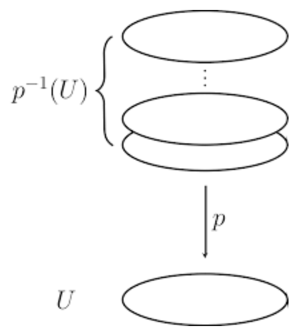
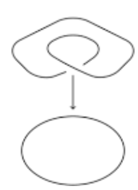
3. **Lie groups, II**

3.1. **A crash course on coverings.** Now we need to review some more topology. Let X, Y be Hausdorff topological spaces, and $p : Y \to X$ a continuous map. Then p is called a **covering** if every point $x \in X$ has a neighborhood U such that $p^{-1}(U)$ is a union of disjoint open sets (called **sheets** of the covering) each of which is mapped homeomorphically onto U by p:



In other words, there exists a homeomorphism $h : U \times F \to p^{-1}(U)$ for some discrete space F with $(p \circ h)(u, f) = u$ for all $u \in U$, $f \in F$. I.e., informally speaking, a covering is a map that locally on X looks like the projection $X \times F \to X$ for some discrete F.

We will consider only coverings with countable fibers, and just call them coverings. It is clear that a covering of a manifold (C^k, real or complex analytic) is a manifold of the same type, and the covering map is regular.



Two paths $x_0, x_1 : [0, 1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are said to be **homotopic** if there is a continuous map

$$x : [0, 1] \times [0, 1] \to X,$$

called a **homotopy** between x_0 and x_1, such that $x(t, 0) = x_0(t)$ and $x(t, 1) = x_1(t)$, $x(0, s) = P, x(1, s) = Q$. See a movie here:
`https://commons.wikimedia.org/wiki/File:Homotopy.gif#/media/File:HomotopySmall.gif`

For example, if $x(t)$ is a path and $g : [0, 1] \to [0, 1]$ is a change of parameter with $g(0) = 0$, $g(1) = 1$ then the paths $x_1(t) = x(t)$ and $x_2(t) = x(g(t))$ are clearly homotopic.

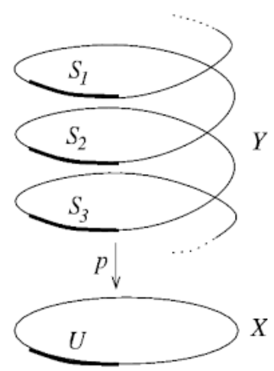
A path-connected Hausdorff space X is said to be **simply connected** if for any $P, Q \in X$, any paths $x_0, x_1 : [0,1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are homotopic.

Example 3.1. S^1 is not simply connected but S^n is simply connected for $n \geq 2$.

It is easy to show that any covering has a **homotopy lifting property**: if $b \in X$ and $\widetilde{b} \in p^{-1}(b) \subset Y$ then any path γ starting at b admits a unique lift to a path $\widetilde{\gamma}$ starting at $\widetilde{b}$, i.e., $p(\widetilde{\gamma}) = \gamma$. Moreover, if γ_1, γ_2 are homotopic paths on X then $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are homotopic on Y (in particular, have the same endpoint). Thus, if Z is a simply connected space with a point z then any continuous map $f : Z \to X$ with $f(z) = b$ lifts to a unique continuous map $\widetilde{f} : Z \to Y$ satisfying $\widetilde{f}(z) = \widetilde{b}$; i.e., $p \circ \widetilde{f} = f$. Namely, to compute $\widetilde{f}(w)$, pick a path β from z to w, let $\gamma = f(\beta)$ and consider the path $\widetilde{\gamma}$. Then the endpoint of $\widetilde{\gamma}$ is $\widetilde{f}(w)$, and it does not depend on the choice of β.

If Z, X are manifolds (of any regularity type), Z is simply connected, and $f : Z \to X$ is a regular map then the lift $\widetilde{f} : Z \to Y$ is also regular. Indeed, if we introduce local coordinates on Y using the homeomorphism between sheets of the covering and their images then $\widetilde{f}$ and f will be locally expressed by the same functions.

A covering $p : Y \to X$ of a path-connected space X is called **universal** if Y is simply connected.



If X is a sufficiently nice space, e.g., a manifold, its universal covering can be constructed as follows. Fix $b \in X$ and let $\widetilde{X}_b$ be the set of homotopy classes of paths on X starting at b. We have a natural map $p : \widetilde{X}_b \to X$, $p(\gamma) = \gamma(1)$. If $U \subset X$ is a small ball around a point $x \in X$ then U is simply connected, so we have a natural identification $h : U \times F \to p^{-1}(U)$ with $(p \circ h)(u, f) = u$, where $F = p^{-1}(x)$ is the set of homotopy classes of paths from b to x; namely, $h(u, f)$ is the concatenation of f with any path connecting x with u inside U.

Here the **concatenation** $\gamma_1 \circ \gamma_2$ of paths $\gamma_1, \gamma_2 : [0,1] \to X$ with $\gamma_2(1) = \gamma_1(0)$ is the path $\gamma = \gamma_1 \circ \gamma_2 : [0,1] \to X$ such that $\gamma(t) = \gamma_2(2t)$ for $t \le 1/2$ and $\gamma(t) = \gamma_1(2t-1)$ for $t \ge 1/2$.

The topologies on all such $p^{-1}(U)$ induced by these identifications glue together into a topology on $\widetilde{X}_b$, and the map $p : \widetilde{X}_b \to X$ is then a covering. Moreover, the homotopy lifting property implies that $\widetilde{X}_b$ is simply connected, so this covering is universal.

It is easy to see that a universal covering $p : Y \to X$ covers any path-connected covering $p' : Y' \to X$, i.e., there is a covering $q : Y \to Y'$ such that $p = p' \circ q$; this is why it is called universal. Therefore a universal covering is unique up to an isomorphism (indeed, if Y, Y' are universal then we have coverings $q_1 : Y \to Y'$ and $q_2 : Y' \to Y$ and $q_1 \circ q_2 = q_2 \circ q_1 = \mathrm{Id}$).

Example 3.2. 1. The map $z \mapsto z^n$ defines an n-sheeted covering $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering $\mathbb{R} \to S^1$.

Now denote by $\pi_1(X, x)$ the set of homotopy classes of *closed* paths on a path-connected space X, starting and ending at x. Then $\pi_1(X, x)$ is a group under concatenation of paths (concatenation is associative since the paths $a(bc)$ and $(ab)c$ differ only by parametrization and are hence homotopic). This group is called the **fundamental group** of X relative to the point x. It acts on the fiber $p^{-1}(x)$ for every covering $p : Y \to X$ (by lifting $\gamma \in \pi_1(X, x)$ to Y), which is called the action by **deck transformations**. This action is transitive iff Y is path-connected and moreover free iff Y is universal.

Finally, the group $\pi_1(X, x)$ does not depend on x up to an isomorphism. More precisely, conjugation by any path from x_1 to x_2 defines an isomorphism $\pi_1(X, x_1) \to \pi_1(X, x_2)$ (although two non-homotopic paths may define different isomorphisms if π_1 is non-abelian).

Example 3.3. 1. $\pi_1(S^1) = \mathbb{Z}$.

2. $\pi_1(\mathbb{C} \setminus \{z_1, ..., z_n\}) = F_n$ is a free group in n generators.

3. We have a 2-sheeted universal covering $S^n \to \mathbb{RP}^n$ (real projective space) for $n \ge 2$. Thus $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ for $n \ge 2$.

Exercise 3.4. Make sure you can fill all the details in this subsection!

3.2. **Coverings of Lie groups.** Let G be a connected (real or complex) Lie group and $\widetilde{G} = \widetilde{G}_1$ be the universal covering of G, consisting of homotopy classes of paths $x : [0,1] \to G$ with $x(0) = 1$. Then $\widetilde{G}$ is a group via $(x \cdot y)(t) = x(t)y(t)$, and also a manifold.

Proposition 3.5. *(i) $\widetilde{G}$ is a simply connected Lie group. The covering $p : \widetilde{G} \to G$ is a homomorphism of Lie groups.*

(ii) $\mathrm{Ker}(p)$ *is a central subgroup of $\widetilde{G}$ naturally isomorphic to $\pi_1(G) = \pi_1(G,1)$. Thus, $\widetilde{G}$ is a central extension of G by $\pi_1(G)$. In particular, $\pi_1(G)$ is abelian.*

Proof. We will only prove (i). We only need to show that $\widetilde{G}$ is a Lie group, i.e., that the multiplication map $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is regular. But $\widetilde{G} \times \widetilde{G}$ is simply connected, and $\widetilde{m}$ is a lifting of the map

$$m' := m \circ (p \times p) : \widetilde{G} \times \widetilde{G} \to G \times G \to G,$$

so it is regular. In other words, $\widetilde{m}$ is regular since in local coordinates it is defined by the same functions as m. $\square$

Exercise 3.6. Prove Proposition 3.5(ii).

Remark 3.7. The same argument shows that more generally, the fundamental group of any path-connected topological group is abelian.

Example 3.8. 1. The map $z \mapsto z^n$ defines an n-sheeted covering of Lie groups $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering of Lie groups $\mathbb{R} \to S^1$.

Exercise 3.9. Consider the action of $SU(2)$ on the 3-dimensional real vector space of traceless Hermitian 2-by-2 matrices by conjugation.

(i) Show that this action preserves the positive inner product $(A, B) = \mathrm{Tr}(AB)$ and has determinant 1. Deduce that it defines a homomorphism $\phi : SU(2) \to SO(3)$.

(ii) Show that ϕ is surjective, with kernel ± 1, and is a universal covering map (use that $SU(2) = S^3$ is simply connected). Deduce that $\pi_1(SO(3)) = \mathbb{Z}/2$ and that $SO(3) \cong \mathbb{RP}^3$ as a manifold.

This is demonstrated by the famous **Dirac belt trick**, which illustrates the notion of a **spinor**; namely, spinors are vectors in $\mathbb{C}^2$ acted upon by matrices from $SU(2)$. Here are some videos of the belt trick:
`https://www.youtube.com/watch?v=17Q0tJZcsnY`
`https://www.youtube.com/watch?v=Vfh21o-JW9Q`

3.3. **Closed Lie subgroups.**

Definition 3.10. A **closed Lie subgroup** of a (real or complex) Lie group G is a subgroup which is also an embedded submanifold.

This terminology is justified by the following lemma.

Lemma 3.11. *A closed Lie subgroup of G is closed in G.*

Exercise 3.12. Prove Lemma 3.11.

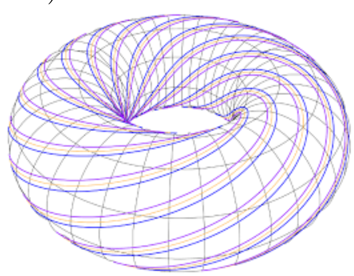
We also have

Theorem 3.13. *Any closed subgroup of a real Lie group G is a closed Lie subgroup.*

This theorem is rather nontrivial, and we will not prove it at this time (it will be proved much later in Exercise 36.13), but we will soon prove a weaker version which suffices for our purposes.

Example 3.14. 1. $SL_n(\mathbb{K})$ is a closed Lie subgroup of $GL_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Indeed, the equation $\det A = 1$ defines a smooth hypersurface in the space of matrices (show it!).

2. Let $\phi : \mathbb{R} \to S^1 \times S^1$ be the irrational torus winding given by the formula $\phi(x) = (e^{ix}, e^{ix\sqrt{2}})$:



Then $\phi(\mathbb{R})$ is a subgroup of $S^1 \times S^1$ but not a closed Lie subgroup, since it is not an embedded submanifold: although ϕ is an immersion, the map $\phi^{-1} : \phi(\mathbb{R}) \to \mathbb{R}$ is not continuous.

3.4. **Generation of connected Lie groups by a neighborhood of the identity.**

Proposition 3.15. *(i) If G is a connected Lie group and U a neighborhood of 1 in G then U generates G.*

(ii) If $f : G \to K$ is a homomorphism of Lie groups, K is connected, and $df_1 : T_1G \to T_1K$ is surjective, then f is surjective.

Proof. (i) Let H be the subgroup of G generated by U. Then H is open in G since $H = \cup_{h\in H} hU$. Thus H is an embedded submanifold of G, hence a closed Lie subgroup. Thus by Lemma 3.11 $H \subset G$ is closed. So $H = G$ since G is connected.

(ii) Since df_1 is surjective, by the implicit function theorem $f(G)$ contains some neighborhood of 1 in K. Thus it contains the whole K by (i). $\square$

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