Spring 2024

18.200 Homework 4

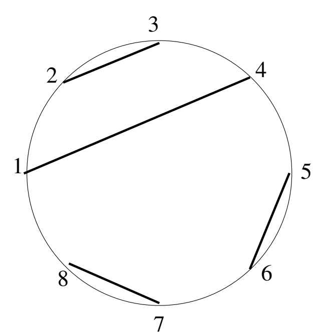
Instructions: Indicate your recitation and collaborators or state that you worked only on your own. In any case, you must write up your own proofs.

1. Consider the following game: n cards numbered 1 through n are dealt face down. Your goal is to guess the last card. To help you, a coin is flipped $n-1$ times, and if the ith coin flip is heads, the ith card is turned face up.

   (a) What is the probability that you guess the card correctly (assuming you make a reasonable guess)?

   (b) Conditioned on the event that you guess the card correctly, what is the probability, as a function of k that you flipped exactly k heads?

2. Consider $2n$ points on the plane labelled $1, 2, \cdots, 2n$, all spaced equally on a circle. A *matching* of these points is a collection of n straight line segments, with every point being the endpoint of precisely one of the line segments. A matching is *noncrossing* if no two of its line segments cross. Here is an example of a noncrossing matching on 8 points (so $n = 4$).



   Determine (with proof) the number of noncrossing matchings of $2n$ points, as a function of n.
   Hint: You might want to look for an appropriate bijection.

3. Let $\mathcal{C}$ be the set of all sequences of letters $\{a, b, c, 1, 2\}$ where all the letters $\{a, b, c\}$ appear before all the letters $\{1, 2\}$. For instance, $\mathcal{C}$ contains the seqneuces $bacca211$ and ab and 12 and $aa221$, but not the sequence $bac2a11$. Let c_n be the number of sequences of length n (n letters), and let $C(x) = \sum_{n=0}^{\infty}$ be the generating function for c_n.

   (a) Determine an expression for $C(x)$.

   (b) Determine an explicit expression for c_n.

4. Suppose you have a sequence that satisfies the recurrence relation $f_k = f_{k-1} + 6f_{k-2}$, with $f_0 = 1$, $f_1 = 2$, $f_2 = 8$. Use generating functions to find a formula for f_k.

5. Find a recurrence relation for the number of ways of tiling a $3 \times n$ strip with tiles of size 2×1 (which may be rotated). A 3×2 strip can be tiled in three ways, and a 3×4 strip can be tiled in eleven ways. Note that this tiles exactly only for even n. Roughly how fast does this sequence grow?
   (Note: you may want to use Mathematica or other software to evaluate the roots of a polynomial.)