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CASEY RODRIGUEZ: All right. So last lecture, we concluded our discussion about measurable functions, I mean, measurable sets. And remember our original motivation was that we're trying to build an integral that somehow surpasses the properties of that of the Riemann integral in that, hopefully, this larger class of functions that are integral with respect to this integral form a Banach space as opposed to Riemann integral functions.

And we started off by asking the question, how would we integrate the simplest types of functions, which are just one on a set and zero off of it. And that led us to how to define measure. And so then we define Lebesgue measurable sets, proved that they form this special type of collection of sets called the sigma algebra, and that they contain a lot of sets, a lot of interesting sets, open, closed, unions, countable unions of closed, which are not necessarily closed, countable intersections of opens, which are not necessarily open, and so on.

Now not every set is Lebesgue measurable. We will not go through that construction of a nonmeasurable set. The construction of the set or how one obtains it also would provide a proof that we could not build a measure that's defined on all subsets of real numbers having these three properties that is translation invariant.

The measure of an interval is the length of the interval. And the measure of a disjoint union is the sum of the measures. But this is a functional analysis class. Our goal is to build up a Banach space of integral functions so we needed to define a better notion of integral than Riemann. I imagine in a measured theory class, you would see such a construction but we won't cover that here.

So in some sense, if we've built up measure, we kind of know, roughly speaking, how we would integrate the simplest type of function, which is one on a measurable set and zero off of it. Now what about what would be a method of trying to integrate more general functions? So this is, I think-- so now we're going to be talking about measurable-- this is how it's spelled-- measurable functions.

So to motivate the definition of a measurable function, let me just give a few minutes of why we introduced this definition, what's really behind it. So historically, when Lebesgue thought of his theory of integration, if we had a function on a closed interval, AB-- and let me draw it. Let's say it's just increasing.

So what Riemann does, of course, is you partition up ab. And then you form, essentially, these boxes that have width given by how you've chopped up ab and height given by f, the function f, evaluated at some point in those subintervals. Right. And you form a Riemann sum and you take a limit. Yeah. And that gives you the Riemann integral.

And so what Lebesgue thought of doing was, instead of chopping up the domain, chopping up the range. So let's imagine-- so this function is only this height. So what he would do, or what one would do, is let's-- and let's say we're just interested in integrating non-negative things-- chop up the range and so on up to some finite point.

And now how do you form your boxes, if you like, that you're going to take the size of? Well, you look at the piece of f that's in between a given partition. And this portion of f will-- OK. In this picture, we'll call this point c.

So then you take this portion of f that's in here. And then ac would be given by is the set f inverse of, in this picture-- and I'm just going to write it in this way-- yi, y, yi minus 1, yi. And you could build up a box by now taking it to be the lower value y2 and having width given by a to c. And I can actually do this because this is an increasing function so this will be an actual box.

Now what am I getting at? And then you would say something like we would like to define the integral of f over ab. So this is all motivation, informal discussion. Don't take this to heart too much. Then we would like to define the integral from ab to f to be somehow this limit.

And I'm not even going to write as something goes to zero or infinity you can think of as partitions get smaller of a sum now where I'm summing from i equals 1 to n of y minus 1, so the lower part, times now the length of, at least for this picture-- so this would be the length of the interval given by f inverse of yi minus 1, yi.

OK? So this would be kind of an analogous procedure to what Reimann does on the domain except now focusing on the range. Now I wrote length here because this function I drew is increasing so the inverse image of one of these intervals is going to be another interval.

So the length is meaningful. But if f is more general, f inverse doesn't need to be of yi minus 1, yi, need not be an interval. And so taking its length would not be a meaningful thing. But remember we now have this notion of measure, which should be the substitute for length for more general sets, measurable sets.

So this procedure could still work if, instead of this requiring that this is an interval, requiring that this is a Lebesgue measurable set. And then perhaps one could define the integral in this way. And there these kind of things would no longer be boxes. So that should motivate why perhaps we should look at functions so that the inverse image of closed intervals is a measurable set so that we can take its measure and maybe do this procedure of defining an integral.

Now all of that is a bit-- again, this is all informal discussion just meant to be motivation. In fact, in how we'll define the Lebesgue integral, we're not going to define it in this way, because what this way of doing it suffers is it's not clear that this is independent of how I partitioned up the range.

So maybe if I take a limit as the partitions get smaller along some sequence of partitions, I get a different number from the others so that would have to be checked. But as we'll see when we do define the Lebesgue integral, this number, you can compute it by essentially this procedure I gave here where you chop up the range of f and take approximations to f that are more general than just step functions, which is what this is.

It's a function that's a given value off on an interval and then zero outside. And that'll be a way of seeing that this motivation connects to actually how we define the Lebesgue integral.

But again, the whole point of what I'm saying is that-- we're trying to motivate is that we should, and at least historically this is why one considers these things, consider functions so that the inverse image of closed intervals is measurable. OK. And that's the motivation for measurable functions.

Now, as we saw when we were discussing what sets are measurable, we didn't exactly conclude directly that closed intervals were measurable. We started with something a little more basic, which was half infinite intervals, proved those were measurable, and then concluded that close intervals are measurable. Any interval is measurable.

And so when we actually define measurable functions, we'll be using as input more of these half infinite intervals being measurable rather than this. So let's get to it. Let's define measurable function.

So now just this comes with the territory. But we're going to be considering extended real valued functions in what we do. So I shouldn't have written that down. I should have written down extended real numbers. What does this mean? This just means the set of real numbers along with plus and minus infinity. So when I write the interval minus infinity, infinity, this is our union, the two symbols plus or minus infinity.

Now we're going to have expressions where we allow a function to take on the value plus infinity or minus infinity. So we should set down what we mean when we multiply some of these things together. So sums are defined as x plus or minus infinity equals plus or minus infinity for all x in r.

So if I have two functions whose values could be in the extended real numbers and I have one is finite and the other is infinite, then I just, by convention, their sum is defined to be plus infinity. But I'm not allowed to take infinity minus infinity or infinity plus minus infinity. So this and products are defined as we take the convention that zero times plus or minus infinity equals zero.

And we won't come up against this until we discuss the integral, why we would need such expressions. But you could have a function that's, let's say, identically equal to infinity, identically gives this symbol plus infinity. If I multiply that function by zero, then I should get zero.

This has nothing to do with limiting processes though. These are purely algebraic expressions we'll be dealing with. I'm not saying that everything you learned in 18100 Real Analysis about being careful when you have infinity over infinity or zero times infinity is-- I'm not saying throw that away. I'm just saying when we have certain algebraic expressions, these are the conventions that we adopt. And x times plus or minus infinity equals plus or minus infinity for all x in R take away zero.

OK. And let me just recall for you what it means for-- we'll be making some limiting statements about certain numbers approaching other numbers. So what would it mean for a sequence of numbers to approach plus infinity or minus infinity?

I want you to recall that a sequence a n of real numbers converges to infinity. And then you can make a similar definition for minus infinity if, for every R positive, there exists a natural number N such that for all N bigger than or equal to N a sub n is bigger than R. OK.

OK. And I mean I've been using something equals infinity already when we've been just discussing measure, the outer measure of something being equal to infinity. In what follows, we'll be having expressions where we allow something to equal infinity.

And we'll have algebraic expressions of this type so I just need to set down convention that if I have a real number plus or minus infinity, that is by definition equal to plus or minus infinity. If I have zero times something that equals plus or minus infinity, then that is by convention equal to zero and so on.

All right. So measurable functions I said should be those functions at least motivated by this discussion before. Those functions or the functions we're interested in are those functions so that the inverse image of closed intervals are measurable. So that's how we'll define measurable functions almost. And the equivalent way, which is a little bit easier to work with, is the following. So let E be a measurable set and f from E to the extended reels.

You say f is Lebesgue measurable-- that's new terminology-- if, for all alpha in R, if I take the inverse image of the half infinite interval, alpha to infinity, this is measurable I should say. We had some notation last time, the script M being the collection of measurable sets, i.e. is Lebesgue measurable. OK.

As we'll see in a minute, this is an equivalent definition really of requiring that the inverse image of a closed and bounded interval is measurable, or at least we'll see one direction of why that's equivalent. And then in your own time, you can figure out why it's actually equivalent.

So now why not include alpha? Why do I have to look at the inverse image of this half open interval? And why am I going from alpha to infinity and not, say, alpha to minus infinity or include alpha there? Why this specific kind?

And then so what I want to first tell you is that looking at those types of sets and seeing if those are measurable is equivalent to this definition that I'm giving here. So let's take a function from a measurable set E, a subset of R to the extended reals. Then the following are equivalent.

One is kind of the property that's in the definition of being measurable. For all alpha in R, f inverse of alpha infinity is measurable. Two. For all alpha in R, f inverse including alpha is measurable.

Three. All alpha in R, the inverse image of minus infinity to alpha is measurable. And the last condition is, for all alpha in R, the inverse image of now include alpha is measurable.

So to check what you get from this, to check that a function is Lebesgue measurable, it suffices not to just solely prove this property or check this property. You can check it on other types of sets.

It suffices to only check it-- it suffices to check it on either these types of sets, or these types of sets, or these types of sets. So if you can prove that for all alpha, this type of set or the inverse image of these types of intervals are measurable, then it's equivalent to saying that f is Lebesgue measurable. OK.

OK. Now the proof is not hard based on what we know about Lebesgue measurable sets which, again, we proved last time that they form a sigma algebra. They're closed under taking countable unions, intersections, and complements, and so on.

So let's first prove that one implies two. So I'm saying they're all equivalent, which means they should all imply each other. So let's first prove one implies two. So suppose one holds, then for all alpha in R, now I want to check that two holds.

I can write alpha to infinity as the intersection over natural numbers n of alpha minus 1 over n infinity, which implies the great thing about taking inverse images is that they respect all operations you can do on set. So the inverse image of any type of intersection is the intersection of the inverse images.

And if I'm assuming each of these is measurable by 1, then this is a countable intersection of Lebesgue measurable sets. Again, when I say what is the Lebesgue measurable sets, I'm talking about the inverse image of this set, not this set in here. I'm talking about the inverse image of this interval. So each of those is a Lebesgue measurable set so their intersection is Lebesgue measurable. OK. And therefore for every alpha in R, the inverse image of these closed intervals are measurable.

And how do I show two implies one? By a similar game. Suppose two holds. Then for all alpha in R, the half open interval now I can write as the union over n alpha plus 1 over n to infinity. And therefore the inverse image of alpha to infinity is equal to the union of the inverse images of these sets.

And again, if I'm assuming two, then I'm assuming each of these for each n is a Lebesgue measurable set. And we know countable unions of Lebesgue measurable sets are again Lebesgue measurable. So this is Lebesgue measurable. OK. So that proves one and two are equivalent.

OK. And we get three and four. So first off, two is equivalent to three simply because if I take-- let's see-- minus infinity to alpha, this is equal to alpha infinity complement. And therefore again, when I take inverse images, and knowing that that the complement of a Lebesgue measurable set is Lebesgue measurable, I get that two is equivalent to three and one is equivalent to four.

Again, since I have this last type of set, minus infinity to alpha, this is equal to-- so this is for all alpha in R since for all alpha in R, this is equal to complement. So again, if I take the inverse image of this set, I will get the complement of the inverse image of this set. And if that's measurable, that inverse image is measurable, then its complement is measurable, implying that's measurable. So it's a simple game or it's a simple proof just based on the fact that we know that Lebesgue measurable sets are closed undertaking countable unions, intersections, and complements. OK.

So then we get the following theorem that, if E is measurable, so E is a subset of R is measurable, and f from E to R is a measurable function-- in the future, I will probably just say f is measurable, not a measurable function-- then for all f in the Borel sigma algebra, f inverse of F is measurable. The inverse image of f is measurable. OK.

So what's the proof? f measurable implies that, for all ab, a less b, f inverse of the open interval ab, which is equal to the f inverse of the two intersect equals f inverse of--

So if I take the inverse image of this open interval ab, then it's equal to the inverse image of this intersection, which is equal to the intersection of these two inverse images. And if I'm assuming f is measurable, then each of these sets-- each of these pre-images are measurable by the previous theorem that I just proved and therefore that's measurable.

So I've shown that open intervals are measurable. And similar to how we concluded that open sets are measurable, you can then-- we use the fact that every open set, which you proved in assignment 3, that every open set can be written as a countable union of disjoint open intervals, thus f inverse of u is measurable for all open sets, subsets of R. OK.

And since the set-- let's call it A, which is the set of all F such that this is a-- so you're also proving in the assignment this collection of sets, such that the inverse images are Lebesgue measurable, this is a sigma algebra containing-- since we've now proved that all open sets are in the set, this implies that the Borel sigma algebra, which is, again, the smallest sigma algebra containing all open sets, is a subset of this here, which is the statement of the theorem. OK.

All right. And so let me just add on here another thing that-- OK. So now we know that if we have a measurable function, it takes the inverse image of Borel sets. So sets that belong in the Borel sigma algebra are measurable. This includes all open sets.

What about throwing plus or minus infinity into the mix? So if f goes from one measurable set E to R is measurable, then the sets-- well, let me write it this way. The inverse image of where f is infinite or equal to minus infinity, these are measurable as well. And how does one prove that? So proof.

Again, we could just use the definition of how this works and the fact that we know Lebesgue measurable functions, or Lebesgue measurable sets, are closed undertaking countable intersections, unions, and complements. We have that f inverse of infinity, this is equal to the intersection overall n of f inverse, the inverse image of n to infinity.

And if I'm assuming f is measurable, then each of these is a measurable set. The inverse image of n to infinity is a measurable set. I will often use inverse image or pre-image. These are meant to be taken as saying the same thing. Each of these is a measurable set so it's the countable intersection of measurable sets is measurable. So that's measurable.

And similarly, if I look at the set where it's equal to minus infinity, this is equal to the intersection of the inverse image of minus infinity to n. Let's take these positives, so then I'll put minus n here. And by this theorem that we just proved a minute ago, each of these-- well, I mean we just proved that the inverse image of Borel sets are Lebesgue measurable so-- well, that won't apply.

Never mind. Forget what I just said. The theorem we just proved a minute ago that Lebesgue measurable functions take sets of this type to Lebesgue measurable sets implies that each of these is Lebesgue measurable and therefore it's a union of Lebesgue measurable sets, so for all n, so that's measurable.

So if I have an extended real value function, and the inverse image of every Borel set-- you can even toss in the two infinities if you like. The inverse image of those sets is Lebesgue measurable. So in particular, we conclude that the inverse images of closed and bounded intervals is Lebesgue measurable for measurable functions.

OK. Now what are the simplest types? Again, you see a definition. You should ask for an example. What are the simplest types of measurable functions? So if f from R to R is continuous, then that implies that f is measurable.

So in the end when we build our definition of the Lebesgue integral, we should encompass Reimann integration as well. In other words, we should be able to also integrate continuous functions. And later we'll see, and something that we want as well, is that the Lebesgue integral of a continuous function should reduce down to the Reimann integral of a continuous function.

So at a minimum, when we're building these concepts, we should check as a kind of sanity check that we are including continuous functions in these possible functions which we will integrate. So if f is continuous, f is measurable, why is this? This is because for all alpha in R, if I look at the inverse image of alpha infinity-- so first off, this is equal to this set. And this is an open set so this is open. So the inverse image of an open set for a continuous function is open. And therefore it's measurable. Right? OK.

So how about a different example?

Let's take a measurable subset E of R. Let F be another measurable subset. And define the indicator function chi of F of x to be 1 if x is in F, 0 if x is not in F. Then this function chi F-- now if I think of it as being a function from E to R, this is measurable.

Now, why is that? Well, we can just compute. If alpha is in R, if I look at the inverse image of alpha to infinity, this is equal to one of three things. Since chi takes on only the values 1 and 0, if alpha is bigger than 1, this is the empty set, which is measurable if alpha's bigger than or equal to 1 since this set does not include-- would not include 1.

If alpha is between 0 and less than 1, then the inverse image of this set is equal to E intersect F. E is measurable. F is measurable. So their intersection is measurable. And if alpha's less than 0, then the inverse image of this set-- in other words, what sets map into-- from one negative number to infinity-- well, both 1 and 0 map into there.

And that's the entire set E. And so whatever you-- no matter what alpha is, the inverse image of these-- of that set is measurable.

So these are basic properties of measurable functions. Let's continue what would we like more-- so what other properties of measurable functions, again, which we would hope to integrate or we will be integrating at least a class of these in the end to satisfy. Well, we would like them to be closed, undertaking linear combinations, and also products, because in the end, we'll have LP spaces, which is products of-- integrals of products of integral functions.

So we have the following theorem. So let's suppose E is measurable. And I have two functions going from E to R that are measurable. And I have a scalar in R. Then c times f, f plus g, and f times g-- so these are now all functions from E to R-- are measurable, are measurable functions.

So what's the proof? Let's start with scalar multiplication. This equals 0. And c times f equals 0, which is a continuous function. It's a constant function. 0 is continuous, hence measurable.

So let's suppose c is non-zero. So the-- we're out of the silly case. If c does not equal 0 and alpha is an element of R, then c times f of x greater than alpha-- this is equivalent to f of x is greater than alpha over c.

So this implies that if I want the inverse image, if I look at c times f and I look at the inverse image of alpha to infinity, this is equal to the inverse image of-- by f of the set alpha over c infinity. And now if I'm assuming f is measurable, then the inverse image of this set is measurable. So that's a measurable set. And therefore, c times f is a measurable function.

So now, let's look at the function f plus g. Suppose alpha's in R. Then let's do something similar. Then f of x plus g of x is greater than alpha. This is if and only if f of x is greater than alpha minus g of x.

So I didn't say this. But when I'm looking at such a condition, I'm looking at those x's that will be in this inverse image. So I'm just trying to figure out an equivalent way of expressing this condition, which we saw was this condition. So what I'm really looking at is those x's which would lie in the-- this pre-image here.

And what I did a minute ago was show it's equal to this pre-image here. So this is why I'm considering f of x plus g of x plus alpha. This is the condition that-- so maybe I'll just write this out. Then x is in this set if and only if f of x plus g of x is bigger than alpha, which is equivalent to f of x plus-- f of x is greater than alpha minus g of x.

Now, you learned in 18.100-- A, B, P, Q, whatever-- that if I have any two real numbers, one bigger than the other, then I can find a rational number in between them. So if this number is bigger than this number, there exists a rational number, r, such that f of x is greater than r is greater than alpha minus g of x.

So this-- assuming this does imply this. And of course, this condition also implies this condition. If there exists a rational number so that f of x is bigger than r is bigger than alpha minus g of x, then, of course, f of x is bigger than alpha minus g of x. So these two conditions are equivalent.

So this is equivalent to saying there exists an r in Q such that x is in the inverse image of f r to infinity intersect and r is bigger than alpha minus g of x, which one can state as x is in the inverse image of alpha minus r to infinity.

So this last expression here-- so let me come over here and started erasing. So we've shown that x is in the inverse image of this set, f plus g, the inverse image by f plus g of this set, if and only if there exists a rational number so that x is in the intersection of these two types of sets, which we know are measurable.

So we can express this, or summarizing what we just found-- that f plus g inverse image of alpha infinity-- this is equal to the union over rational numbers Q, r and Q, such that-- or the inverse image of f inverse or the inverse image of r infinity intersect alpha minus r infinity-- so that's just, again, expressing what we did over there.

And now what do we know? If we're assuming f is measurable, then this whole set is measurable. And we're assuming g is measurable. So this whole set is measurable. And therefore, this intersection of these two sets is measurable. And now I have countable union because, again, the rational numbers are countable. One of the first things you prove in analysis is that the set of rational numbers is countable. This is a countable union of measurable sets. So that's measurable.

Now, what about f times g? Here, we'll pull a little trick. So we'll prove that if f is measurable, then its square is measurable. And then we'll use a simple identity to conclude that f times g is measurable.

So now I claim that f squared is measurable. Let alpha be a subset of R. If alpha is bigger than or equal to 0, well, let's do this stupid case first. If alpha is less than 0, then f squared-- this is a non-negative function. So if I take the inverse image of alpha to infinity, this is just equal to the domain E. So this is measurable.

Remember, this is a set of all x's that get mapped by f squared to alpha to infinity. And if alpha is negative, then no matter what x is in E, f squared of x is going to be in alpha to infinity, again, for alpha less than 0. So this inverse image equals E, which is measurable by assumption.

And then the other less trivial case is if alpha is bigger than or equal to 0, then f squared of x is bigger than alpha if and only if either what? f of x is bigger than the square root of alpha or f of x is less than minus square root of alpha. And therefore-- so, again, this expression here is-- this expression is expressing. But this relation here is expressing x is in the inverse image of alpha to infinity.

So this says that the inverse image of alpha to infinity is equal to the set of all x's satisfying this condition on the right of this, if and only if, which is-- can be written as x is in the inverse image of-- f is bigger than square root of alpha, union, again, coming from the or, f inverse of-- and again, if we have a measurable function, then not only are these types of the pre-images of these types of sets measurable, but the pre-images of these types of sets are also measurable.

That was the first thing we prove. And so since each of these pre-images are measurable, their union is measurable. So that's measurable.

So we proved that f squared is measurable. And now we conclude that f times g is measurable by a simple identity that f times g-- this is equal to 1/4 f plus g squared minus f minus g squared. So f times g is equal to this function squared. If f and g are measurable, their sum is measurable. And therefore, by what we just proved, the square is measurable.

And again, over here, this is going to be measurable. And by scalar multiple of minus 1, that thing's measurable. Scalar multiple of 1/4 on the outside is fine as well. So we conclude that this thing is measurable because every operation in this expression preserves the function being measurable, as we proved before this expression. And that's-- that concludes the proof.

So the sum of two measurable functions is measurable. Scalar products are measurable. Products are measurable-- big deal because as far as Riemann integration goes, that's-- that still holds.

So what's something that sets apart a function being measurable and, eventually, Lebesgue integrable is that it has better-- it's closed under taking limits as opposed to being Riemann integrable. So this is the following theorem.

If I have a measurable set and then I have a sequence of functions, fn, going from minus infinity to infinity-- oh, and so I should have said a minute ago-- I'm just catching myself. Oh, no. Everything's fine. So I just wanted to make sure I had gone to R because f plus g is only defined going from E to R.

But again, if I have one function that is finite everywhere and another function that's an extended real-valued function, you can check that f plus g is then going to be measurable as well. I can't make sense of the sum of two extended real-valued functions because I might be in a situation where I have plus infinity minus infinity, which is an undefined expression.

But what I'm saying is I can take a-- for all of this, for one of them to be extended real-valued is fine. And also, the product of extended real-valued is fine, although I left it out of the list of rules, plus infinity times minus infinity is defined to be minus infinity, and with the usual sign rules.

So back to the theorem-- if I have a sequence of measurable functions-- so then some other functions are measurable. Then the function g1 of x equal to sup n fn of x-- so that's now a function defined on E. This is also measurable.

So I'll just list the functions and then say they're all measurable. g2 of x equals the inf over n fn of x. 3 x equals limsup as n goes to infinity of fn of x, which we'll recall the definition you can write as the inf over all n of the sup over all k bigger than or equal to n fk of x.

And 4-- so I don't know why I'm double-labeling them. I have 1, 2, 3, and 4. But I also have 1, 2, 3, and 4 here. And the liminf of-- the point-wise liminf of these functions, which I will recall is equal to the sup over m of inf-- these are all measurable functions.

So let's prove this. So the proof is not too difficult. So let's start with the first one. Actually, the third and the fourth follow from the first and the second. But so x is in the inverse image by g1 alpha to infinity if and only if, of course, sup n fn of x is greater than alpha.

And this is equivalent to the sup over all n of fn of x is bigger than alpha if and only if there exists some n so that fn of x is bigger than alpha. If all the fns stay below alpha when evaluated at x, then the sup is less than or equal to alpha.

So since the sup is the least upper bound-- so if and only if there exists an n such that-- if and only if there exists an n so that x is in fn inverse of-- and therefore, we've proven that the inverse image of the set by g1 is equal to the union over all n of f inverse f sub n inverse the inverse image of alpha to infinity.

And now we're assuming the fn's are all measurable. So each of these is measurable. It's a countable union of measurable sets. So it's, again, a measurable set. So we've proven that for all alpha, this is a measurable set. So g1 is measurable.

Now, if we go on to g2, it's the same thing. I can prove that the inverse image of-- let's see. What do I do here? Now I'm going to include alpha here. The inverse image of alpha to infinity-- so this is equal to the requiring that the inf over fn of x is bigger than or equal to alpha. This is equivalent to the intersection now of f inverse of these guys.

Now, each of these is, again, measurable by assumption. And therefore, it's countable. Intersection is measurable. So we've proven that taking sups and infs of sequences of functions are measurable.

But by how the limsup and liminf are defined-- so now we've proven that for any sequence of measurable functions, the sup and the infs are measurable functions. But the limsup is the sup-- is this sup first, followed by an inf. Now, if all of the f's are measurable, this sup is measurable for all n. And therefore, this inf is measurable.

So g3 being measurable follows immediately from proving that sups and infs of measurable functions are measurable, and the same thing with the liminf. If f sub k is measurable for all k, then we've already proven that the infs over the k's-- bigger than or equal to n doesn't matter-- is measurable, a measurable function. And therefore, the sup over the n's of all these measurable functions are, again, measurable.

So the fact that g3 and g4 are measurable follow from the expressions for-- and the previous two cases, meaning that we proved infs and sups of measurable functions are measurable functions.

Now, what do we get from this? So an immediate theorem is the following. If E is measurable, fn goes from-- and fn is measurable for all n. And they converge pointwise to some function f of x. Then f is measurable.

I shouldn't write it as a theorem. I should write it as a corollary. Why? Because if these converge, then f is equal to the limsup. It's also equal to the liminf. But f is equal to the limsup.

So f-- so the proof is one line. For all x in E, f of x is equal to limsup of fn of x. And it's also equal to the liminf if the fns are converging to f.

You covered this in 18.100. You have a limit of a sequence if and only if the limsup equals the liminf. And this also holds if I include plus or minus infinity as being possible elements of the extended real numbers that are in the sequence or the possible limit. Since this is measurable by the previous theorem, f is measurable.

So this says something that is an indicator that we're doing something that's or we're building something that's stronger than being Riemann integrable. As I've said, the functions which we will define a Lebesgue integral for will be a certain subset of measurable functions.

And so taking them as our possible candidates, let me just make the following remark, which now separates measurable functions, which are-- again, are candidates for being Lebesgue integrable from Riemann integrable is that this fails if I replace measurable by Riemann integrable.

If fn-- let's go from a, b to even R-- is Riemann integrable. And fns converge to f pointwise, which just means this. For all x in a, b, limit as n goes to infinity of fn of x equals f of x. Then f need not be Riemann integrable. In other words, when I look at Riemann integrable functions, they're not closed under taking pointwise limits.

So yet for the Lebesgue integral, at least for our candidates of Lebesgue measurable functions, those are closed under taking pointwise limits, as we just proved. Now-- which says that we're on track to proving something that's a-- that has better qualities than the Riemann integral.

Now, as it turns out, the pointwise limit of Lebesgue integrable functions need not be Lebesgue integrable. But if you add another very minor condition, the answer is, yes, it is Lebesgue integrable. But at the very least, we would like for our candidates-- again, Lebesgue measurable functions-- to be closed under taking pointwise limits or, at least, that should indicate to us that we're doing something that will have better properties than Riemann integrable functions.

So why is this? What's a sequence of functions that are Riemann integrable, but not-- but their pointwise limit is not Riemann integrable? So for example, you could take the function fn to be-- so you know what? I'm actually going to add this to the assignment. And it won't be-- no. So I'll go ahead and say why this is the case.

Let's write-- so the rational numbers Q and-- let's say intersect 0, 1-- this is a countable set. It's a subset of the rational numbers. So this is countable set. So we can list them. This is equal to r1, r2, and so on. If I define fn of x to be 1 if x is in r1 up to rn and 0 otherwise, now this is a function which is piecewise continuous on 0, 1. So it's Riemann integrable.

But what happens as I take n goes to infinity? The pointwise limit of these functions converges to the indicator function of the rational numbers-- intersect 0, 1. And hopefully, this was something that you checked when you learned about Riemann integration.

If you just learned about the Riemann integral of continuous functions, then that's fine, too. But anyways, this function is 1 on the rationals and 0 elsewhere. And-- which is a function which is discontinuous everywhere, or-- let's see-- not everywhere, or-- yeah, everywhere.

And you can convince yourself that that's too crazy of a function to be Riemann integrable if all you knew was that continuous functions are Riemann integrable. But if you learned more about the Riemann integral, then one of the things that you learned was that this function is not Riemann integrable.

So taking the pointwise limit of Riemann integrable functions may not be Riemann integrable. However, what we've proven is that the candidates for being Lebesgue integrable-- namely, those measurable functions-- are closed under taking pointwise limits, which is, like I said, an indication that this theory of the integral that we're building up is going to be a stronger theory in the sense that we can prove more things than that of Riemann.

And one of those things we'll prove is that the space of Riemann-- I mean, Lebesgue integrable functions with a norm being the integral of the absolute value is, in fact, a Banach space as opposed to the Riemann integral.

So now let me just make a couple of final statements. And then we'll call it a day for this lecture. So this is just really some terminology that when I make a statement P of x and I say that with x to-- an element of a measurable set E-- If I say this statement holds almost everywhere on E, and I'll usually say-- I'll shorten that to just "ae" on E or just "ae" if I'm deleting even more from what I write-- this holds almost everywhere if the set where it doesn't-- where it's-- doesn't hold has measure 0.

So I'm saying-- you may think I'm saying two things here, that this set is measurable and its measure is 0. But remember, when we developed the theory of measure, first thing we learned about measurable sets is that if it has outer measure equal to 0, then it's measurable. And remember, the outer-- the measure of a measurable set is just its outer measure. So perhaps I'll say with-- in parentheses--

So a statement holds almost everywhere if the set where it doesn't hold has a Lebesgue measure 0, which is equivalent to saying it has outer measure equal to 0 because we proved that sets that have outer measure equal to 0 are Lebesgue measurable.

And so the final theorem that we'll prove today is that if I have two functions-- one measurable, the other differing by that measurable function off a-- on a set of measure 0-- then the second function is also measurable. So somehow, sets of measure 0 don't affect being a measurable function.

So if fg go from a measurable set E to the extended real numbers, f is measurable. And f equals g almost everywhere on E, meaning the set of x's where f does not equal g is a set of measures 0 in E. Then g is measurable.

So if you take a measurable function f, change it on a set of Lebesgue measure 0, you still get a measurable function. And again, this just follows from the fact that all sets of Lebesgue measure 0 or outer measure equal to 0 are Lebesgue measurable.

And then, again, the fact that we know measurable sets are closed under taking unions and complements-- so let n be the set of x's in E such that f of x does not equal g of x. Then this is a set of outer measures 0. And therefore, it's Lebesgue measurable.

So now if I take an alpha in R and define another set-- call it N sub alpha-- to be the set of all x's in N such that g of x is greater than alpha, this is a subset of N just by definition. So this is how I'm defining this set. So it has outer measure equal to 0. And therefore, it's measurable because N has outer measure 0.

Then if I take the inverse image of alpha to infinity, this is equal to the inverse image by f of alpha to infinity intersect, where they equal each other. So we'll intersect the complement of N union this set, N sub alpha.

Now N has outer measure 0. So it's Lebesgue measurable. And therefore, its complement is measurable. This is measurable by assumption. And therefore, this intersection is measurable. And this set here is measurable. So the union of this measurable set with this measurable set is, again, measurable. And we conclude that this is measurable.

So we have defined measurable extended real-valued functions and proven some properties of them, the most striking of which being that being measurable is closed under taking pointwise limits and that you can change a function on a set of measure 0 and still be measurable.

Next time, we'll extend this notion in a trivial way to functions which take on complex values, not extended, just finite complex numbers, not including the-- pointed at infinity. And then we will move on to defining-- so once we have these-- so here's the game plan. So we'll extend this notion of being measurable to complex-valued functions.

And then we'll show that there's a particular class of functions called simple functions which, again, are simplest in the sense that they just take on finitely many values, and show that those are the universal measurable functions.

And from there, we will then define the Lebesgue integral of a non-negative measurable function and prove some properties of that. And then once we've done that-- so you can define the-- an integral of a non-negative-- an arbitrary non-negative measurable function.

We will then restrict to those functions which we call Lebesgue integrable, which now can-- not necessarily non-negative, but will have finite Lebesgue integral, and then prove those are-- and then prove some properties of that, including the big convergence theorems, and then close out our section on or chapter on measure and integration by proving that the Lebesgue integrable functions form a Banach space with a natural norm put on it as opposed to Riemann integrable functions. We'll stop there.