H$ for $g \in G$ and $h \in H$. Thus, taking $h = \exp(sy)$, $y \in \mathfrak{h}$ and taking the derivative in s at zero, we get $\mathrm{Ad}_g(y) \in \mathfrak{h}$. Now taking $g = \exp(tx)$, $x \in \mathfrak{g}$ and taking the derivative in t at zero, by Proposition 7.11(iv) we get $[x, y] \in \mathfrak{h}$, i.e., $\mathfrak{h}$ is a Lie ideal.

(iii) If $x \in \mathfrak{g}$, $y \in \mathfrak{h}$ are small then

$$\exp(x)\exp(y)\exp(x)^{-1} =$$

$$\exp(\mathrm{Ad}_{\exp(x)}y) = \exp(\exp(\mathrm{ad}x)y) = \exp(\sum_{n=0}^{\infty} \tfrac{(\mathrm{ad}x)^n y}{n!}) \in H$$

since $\sum_{n=0}^{\infty}\frac{(\mathrm{ad}x)^n y}{n!}\in\mathfrak{h}$. So G acting on itself by conjugation maps a small neighborhood of 1 in H into H (as G is generated by its neighborhood of 1 by Proposition 3.15, since it is connected). But H is also connected, so is generated by its neighborhood of 1, again by Proposition 3.15. Hence H is normal. $\square$

8.4. **The Lie algebra of vector fields.** Recall that a vector field on a manifold X is a compatible family of derivations $\mathbf{v}: O(U)\to O(U)$ for open subsets $U\subset X$.

Proposition 8.10. *If $\mathbf{v},\mathbf{w}$ are derivations of an algebra A then so is $[\mathbf{v},\mathbf{w}] := \mathbf{v}\mathbf{w}-\mathbf{w}\mathbf{v}$.*

Proof. We have

$$(\mathbf{vw}-\mathbf{wv})(ab)=\mathbf{v}(\mathbf{w}(a)b+a\mathbf{w}(b))-\mathbf{w}(\mathbf{v}(a)b+a\mathbf{v}(b))=$$

$$\mathbf{vw}(a)b+\mathbf{w}(a)\mathbf{v}(b)+\mathbf{v}(a)\mathbf{w}(b)+a\mathbf{vw}(b)$$

$$-\mathbf{wv}(a)b-\mathbf{v}(a)\mathbf{w}(b)-\mathbf{w}(a)\mathbf{v}(b)-a\mathbf{wv}(b)=$$

$$(\mathbf{vw}-\mathbf{wv})(a)b+a(\mathbf{vw}-\mathbf{wv})(b).$$

$\square$

Thus, the space $\mathrm{Vect}(X)$ of vector fields on X is a Lie algebra under the operation

$$\mathbf{v},\mathbf{w}\mapsto[\mathbf{v},\mathbf{w}],$$

called the **Lie bracket of vector fields**.[7]

In local coordinates we have

$$\mathbf{v}=\sum_i v_i\frac{\partial}{\partial x_i},\ \ \mathbf{w}=\sum w_j\frac{\partial}{\partial x_j},$$

so

$$[\mathbf{v},\mathbf{w}]=\sum_i\left(\sum_j(v_j\tfrac{\partial w_i}{\partial x_j}-w_j\tfrac{\partial v_i}{\partial x_j})\right)\tfrac{\partial}{\partial x_i}.$$

This implies that if vector fields $\mathbf{v},\mathbf{w}$ are tangent to a k-dimensional submanifold $Y\subset X$ then so is their Lie bracket $[\mathbf{v},\mathbf{w}]$. Indeed, in local coordinates Y is given by equations $x_{k+1}=...=x_n=0$, and in such coordinates a vector field is tangent to Y iff it does not contain terms with $\frac{\partial}{\partial x_j}$ for $j>k$.

[7]Note that this Lie algebra is infinite dimensional for all real manifolds and many (but not all) complex manifolds of positive dimension.

Exercise 8.11. Let $U \subset \mathbb{R}^n$ be an open subset, $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(U)$ and g_t, h_t be the associated flows, defined in a neighborhood of every point of U for small t. Show that for any $\mathbf{x} \in U$

$$\lim_{t,s\to 0} \frac{g_t h_s g_t^{-1} h_s^{-1}(\mathbf{x}) - \mathbf{x}}{ts} = [\mathbf{v}, \mathbf{w}](\mathbf{x}).$$

Now let G be a Lie group and $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ be the subspaces of left and right invariant vector fields.

Proposition 8.12. $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ *are Lie subalgebras which are both canonically isomorphic to* $\mathfrak{g} = \mathrm{Lie}G$*.*

Proof. The first statement is obvious, so we prove only the second statement. Let $\mathbf{x}, \mathbf{y} \in \mathrm{Vect}_L(G)$. Then $\mathbf{x} = \mathbf{L}_x$, $\mathbf{y} = \mathbf{L}_y$ for $x = \mathbf{x}(1), y = \mathbf{y}(1) \in \mathfrak{g}$, where $\mathbf{L}_z$ denotes the vector field on G obtained by left translations of $z \in \mathfrak{g}$. Then $[\mathbf{L}_x, \mathbf{L}_y] = \mathbf{L}_z$, where $z = [\mathbf{L}_x, \mathbf{L}_y](1)$. So let us compute z.

Let f be a regular function on a neighborhood of $1 \in G$. We have shown that for $u \in \mathfrak{g}$

$$(\mathbf{L}_u f)(g) = \tfrac{d}{dt}|_{t=0} f(g \exp(tu)).$$

Thus,

$$z(f) = x(\mathbf{L}_y f) - y(\mathbf{L}_x f) = x(\tfrac{\partial}{\partial s}|_{s=0} f(\bullet \exp(sy))) - y(\tfrac{\partial}{\partial t}|_{t=0} f(\bullet \exp(tx))) =$$
$$\tfrac{\partial}{\partial t}|_{t=0} \tfrac{\partial}{\partial s}|_{s=0} f(\exp(tx)\exp(sy)) - \tfrac{\partial}{\partial s}|_{s=0} \tfrac{\partial}{\partial t}|_{t=0} f(\exp(sy)\exp(tx)) =$$
$$\tfrac{\partial^2}{\partial t \partial s}|_{t=s=0} (F(tx + sy + \tfrac{1}{2} ts[x,y] + ...) - F(tx + sy - \tfrac{1}{2} ts[x,y] + ...)),$$

where $F(u) := f(\exp(u))$. It is easy to see by using Taylor expansion that this expression equals to $[x,y](f)$. Thus $z = [x,y]$, i.e., the map $\mathfrak{g} \to \mathrm{Vect}_L(G)$ given by $x \mapsto \mathbf{L}_x$ is a Lie algebra isomorphism. Similarly, the map $\mathfrak{g} \to \mathrm{Vect}_R(G)$ given by $x \mapsto -\mathbf{R}_x$ is a Lie algebra isomorphism, as claimed. $\square$