18.408 Topics in Theoretical Computer Science Fall 2022
Problem Set 4

Basics of Discrete Fourier Analysis

1. Write down the discrete Fourier expansion of the following functions:

   (a) $f\colon \{0,1\}^2 \to \{0,1\}$ defined by $f(x,y) = x \vee y$.

   (b) $f\colon \{0,1\}^3 \to \{0,1\}$ defined by $f(x,y,z) = x \vee y \vee z$.

   (c) $f\colon \{0,1\}^3 \to \{0,1\}$ defined by $f(x,y,z) = \mathsf{Not}-\mathsf{All}-\mathsf{Equal}(x,y,z)$. That is, $f(x,y,z) = 1$ if and only if not all of x, y, z are equal.

   (d) $f\colon \{0,1\}^t \to \{0,1\}$ defined by $f(x_1,\ldots,x_t) = x_1 \wedge \ldots \wedge x_t$.

2. A function $f\colon \{0,1\}^n \to \{-1,1\}$ is called odd if $f(x) = -f(1-x)$ for all $x \in \{0,1\}^n$. Similarly, a function is called even if $f(x) = f(1-x)$ for all $x \in \{0,1\}^n$.

   (a) Show that if f is odd, then for each $\alpha \in \mathbb{F}_2^n$ of even Hamming weight it holds that $\widehat{f}(\alpha) = 0$.

   (b) Show that every function $f\colon \{0,1\}^n \to \{-1,1\}$ can be written as a sum of two functions, $f_{\mathsf{even}}\colon \{0,1\}^n \to \{-1,0,1\}$ which is even and $f_{\mathsf{odd}}\colon \{0,1\}^n \to \{-1,0,1\}$ which is odd.

3. Let $f\colon \{0,1\}^n \to \{-1,1\}$ be a function, and consider the following variant of the linearity tester with 4-queries: sample $x, y, z \in \{0,1\}^n$ uniformly and check that $f(x+y+z) = f(x)f(y)f(z)$.

   (a) Prove that for any $f\colon \{0,1\}^n \to \{-1,1\}$, the probability that f passes the test is at least $1/2$.

   (b) Show that if $f\colon \{0,1\}^n \to \{-1,1\}$ passes the test with probability at least $1/2+\varepsilon$, then there is $\alpha \in \mathbb{F}_2^n$ such that $\left|\widehat{f}(\alpha)\right| \geqslant \sqrt{2\varepsilon}$.

4. Let $f\colon \{0,1\}^n \to \{-1,1\}$ be a function, and let $i \in [n]$. The influence of variable i is defined as $I_i[f] = \Pr_x\left[f(x) \neq f(x \oplus e_i)\right]$, and the total influence of f is defined as $I[f] = \sum\limits_{i=1}^{n} I_i[f]$.

   (a) Show that $I_i[f] = \sum\limits_{\alpha \in \mathbb{F}_2^n : \alpha_i = 1} \widehat{f}(\alpha)^2$ and $I[f] = \sum\limits_{\alpha \in \mathbb{F}_2^n} |\alpha|\, \widehat{f}(\alpha)^2$.

   (b) The variance of f, $\mathsf{var}(f)$, is defined as the variance of $f(x)$ as a random variable where x is chosen uniformly, that is, $\mathsf{var}(f) = \mathbb{E}_x\left[(f(x) - \mathbb{E}[f])^2\right]$. Show that $\mathsf{var}(f) = \sum\limits_{\alpha \in \mathbb{F}_2^n \setminus \{\vec{0}\}} \widehat{f}(\alpha)^2$ and deduce Poincare's inequality, stating that $I[f] \geqslant \mathsf{var}(f)$.

Efficient Amplification of Linearity Tests

Recall the linearity test from class checking that $f(x+y) = f(x)f(y)$ where x, y are sampled uniformly. We proved that the soundness of this test is $1/2$, hence sampling t pairs $(x(i), y(i))$ and checking that $f(x(i)+y(i)) = f(x(i))f(y(i))$ yields a test with soundness 2^{-t} and $3t$ queries. In this question, we will show that there is a test that gets a better, essentially optimal, tradeoff between the number of queries and soundness: it makes $\binom{t}{2} + O(t)$ queries and has soundness $2^{-\binom{t}{2}}$.

5. (*) Let $G = ([t], E)$ be the complete undirected graph on t vertices, and let $f\colon \{0,1\}^n \to \{-1,1\}$ be a function.

   (a) Let $x(1), \dots, x(t), y, z$ be sampled uniformly from $\{0,1\}^n$. Show that

   $$\mathop{\mathbb{E}}_{x(1),\dots,x(t),y,z}\left[f(y)f(z)\prod_{i=1}^{t} f(x(i)+y)f(x(i)+z)\right] \leqslant \max_{\alpha\in\mathbb{F}_2^n} \widehat{f}(\alpha)^2$$

   (b) Let $x(1), \dots, x(t)$ be sampled uniformly from $\{0,1\}^n$. Show that

   $$\mathop{\mathbb{E}}_{x(1),\dots,x(t)}\left[\prod_{\{i,j\}\in E} f(x(i))f(x(j))f(x(i)+x(j))\right] \leqslant \max_{\alpha\in\mathbb{F}_2^n} \widehat{f}(\alpha) .$$

   (c) Show that the result of the previous item holds for all non-empty sets of edges $S \subseteq E$. That is, if $S \subseteq E$ is non-empty, then

   $$\mathop{\mathbb{E}}_{x(1),\dots,x(t)}\left[\prod_{\{i,j\}\in S} f(x(i))f(x(j))f(x(i)+x(j))\right] \leqslant \max_{\alpha\in\mathbb{F}_2^n} \widehat{f}(\alpha) .$$

   (d) Consider the following linearity test for f:
   - Sample $x(1), \dots, x(t)$ uniformly from $\{0,1\}^n$.
   - Check that $f(x(i)+x(j)) = f(x(i))f(x(j))$ for all $i \neq j$.

   Establish the following properties of the test:

   i. Show that this test has perfect soundness. Namely, if $f = \chi_\alpha$ for $\alpha \in \mathbb{F}_2^n$, then f passes the test with probability close to 1.

   ii. Show that the probability that $f\colon \{0,1\}^n \to \{-1,1\}$ passes this test is equal to

   $$\mathop{\mathbb{E}}_{(x(1),\dots,x(t))}\left[\sum_{S\subseteq E}\frac{1}{2^{\binom{t}{2}}}\prod_{\{i,j\}\in S} f(x(i)+x(j))f(x(i))f(x(j))\right].$$

   Deduce that

   $$\Pr\left[f \text{ passes the test}\right] - 2^{-\binom{t}{2}} \leqslant \max_{\alpha\in\mathbb{F}_2^n} \widehat{f}(\alpha) .$$

Remark 0.1. *The result above can be used to prove near optimal hardness result for clique, as follows. Based on this exercise, one can construct a PCP with $T = \binom{t}{2} + 2t$ queries and soundness $2^{-\binom{t}{2}}$; this reduction uses the same outline as the 3-Lin result shown in class (using a noisy version of the above test instead of the noisy linearity test). Running the technique from Problem 5 in the 3rd problem set on this PCP, one can prove NP-hardness of the Max-Clique problem within factor $N^{1-\delta}$ for all $\delta > 0$.*

18.408 Topics in Theoretical Computer Science: Probabilistically Checkable Proofs
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