18.408 Topics in Theoretical Computer Science Fall 2022
Problem Set 5

Not for submission

1. In this problem we will show that for every $\varepsilon, \delta > 0$ the problem gap-3-SAT$[1-\varepsilon, \frac{7}{8}+\delta]$ is NP-hard.

   (a) Let Σ_L, Σ_R be finite sets and suppose that $\pi\colon \Sigma_L \to \Sigma_R$ is a projection map. Consider the distribution μ over (x,y,z) where $x, y \in \{0,1\}^{\Sigma_L}$ and $z \in \{0,1\}^{\Sigma_R}$ sampled in the following way:

   - Sample $z \in \{0,1\}^{\Sigma_R}$ and $x \in \{0,1\}^{\Sigma_L}$ uniformly.
   - Let $y' = \pi_{u,v}^{-1}(1-z)$ and sample $g \in \{0,1\}^{\Sigma_L}$ be sampled by taking for each $i \in \Sigma_L$ independently $g_i = 1$ with probability ε and otherwise $g_i = 0$.
   - Take $y = x + y' + g$.

   Show that the marginal distribution of each one of x, y, z is uniform over its respective domain, and also that the marginal distribution over each one of (x,y), (y,z) and (x,z) is uniform over its respective domain.

   (b) Let $f\colon \{0,1\}^{\Sigma_L} \to \{0,1\}$ and $g\colon \{0,1\}^{\Sigma_R} \to \{0,1\}$ be dictatorship functions, that is, $f(x) = x_i$ and $g(z) = z_j$ where the dictators satisfy the projection constraint $\pi(i) = j$. Show that

   $$\mathop{\mathbb{E}}_{(x,y,z)\sim\mu}\left[f(x) \vee f(y) \vee g(z)\right] \geqslant 1 - \varepsilon.$$

   (c) Let $f\colon \{0,1\}^{\Sigma_L} \to \{0,1\}$ and $g\colon \{0,1\}^{\Sigma_R} \to \{0,1\}$ be functions such that $\mathbb{E}[f] = \mathbb{E}[g] = 1/2$, and denote $F(x) = (-1)^{f(x)}$ and $G(z) = (-1)^{g(z)}$. Show that

   $$\mathop{\mathbb{E}}_{(x,y,z)\sim\mu}\left[f(x) \vee f(y) \vee g(z)\right] - \frac{7}{8} \leqslant \sum_{\alpha \in \mathbb{F}_2^{\Sigma_L}} (1-\varepsilon)^{|\alpha|} \widehat{F}(\alpha)^2 \ \widehat{G}(\pi_{\mathsf{odd}}^{-1}(\alpha)) \ .$$

   (d) Show that for every $\varepsilon, \delta > 0$ there is $\eta > 0$ such that there is a polynomial time reduction from gap-LabelCover$[1, \eta]$ over an alphabet of constant size to gap-3SAT$[1-\varepsilon, \frac{7}{8}+\delta]$. Deduce that gap-3SAT$[1-\varepsilon, \frac{7}{8}+\delta]$ is NP-hard.

2. An instance of the problem 2-Lin$_{\mathbb{F}_2}$ consists of a set of variables X and a set of equations E, each equation of the form $x - y = b$ where $x, y \in X$ are variables and $b \in \mathbb{F}_2$ is a constant. In this problem, we will design a surprisingly good approximation algorithm for 2-Lin, showing that for sufficiently small $\varepsilon > 0$, given an instance which is at least $1-\varepsilon$ satisfiable, finds (in polynomial time) a solution satisfying at least $1 - \Theta(\sqrt{\varepsilon})$ fraction of the equations. For simplicity, we assume that half of the equations have $b = 0$, and half of them have $b = 1$.

   (a) Write down an integer program formulation of the Max-2-Lin$_{\mathbb{F}_2}$.

(b) Write down a semi-definite program relaxation of Max-2-Lin$_{\mathbb{F}_2}$ in the variables $\{V_x\}_{x\in X}$, and argue that if (X,E) is at least $1-\varepsilon$ satisfiable then there is a vector valued solution satisfying that

$$\sum_{e\in E: x(e)-y(e)=0} \langle V_{x(e)}, V_{y(e)}\rangle - \sum_{e\in E: x(e)-y(e)=1} \langle V_{x(e)}, V_{y(e)}\rangle \geqslant m(1-\varepsilon).$$

Conclude that

$$\sum_{e\in E: x(e)-y(e)=0} \langle V_{x(e)}, V_{y(e)}\rangle \geqslant \frac{m}{2}(1-2\varepsilon), \qquad -\sum_{e\in E: x(e)-y(e)=1} \langle V_{x(e)}, V_{y(e)}\rangle \geqslant \frac{m}{2}(1-2\varepsilon).$$

(c) Using the fact that such vector valued solution may be found efficiently, design a rounding procedure which produces an integral solution to (X,E). Namely, sampling a vector $h\in\mathbb{R}^n$ and defining the assignment $A(x)=1$ if $\langle h, V_x\rangle > 0$ and $A(x)=0$ otherwise, show that for all $e\in E$ of the form $x(e)-y(e)=0$,

$$\Pr_h[A \text{ satisfies e}] = 1 - \frac{1}{\pi}\mathsf{Arccos}(\langle V_{x(e)}, V_{y(e)}\rangle),$$

and for all $e\in E$ of the form $x(e)-y(e)=1$,

$$\Pr_h[A \text{ satisfies e}] = \frac{1}{\pi}\mathsf{Arccos}(\langle V_{x(e)}, V_{y(e)}\rangle).$$

(d) Show that for $z\in[0,1]$ it holds that $\mathsf{Arccos}(1-z)\leqslant 2\sqrt{z}$, and deduce that if $\alpha_1,\ldots,\alpha_r\in[0,1]$ are such that $\mathbb{E}_i[\alpha_i]\geqslant 1-2\varepsilon$, then $\mathbb{E}_i[\mathsf{Arccos}(\alpha_i)]\leqslant 2\sqrt{2\varepsilon}$.

(e) Deduce that

$$\sum_{e\in E}\Pr_h[A \text{ satisfies e}] \geqslant m(1-O(\sqrt{\varepsilon})),$$

3. Show a polynomial time reduction from gap-d-to-1-Games$[1,\varepsilon]$ to gap-UniqueGames$[1/d,\varepsilon]$.

4. (*) In this question, we will consider the following seemingly stronger form of the Unique-Games Conjecture, and show that it is implied by the standard formulation of it.

Conjecture 0.1 (Strong UGC). *For all $\varepsilon,\delta>0$ there is $k\in\mathbb{N}$, such that given an instance $\psi=(G=(L\cup R,E),\Sigma,\Phi=\{\phi_e\})$ over a regular graph G with alphabet size k it is NP-hard to distinguish between the following two cases:*

- *YES case: there is $L'\subseteq L$ of fractional size at least $1-\varepsilon$ and assignments $A_{L'}\colon L'\to\Sigma$ and $A_R\colon R\to\Sigma$ that satisfy all of the constraints between L' and R.*
- *NO case: no pair of assignments $A_L\colon L\to\Sigma$ and $A_R\colon R\to\Sigma$ satisfy more than δ fraction of the constraints.*

We will show that for all $\varepsilon,\delta>0$ and k there are $\eta,\xi>0$ and k' such that gap-UG$[1-\eta,\xi]$ with alphabet size k is polynomial time reducible to gap-StrongUGC$[1-\varepsilon,\delta]$ with alphabet size k'. Denote $H=1/\eta^{1/4}$.

(a) Suppose that we have an instance Ψ of Unique-Games such that $\mathsf{val}(\Psi) \geqslant 1 - \eta$ and pair of assignments A_L, A_R that satisfy at least $1 - \eta$ fraction of the constraints. Show that for at least $1 - \sqrt{\eta}$ of the vertices $u \in L$, we have that

$$\Pr_{v_1,\ldots,v_H \text{ neighbours of } u} [(u, v_i) \text{ is satisfied by } A_L, A_R \text{ for all } i] \geqslant 1 - \eta^{1/4}.$$

(b) Suppose that we have an instance Ψ of Unique-Games such that $\mathsf{val}(\Psi) \leqslant \xi$, and we fix a pair of assignments A_L, A_R. Show for all but at most $\sqrt{\xi}$ of the vertices $u \in L$, we have that

$$\Pr_{v_1,\ldots,v_H \text{ neighbours of } u} \left[\text{ more than } (\sqrt{\xi} + \varepsilon)H \text{ of } (u, v_i) \text{ are satisfied by } A_L, A_R \right] \leqslant 2^{-\Omega(\varepsilon^2 H)}.$$

(c) Based on the previous two question, construct a poly-time reduction to show that the standard formulation of UGC implies the above strong formulation of UGC.

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