7. **The exponential map of a Lie group**

7.1. **The exponential map.** We will now generalize the exponential and logarithm maps from matrix groups to arbitrary Lie groups.

Let G be a real Lie group, $\mathfrak{g} = T_1G$.

Proposition 7.1. *Let $x \in \mathfrak{g}$. There is a unique morphism of Lie groups $\gamma = \gamma_x : \mathbb{R} \to G$ such that $\gamma'(0) = x$.*

Proof. For such a morphism we should have

$$\gamma(t + s) = \gamma(t)\gamma(s),\ \ t, s \in \mathbb{R},$$

so differentiating by s at $s = 0$, we get[6]

$$\gamma'(t) = \gamma(t)x.$$

Thus $\gamma(t)$ is a solution of the ODE defined by the left-invariant vector field $\mathbf{L}_x$ corresponding to $x \in \mathfrak{g}$ with initial condition $\gamma(0) = 1$. By the existence and uniqueness theorem for solutions of ODE, this equation has a unique solution with this initial condition defined for $|t| < \varepsilon$ for some $\varepsilon > 0$. Moreover, if $|s| + |t| < \varepsilon$, both $\gamma_1(t) := \gamma(s + t)$ and $\gamma_2(t) := \gamma(s)\gamma(t)$ satisfy this differential equation with initial condition $\gamma_1(0) = \gamma_2(0) = \gamma(s)$, so $\gamma_1 = \gamma_2$. Thus

$$\gamma(s + t) = \gamma(s)\gamma(t),\ \ |s| + |t| < \varepsilon;$$

hence $\gamma(t)x = x\gamma(t)$ for $|t| < \varepsilon$.

We claim that the solution $\gamma(t)$ extends to all values of $t \in \mathbb{R}$. Indeed, let us prove that it extends to $|t| < 2^n\varepsilon$ for all $n \geq 0$ by induction in n. The base of induction ($n = 0$) is already known, so we only need to justify the induction step from $n - 1$ to n. Given t with $|t| < 2^n\varepsilon$, we define

$$\gamma(t) := \gamma(\tfrac{t}{2})^2.$$

This agrees with the previously defined solution for $|t| < 2^{n-1}\varepsilon$, and we have

$$\gamma'(t) = \tfrac{1}{2}(\gamma'(\tfrac{t}{2})\gamma(\tfrac{t}{2})+\gamma(\tfrac{t}{2})\gamma'(\tfrac{t}{2})) = \tfrac{1}{2}\gamma(\tfrac{t}{2})x\gamma(\tfrac{t}{2})+\tfrac{1}{2}\gamma(\tfrac{t}{2})^2x = \gamma(\tfrac{t}{2})^2x = \gamma(t)x,$$

as desired.

Thus, we have a regular map $\gamma : \mathbb{R} \to G$ with $\gamma(s+t) = \gamma(s)\gamma(t)$ and $\gamma'(0) = x$, which is unique by the uniqueness of solutions of ODE. □

Definition 7.2. The **exponential map** $\exp : \mathfrak{g} \to G$ is defined by the formula $\exp(x) = \gamma_x(1)$.

Thus $\gamma_x(t) = \exp(tx)$. So we have

[6] For brevity for $g \in G$, $x \in \mathfrak{g}$ we denote L_gx by gx and R_gx by xg.

Proposition 7.3. *The flow defined by the right-invariant vector field* $\mathbf{R}_x$ *is given by* $g \mapsto \exp(tx)g$*, and the flow defined by the left-invariant vector field* $\mathbf{L}_x$ *is given by* $g \mapsto g\exp(tx)$.

Example 7.4. 1. Let $G = \mathbb{K}^n$. Then $\exp(x) = x$.

2. Let $G = GL_n(\mathbb{K})$ or its Lie subgroup. Then $\gamma_x(t)$ satisfies the matrix differential equation

$$\gamma'(t) = \gamma(t)x$$

with $\gamma(0) = 1$, so

$$\gamma_x(t) = e^{tx},$$

the matrix exponential. For example, if $n = 1$, this is the usual exponential function.

The following theorem describes the basic properties of the exponential map. Let G be a real or complex Lie group.

Theorem 7.5. *(i)* $\exp : \mathfrak{g} \to G$ *is a regular map which is a diffeomorphism of a neighborhood of* $0 \in \mathfrak{g}$ *onto a neighborhood of* $1 \in G$*, with* $\exp(0) = 1$, $\exp'(0) = \mathrm{Id}_{\mathfrak{g}}$.

(ii) $\exp((s+t)x) = \exp(sx)\exp(tx)$ *for* $x \in \mathfrak{g}$, $s, t \in \mathbb{K}$.

(iii) For any morphism of Lie groups $\phi : G \to K$ *and* $x \in T_1G$ *we have*

$$\phi(\exp(x)) = \exp(\phi_* x);$$

i.e., the exponential map commutes with morphisms.

(iv) For any $g \in G$, $x \in \mathfrak{g}$*, we have*

$$g\exp(x)g^{-1} = \exp(\mathrm{Ad}_g x).$$

Proof. (i) The regularity of exp follows from the fact that if a differential equation depends regularly on parameters then so do its solutions. Also $\gamma_0(t) = 1$ so $\exp(0) = 1$. We have $\exp'(0)x = \frac{d}{dt}\exp(tx)|_{t=0} = x$, so $\exp'(0) = \mathrm{Id}$. By the inverse function theorem this implies that exp is a diffeomorphism near the origin.

(ii) Holds since $\exp(tx) = \gamma_x(t)$.

(iii) Both $\phi(\exp(tx))$ and $\exp(\phi_*(tx))$ satisfy the equation $\gamma'(t) = \gamma(t)\phi_*(x)$ with the same initial conditions.

(iv) is a special case of (iii) with $\phi : G \to G$, $\phi(h) = ghg^{-1}$. □

Thus exp has an inverse $\log : U \to \mathfrak{g}$ defined on a neighborhood U of $1 \in G$ with $\log(1) = 0$. This map is called the **logarithm**. For $GL_n(\mathbb{K})$ and its Lie subgroups it coincides with the matrix logarithm. The logarithm map defines a canonical coordinate chart on G near 1, so a choice of a basis of $\mathfrak{g}$ gives a local coordinate system.

Proposition 7.6. *Let G be a connected Lie group and $\phi : G \to K$ a morphism of Lie groups. Then ϕ is completely determined by the linear map $\phi_* : T_1G \to T_1K$.*

Proof. We have $\phi(\exp(x)) = \exp(\phi_*(x))$, so since exp is a diffeomorphism near 0, ϕ is determined by ϕ_* on a neighborhood of $1 \in G$. This completely determines ϕ since this neighborhood generates G by Proposition 3.15. □

Exercise 7.7. (i) Show that a connected compact complex Lie group is abelian. (**Hint:** consider the adjoint representation and use that a holomorphic function on a compact complex manifold is constant, by the maximum principle.)

(ii) Classify such Lie groups of dimension n up to isomorphism (Show that they are compact complex tori whose isomorphism classes are bijectively labeled by elements of the set $GL_n(\mathbb{C})\backslash GL_{2n}(\mathbb{R})/GL_{2n}(\mathbb{Z})$.)

(iii) Work out the classification explicitly in the 1-dimensional case (this is the classification of complex elliptic curves). Namely, show that isomorphism classes are labeled by points of $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half-plane and $\Gamma = SL_2(\mathbb{Z})$ acting on $\mathbb{H}$ by Möbius transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ (where $\mathrm{Im}(\tau) > 0$).

7.2. **The commutator.** In general (say, for $G = GL_n(\mathbb{K})$, $n \geq 2$), $\exp(x + y) \neq \exp(x)\exp(y)$. So let us consider the map

$$(x, y) \mapsto \mu(x, y) = \log(\exp(x)\exp(y))$$

which maps $U \times U \to \mathfrak{g}$, where $U \subset \mathfrak{g}$ is a neighborhood of 0. This map expresses the product in G in the coordinate chart coming from the logarithm map. We have $\mu(x, 0) = \mu(0, x) = x$ and $\mu_*(x, y) = x+y$. So, since μ is regular, we have the second Taylor approximation

$$\mu(x, y) = x + y + \tfrac{1}{2}\mu_2(x, y) + ...$$

where $\mu_2 = d^2\mu_{(0,0)}$ is the quadratic part and ... are higher terms. Moreover, $\mu_2(x, 0) = \mu_2(0, y) = 0$, hence μ_2 is a bilinear map $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$. It is easy to see that $\mu(x, -x) = 0$, hence μ_2 is skew-symmetric.

Definition 7.8. The map μ_2 is called the **commutator** and denoted by $x, y \mapsto [x, y]$.

Thus we have

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x, y] + ...). \tag{7.1}$$

Example 7.9. Let $G = GL_n(\mathbb{K})$. Then

$\exp(x)\exp(y) = (1+x+\frac{x^2}{2}+...)(1+y+\frac{y^2}{2}+...) = 1+x+y+\frac{x^2}{2}+xy+\frac{y^2}{2}+... =$

$$1+(x+y)+\frac{(x+y)^2}{2}+\frac{xy-yx}{2}+...=\exp(x+y+\frac{xy-yx}{2}+...)$$

Thus

$$[x,y]=xy-yx.$$

This justifies the term "commutator": it measures the failure of x and y to commute.

Corollary 7.10. *If $G\subset GL_n(\mathbb{K})$ is a Lie subgroup then $\mathfrak{g}=T_1G\subset \mathfrak{gl}_n(\mathbb{K})$ is closed under the commutator $[x,y]=xy-yx$, which coincides with the commutator of G.*

For $x\in\mathfrak{g}$ define the linear map $\mathrm{ad}x:\mathfrak{g}\to\mathfrak{g}$ by

$$\mathrm{ad}x(y)=[x,y].$$

Proposition 7.11. *(i) Let G,K be Lie groups and $\phi:G\to K$ a morphism of Lie groups. Then $\phi_*:T_1G\to T_1K$ preserves the commutator:*

$$\phi_*([x,y])=[\phi_*(x),\phi_*(y)].$$

(ii) The adjoint action preserves the commutator.

(iii) We have

$$\exp(x)\exp(y)\exp(x)^{-1}\exp(y)^{-1}=\exp([x,y]+...)$$

where ... denotes cubic and higher terms.

(iv) Let $X(t),Y(s)$ be parametrized curves on G such that $X(0)=Y(0)=1$, $X'(0)=x,Y'(0)=y$. Then we have

$$[x,y]=\lim_{s,t\to 0}\frac{\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1})}{ts}.$$

In particular,

$$[x,y]=\lim_{s,t\to 0}\frac{\log(\exp(tx)\exp(sy)\exp(tx)^{-1}\exp(sy)^{-1})}{ts}$$

and

$$[x,y]=\frac{d}{dt}|_{t=0}\mathrm{Ad}_{X(t)}(y).$$

Thus $\mathrm{ad}=\mathrm{Ad}_*$*, the differential of* Ad *at $1\in G$.*

(v) If G is commutative (=abelian) then $[x,y]=0$ for all x,y.

Proof. (i) Follows since ϕ commutes with the exponential map.

(ii) Follows from (i) by setting $\phi=\mathrm{Ad}_g$.

(iii) By (7.1), modulo cubic and higher terms we have

$$\log(\exp(x)\exp(y))=\log(\exp(y)\exp(x))+[x,y]+...,$$

which implies the statement by exponentiation.

(iv) Let $\log X(t)=x(t),\ \log Y(s)=y(s)$. Then by (iii) we have

$$\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1})=$$

$$\log(\exp(x(t))\exp(y(s))\exp(x(t))^{-1}\exp(y(s))^{-1}) = ts([x,y]+o(1)),\ t,s \to 0.$$

This implies the first two statements. The last statement follows by taking the limit in s first, then in t.

(v) follows from (iii). □

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