Let $x \in \mathcal{X}^n$, $x = (x_1, \ldots, x_n)$. Suppose $A \subseteq \mathcal{X}^n$. Define

$$V(A,x) = \{(I(x_1 \neq y_1), \ldots, I(x_n \neq y_n)) : y = (y_1, \ldots, y_n) \in A\},$$

$$U(A,x) = \text{conv } V(A,x)$$

and

$$d(A,x) = \min\{|s|^2 = \sum_{i=1}^n s_i^2,\ s \in U(A,x)\}$$

In the previous lectures, we proved

Theorem 32.1.

$$\mathbb{P}\left(d(A,x) \geq t\right) \leq \frac{1}{\mathbb{P}\left(A\right)} e^{-t/4}.$$

Today, we prove

Theorem 32.2. *The following are equivalent:*

(1) $d(A,x) \leq t$
(2) $\forall \alpha = (\alpha_1, \ldots, \alpha_n), \exists y \in A,\ s.t.\ \sum_{i=1}^n \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t}$

Proof. **(1)⇒(2):**

Choose any $\alpha = (\alpha_1, \ldots, \alpha_n)$.

$$\min_{y \in A} \sum_{i=1}^n \alpha_i I(x_i \neq y_i) = \min_{s \in U(A,x)} \sum_{i=1}^n \alpha_i s_i \leq \sum_{i=1}^n \alpha_i s_i^0 \tag{1}$$

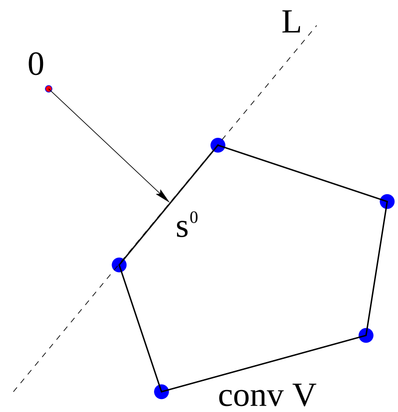
$$\leq \sqrt{\sum_{i=1}^n \alpha_i^2} \sqrt{\sum_{i=1}^n (s_i^0)^2} \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t} \tag{2}$$

where in the last inequality we used assumption (1). In the above, min is achieved at s^0.

(2)⇒(1):

Let $\alpha = (s_1^0, \ldots, s_n^0)$. There exists $y \in A$ such that

$$\sum_{i=1}^n \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum_{i=1}^n \alpha_i^2 \cdot t}$$



Note that $\sum \alpha_i s_i^0$ is constant on L because s^0 is perpendicular to the face.

$$\sum \alpha_i s_i^0 \leq \sum \alpha_i I(x_i \neq y_i) \leq \sqrt{\sum \alpha_i^2 t}$$

Hence, $\sum (s_i^0)^2 \leq \sqrt{\sum (s_i^0)^2 t}$ and $\sqrt{\sum (s_i^0)^2} \leq \sqrt{t}$. Therefore, $d(A,x) \leq \sum (s_i^0)^2 \leq t$. $\square$

We now turn to an application of the above results: Bin Packing.

Example 1. *Assume we have* $x_1, \ldots, x_n$, $0 \leq x_i \leq 1$, *and let* $B(x_1, \ldots, x_n)$ *be the smallest number of bins of size* 1 *needed to pack all* $(x_1, \ldots, x_n)$. *Let* $S_1, \ldots, S_B \subseteq \{1, \ldots, n\}$ *such that all* x_i *with* $i \in S_k$ *are packed into one bin,* $\bigcup S_k = \{1, \ldots, n\}$, $\sum_{i \in S_k} x_i \leq 1$.

Lemma 32.1. $B(x_1, \ldots, x_n) \leq 2 \sum x_i + 1$.

Proof. For all but one k, $\frac{1}{2} \leq \sum_{i \in S_k} x_i$. Otherwise we can combine two bins into one. Hence, $B - 1 \leq 2 \sum_k \sum_{i \in S_k} x_i = 2 \sum x_i$ $\square$

Theorem 32.3.

$$\mathbb{P}\left(B(x_1, \ldots, x_n) \leq M + 2\sqrt{\sum x_i^2 \cdot t} + 1\right) \geq 1 - 2e^{-t/4}.$$

Proof. Let $A = \{y : B(y_1, \ldots, y_n) \leq M\}$, where $\mathbb{P}(B \geq M) \geq 1/2$, $\mathbb{P}(B \leq M) \geq 1/2$. We proved that

$$\mathbb{P}(d(A,x) \geq t) \leq \frac{1}{\mathbb{P}(A)} e^{-t/4}.$$

Take x such that $d(A,x) \leq t$. Take $\alpha = (x_1, \ldots, x_n)$. Since $d(A,x) \leq t$, there exists $y \in A$ such that $\sum x_i I(x_i \neq y_i) \leq \sqrt{\sum x_i^2 \cdot t}$.

To pack the set $\{i : x_i = y_i\}$ we need $\leq B(y_1, \ldots, y_n) \leq M$ bins.

To pack $\{i : x_i \neq y_i\}$:

$$B(x_1 I(x_1 \neq y_1), \ldots, x_n I(x_n \neq y_n)) \leq 2 \sum x_i I(x_i \neq y_i) + 1$$
$$\leq 2\sqrt{\sum x_i^2 \cdot t} + 1$$

by Lemma.

Hence,

$$B(x_1, \ldots, x_n) \leq M + 2\sqrt{\sum x_i^2 \cdot t} + 1$$

with probability at least $1 - 2e^{-t/4}$.

By Bernstein's inequality we get

$$\mathbb{P}\left(\sum x_i^2 \leq n\mathbb{E}x_1^2 + \sqrt{n\mathbb{E}x_1^2 \cdot t} + \frac{2}{3}t\right) \geq 1 - e^{-t}.$$

Hence,

$$B(x_1, \ldots, x_n) \lesssim M + 2\sqrt{n\mathbb{E}x_1^2 \cdot t}$$

$\square$