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[CLICKING]

CASEY

RODRIGUEZ:

All right. So last time, we proved the fundamental theorem of calculus. And as a consequence, integration by parts of formula and the change of variables formula or use substitution. So let me just recall, for integration by parts, it was that if I have two continuously differentiable functions, then I can shift the burden of taking a derivative.

f' times g integrate is equal to f of b , g of b , minus f of a , g of a , minus the integral from a to b of g' times f . So I can shift that derivative over from f to g . And this is apart from the triangle inequality, probably one of the most useful theorems that comes out of calculus other than the fundamental theorem of calculus.

In fact, for those of you who've taken quantum mechanics, which I mean, you don't have to. But or you heard of something called the Heisenberg uncertainty principle, which says something like something to the effect that you cannot measure the position of a particle and its momentum to arbitrarily good degree, you're bound by if you can measure the position of a particle very well, then your measurement of the momentum is going to be not so great and vice versa.

And that's based on an inequality, and how do you prove that inequality? Integration by parts. So integration by parts is, in fact, responsible for one of the great head-scratchers from quantum mechanics. So just to back up my claim a little bit.

Now, I'm not going to give that as an application. I'm going to give a different application related to Fourier series. So the Fourier series, what are these? So suppose we have a function. And I'm not going to say what type. Suppose f from minus π to π to $\mathbb{R}$ is 2π periodic.

And so, the question that arose due to Fourier in his study of heat transfer-- so this is Fourier, I don't know, something like 200 years ago. He made the following claim, that the function f of x can be expanded in terms of simpler building blocks, in terms of simpler functions.

So we haven't talked about Taylor series. We will in just a minute, or power series, which you've come into contact with, which is a way of, if you like, expanding a function in terms of polynomials. And now-- or monomials. Now, Fourier suggested that f of x can be expanded as a superposition of functions which are 2π periodic and kind of the most basic 2π periodic functions.

Now, what is so special about sine x and cosine x ? Well, this is a little bit deeper, the fact that they satisfy certain second order differential equation. And they are all of the solutions to the second order differential equation that are 2π periodic. So you should think of these as kind of being building blocks.

Another way to think of this is analogous to for if you have a vector, so now this is not a partition. This is a vector, x_1, x_2, x_n , then you can expand this vector as a sum of coefficients $a_{sub n}, a_{sub j}, j$ equals 1 to n .

You know what? Let me make this M . Let's make this N . Let's make this M , so it looks a little-- a sub n , e sub, where now, e sub n this is the basis vector given by $0 \ 1 \ 0$ where this is in the end spot.

So you can think of this expansion in terms of sines and cosines as being analogous to expanding a vector in terms of basis elements. Or you can think of it as a different way to expand a function other than Taylor series or power series.

But these components arose in a natural way if one were to study the problem of heat transfer, which is governed by the equation $\frac{du}{dt} = \frac{d^2x}{dx^2} u$. And then, along with an initial condition, that $u(0, x) = f(x)$, so at time 0, equals f of x .

So now, just like for how we expand a vector into basis elements, there's a formula for computing these coefficients. So they should be a_n should be x_n over here. But what's a different way of obtaining these coefficients? So this is a vector in $\mathbb{R}^M$.

And so, how do you obtain the coefficients a_n ? Well, if I take the inner product of both sides, the dot product, say of e_n dot, let's say, e_n prime, let's say-- so I've used M_n , let's say l . This is equal to the sum from n equals 1 to the M of $a_n e_n$ dot e_l .

Now, I wrote these basis vectors this way, because that's kind of a standard choice for $\mathbb{R}^M$. And what makes them standard is, they have unit length. And they're orthogonal to each other. So they form an orthonormal basis.

So when I take the product of e_n with e_l , I pick up what is usually referred to as δ_{nl} . Where here, δ_{nl} , this is 1 if n equals l , and 0 if n does not equal l . And therefore, this just reduces to a_l . So we see that a_l is equal to x dot e_l .

And so, all of this discussion was in the setting of a finite dimensional vector in $\mathbb{R}^M$. And expanding in terms of the standard basis here. But it didn't have to be. It could have been as long as it's an orthonormal basis, then I get this relation, that the coefficient that appears in front of that vector is equal to the thing I'm interested in dotted with that vector, which is written here.

So let's say we try and do the same thing now with $f(x)$, except now and say-- so these are functions. So instead of taking dot products, which is a sum of components, let's take an integral. So if I take $f(x)$ and, if you like, dot it with $\sin(x)$ in sum, which is-- you can think of as I said that the integral is, you should think of as maybe a continuous sum.

What do we get assuming that this expansion holds, this is equal to the sum from n equals 0. So let me make this l . This is the sum from n equals 0 to infinity of $a_n \sin(n, x) \times \sin(l, x)$.

So let me-- forgetting to write the integrals here. Skipping a point I want to make as well. Sum, and just remember the sum is starting from 0 to infinity. I don't want to keep writing it. $a_n \sin(n, x) \sin(l, x)$ plus $b_n \cos(n, x) \sin(l, x)$. And then, n equals 0 to infinity. I'll just write it. Stop being lazy.

Now, assuming I can do what I'm about to do, and that's actually going to be a lot of the motivation for what we're going to discuss in our final chapter, assuming I can take this infinite sum and interchange it with this integral, this is the interchanging of two limits. The sum is the infinite limit. An integration is a limit.

So assuming I can switch these two limiting processes, then I pick up a sub n minus π to π , sine in x , sine in lx , sine lx , sorry, plus b sub n integral minus π to π , cosine in x , sine lx dx .

Now, you can actually sit down and compute this based on trigonometric identities. And what you get is that this is always equal to 0. And that this here equals π times Δ in l . So this equals the sum from n equals 0 to infinity $a_n \pi \Delta$ in l , which equals π times al .

So we get this quantity here is assuming everything we've done is kosher equal to π times a sub l . And then, to pick up the b sub l is the same, except now you integrate against cosine of lx . So similarly, π times b sub l is equal to the integral from minus π to π of $f(x) \cos lx$ dx .

So the b sub l 's and a sub l 's are referred to as a Fourier coefficients of the function f . So if f from minus π to π to R is continuous, and 2π periodic, the numbers a_n equals $\frac{1}{\pi}$, integral from minus π to π of $f(x) \sin nx$ dx , b_n equals $\frac{1}{\pi}$, integral from minus π to π , $f(x) \cos nx$ dx are referred to as the Fourier coefficients of f .

And so, just using integration by parts, so what's the first question one should ask if it's even possible, or in what sense does $f(x)$ equal this infinite sum? Well, we haven't even gotten into that. But one question you can ask is, do these coefficients that come in front of these basic building blocks, sine nx and cosine nx , do they converge to 0?

I mean, if we expect $f(x)$ to be equal to the sum of these basic parts, then the contributions from each should be getting smaller and smaller. So does a_n and b_n tend to 0 as n goes to infinity?

And this is the content of what's usually referred to as the Riemann-Lebesgue lemma. But it's usually stated in a different way. I'm just going to state it this way right now. And which is the following. If f from a to b to R is continuously differentiable, then limit as n goes to infinity of a_n equals the limit as n goes to infinity of b_n equals 0.

Now, the actual way the Riemann-Lebesgue lemma is typically stated is, in fact, I don't need it to be continuously differentiable. I just need it to be continuous. This is still true. But we haven't done-- or won't do in this class approximation theorems for continuous functions. Which says that if you can do this for continuously differentiable functions, then basically you can do it for continuous functions. But this will suffice.

So what this says is that the contributions coming from these building blocks is getting smaller, at least in the sense that the coefficients are getting smaller. But it says nothing about if that sum up there with the a_n 's and b_n 's defined this way actually converges to f .

I do want to emphasize that, in fact, trying to straighten out this question, in what sense this series converges to f is really the motivation for a lot of analysis developed past in the first part of the last century and the last part of the century before that and forms the basis of what's called harmonic analysis. Which is a really beautiful subject and still an active area of ongoing research.

So how do we prove this? Well, I stated the integration by parts formula earlier. So in fact, it'll follow pretty easily from that. Let's prove that the limit as n goes to infinity of b_n equals 0. The one for a_n is similar. There's just an extra piece.

But I'm going to be a little bit lazy and do the easier one. We'll show-- so let's look at b_n . This is equal to the integral from $-\pi$ to π of $f(x) \cos nx$. And in fact, let me write $\cos nx$ times $f(x)$, dx , dx . Why am I writing dx , dx ?

And now what I do is, $\cos nx$, I can write as the derivative of something. $\frac{1}{n}$ times $\sin nx$. If I take the derivative of that with respect to x , I get $\cos nx$. I didn't actually prove that. But you can look back in your calculus textbooks. We've proven enough to be able to make that precise.

So by integration by parts, I can now shift the blame, or shift the burden of this derivative onto f . But look what I've gained. I've gained $\frac{1}{n}$ here. So now, this is equal to $\frac{1}{n} \sin n\pi f(\pi) - \frac{1}{n} \sin n(-\pi) f(-\pi) + \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx$.

And really, what this competition is showing is-- illustrating is the oscillatory nature of what's going on. $\cos nx$ is oscillating as n gets very large between -1 and 1 and equal footing. So on average, you're getting the same amount of positive f as minus-- as negative f . So or you're weighting f in such a way that it's both positive and negative in equal amounts.

Now, $\sin n\pi$, no matter what n is, I get 0 . $\sin n(-\pi)$, I get 0 . So this first part drops off. So this is equal to $\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx$. And therefore, if I take the absolute value of b_n , this is less than or equal to $\frac{1}{n} \int_{-\pi}^{\pi} |f'(x)| \, dx$.

If I bring the absolute value inside, so I can bring the absolute value inside and still get this. So in fact, before when I had the absolute value outside, it's the equality. But now it's a less than or equality-- I mean, an inequality.

So now, $\sin nx$ is always bounded by 1 . So this is less than or equal to the integral of $|f'(x)| \, dx$. So this is equal to $\frac{1}{n} \int_{-\pi}^{\pi} |f'(x)| \, dx$. Now, this is just $|f'|$. This is just a fixed number times $\frac{1}{n}$. So this converges to 0 as n goes to infinity. And b_n , an absolute value, of course, it's always bigger than or equal to 0 . And it's bounded by something converging to 0 as n goes to infinity.

So by the squeeze theorem, we conclude b_n converges to 0 . And that's the proof. The proof for the a_n 's is similar, except now you can't throw away necessarily the endpoints. But it's still not a very big deal.

And in fact, if we have time, I'll show you how much-- so in fact, one can prove-- and this is proven in classes on-- courses on harmonic analysis, that in fact, for a function which is continuously differentiable, this-- and 2π periodic-- this series actually does converge uniformly to f on this interval.

And I haven't even said what uniform convergence means. But actually does converge to the function $f(x)$. So this is the case for continuously differentiable functions. I'll give a proof later that, in fact, this series converges if f is twice continuously differentiable. We can actually do that using the fundamental-- the integration by parts again, essentially.

But so, but there are a few things here that are behind the scene that are kind of swept away. First off, when we computed these formulas-- formulae, I guess-- we interchanged summation, infinite summation of functions with integration.

When can we do that? In what sense does if this Fourier series converges, in what sense does it converge to f ? For convergence of real numbers, there was just one sense of the convergence of real numbers.

Now, when we have a sequence of functions, which is now what we're going to turn to, we'll have different notions of convergence to another function. And depending on in what sense that convergence-- that convergence takes place, some of these limiting operations may not interchange.

So now, we're going to move on to the final chapter of this class. And I know it seems like we're kind of hitting a lot of different things now towards the end of the class. And we took it slow during the first part of class.

But that's, like I said, that's-- I think I even said this at the start of class. We didn't have very much to go off of. We built things from the ground up. And the more technology you have, the more things you can prove, the more interesting questions you can ask.

So now, we're going to go on to sequences of functions. And you could also put sequences and series. Because a series is just a special type of sequence of functions. So I motivated a little bit of why we would be interested in functions converging to other functions or sequences of functions converging to a function. But we could look at something much more basic.

So let's take a step back and look at power series. And this should be thought of as motivation for what's to come, just like our discussion about Fourier series. And again, I'm not going to ask any questions about Fourier series on the homework or on the exam. So a lot of this is just-- this discussion was to motivate this theorem here.

But now, I'm going to make a kind of a more precise motivation, I guess, for what's to come. So although we've had series forever, I never brought up power series. And it's for a reason. It's because I didn't think they belonged anywhere until we got to sequences of functions.

So a power series about a point x_0 is a series of the form $\sum_{j=0}^{\infty} a_j (x - x_0)^j$. So the x_0 is given. And the things that could change are the coefficients or this number x here.

So theorem, which immediately follows from essentially the root test. Suppose, this number R , which is the limit as j goes to infinity of $|a_j|^{1/j}$ exists. So it's a finite number, positive, not negative number. And define ρ to be $1/R$ if R is bigger than 0, and infinity if R equals 0.

Then we have the following conclusion, that this power series $\sum a_j (x - x_0)^j$ converges absolutely if $|x - x_0|$ is less than ρ , and diverges if $|x - x_0|$ in absolute value is bigger than ρ . And this number ρ , we refer to as radius of convergence.

So again, the proof follows immediately from the root test because if we take a limit as j goes to infinity of $|a_j (x - x_0)^j|^{1/j}$, this is equal to $|x - x_0| \lim_{j \rightarrow \infty} |a_j|^{1/j}$. This kills that j . This is just a fixed number. So this pops out of the limit. And this limit exists.

So this equals $|x - x_0| R$. And we have two things happening. This is less than 1 if $|x - x_0|$ is less than ρ , bigger than 1 if $|x - x_0|$ is bigger than ρ -- this number here. And therefore, by the root test, the theorem holds.

So we see that this series, where the, if you like, what's given are the coefficients a_j and x_0 , and what could change is x , that this series converges as long as $x - x_0$ is less than ρ . So as long as we stay-- [SNEEZES] excuse me. As long as we stay in that interval, a symmetric interval about x_0 , then this series converges, absolutely.

So we can define a function where if I take x in this interval, stick it into this series, I get out a number. So define function f now going from this interval. So $x_0 - \rho$, $x_0 + \rho$ to $\mathbb{R}$ by $f(x)$ equals the number that gets spat out by this power series. [INAUDIBLE] there.

So for example, what is $f(x)$? Let's say I take all of the coefficients to be-- let's say, x_0 is 0, and all the coefficients are 1. So let's say I look at sum of x^j . Then $f(x)$, so we've already computed for a geometric series. This is equal to $1/(1-x)$ for x in $(-1, 1)$. So in the simple setting, $1/(1-x)$ is equal to the sum.

Another example is, that if I take $a_j = 1/j!$, $x_0 = 0$, then you've done this in an exercise, that this series here converges absolutely for all x . And meaning $\rho = \infty$.

And this is how we define the exponential function. Exponential function, exponential of x to be what comes out of this series. And then, simply from this definition, you can show things that an exponential function should satisfy, $e^x e^y = e^{x+y}$ is equal to $e^x e^y$ to the e of 1 to the n -th power, and so on, and so on.

So that this really does obey what you believe an exponential function should look like. And also, it grows faster than any power of x as x goes to infinity, goes to 0, those types of things. But this is how the exponential function is defined.

So we have this function that's defined by whatever this power series spits out, for x inside this interval of convergence. So then, I could write $f(x)$ as the limit as n goes to infinity of a sequence of functions. Because this is just how it's defined, where $f_n(x)$ is the partial sum. It's just a polynomial.

So for power series, the limiting function, you can write it as the limit of the partial sums, which are just polynomials. And so, I should say, for all x in this interval, we have this.

So now, some questions arise. So what this function is equal to, the limit of these maybe simpler functions. These simpler functions are just polynomials for the case of power series. So like I said, $1/(1-x)$ is equal to the limit of these polynomials.

Some questions should arise. I mean, analysis is about limits. You can think of that as half the story. First off, what is the limit? What are important limiting processes that we consider?

The second question is, how do different limits interact? So let's pose that as a question now for a three-parted question for power series. And this will motivate-- and this is, again, motivating all of what we're going to be doing now. So is the function that I get as this limit of polynomials as the output from a power series, is it continuous?

The individual pieces that I take a limit to get f of x , these polynomials or the partial sums are certainly continuous. They're just polynomials. So is the limiting thing continuous? Now, if so, is f differentiable? And in particular, since f is equal to the limit as n goes to infinity of the f_n 's, is f prime equal to the limit as n goes to infinity of the f_n prime?

So the derivative is a limiting process. So I'm taking the derivative of the limit. So I'm asking, can I take that derivative inside the limit? Can I swap the two processes? And the same with integration. If one, does the integral of f equal limit as n goes to infinity of the integrals of the f_n 's?

So again, this is a limiting process that we're asking us to flip, because f is equal to the limit as n goes to infinity of f_n . And what I'm asking is, can I take this integral inside that limit?

Now, you can ask these questions not just for power series, but in a more general setting, which is what we're going to turn to now. But this should be in the back of your mind as the motivation for what we're doing.

And apart from being just an academic question, it's also somehow giving you information over whether the formal manipulations that you're doing with Fourier series that are actually somehow modeling some physical phenomenon, are these formal manipulations even meaningful?

So these are the three questions that motivate what we're going to do going forward. But we don't have to just stick to the setting of being in power series. This should be a very important example of a sequence of functions converging to a function, a limiting function. And then, we can ask these questions. But we don't have to just stick to power series.

So let me move on to a more general setting in which we'll answer these three questions. And two modes of convergence for limits of functions or sequences of functions.

So first definition, this is, in fact, what we showed or what we were talking about before for N natural number, let f_n be a function from S to $\mathbb{R}$. S is some non-empty subset of the real numbers. And let f be from S to $\mathbb{R}$.

We say, $f_n \rightarrow f$ so the sequence of functions f_n converges point-wise to the function f if for all x in S , by sticking x into S , so for each fixed x in S , if I stick this into f_n of x , I get a limit. And this limit is f of x .

So for example, going back to power series, if we defined f of $x \rightarrow$ so let me just rewrite that example that I had up there. If I define f of x equals 1 over 1 minus x , f_n of x to be in sum from j equals 0 to n of x^j , then for all x minus 1 to 1 , limit as n goes to infinity of f_n of x equals f of x . I.e. this, the sequence of partial sums corresponding to this power series, converges to 1 over 1 minus x point-wise on minus 1 , 1 .

So I said, whenever you come across a definition, you should negate it. But the negation of this definition is not too difficult. A sequence of functions does not converge to another function point-wise if there exists some point, so that when I stick them into f_n of x , f_n of x does not converge to f of x .

So let's look at another example, which is not a power series. Let's say, we take f_n of x to be x to the n , where x is in the closed interval $[0, 1]$. So what's happening here, as n gets very large, there's $1, 1$, there's, I don't know, f of x . And then, as n gets very large, these guys are dropping down even more and more.

And what is-- are we picking up something in the limit? Well, let's look. Well, if x equals 1, and it's pretty clear that the limit as n goes to infinity of f_n of 1, this equals 1. If I stick in 1 here, I get 1 for all n . And therefore, the limit as n goes to infinity of f_n is 1.

Now, if x is in $(0, 1)$, then, I mean, we've done this limit before. The limit as n goes to infinity of f_n of x , this is equal to the limit as n goes to infinity of x^n . Now, x is less than 1. So x is being raised to a higher and higher power. This equals 0.

Thus, what do we conclude? For all x in $(0, 1)$, this sequence of functions x^n converges point-wise to the function f of x , which is equal to 0 if x is in $(0, 1)$, and 1 if x equals 1.

So I draw another picture here of what the limit looks like. And you can start to see this, as m gets large, again, this is becoming more vertical there. But then going to 0. So for any fixed n , it's converging to this picture on the right.

And so, we can already pick up something, or at least answer one of these questions, if we take them as a question about convergence or functions in the point-wise sense. So we could have asked this question now.

After having this definition, suppose f_n 's are continuous converging point-wise to a function f . Is the function f continuous? And what this example shows is that, no, that's not the case. x^n is always continuous. Yet, the limit as n goes to infinity, the point-wise limit is given by this function, which is 0 from 0 to 1, and 1 at x equals 1, which is not a continuous function.

So already, we're kind of seeing that point-wise convergence, which is this first weakest mode of convergence that we as of now can say about power series and is not good enough to ensure that the limit is even continuous. Because this example shows that we have a sequence of continuous functions whose point-wise limit is not continuous.

So as another example, you should always-- I like this kind of last chapter, because you can draw a lot of pictures. So I'm going to draw pictures of f_n . So it's piece-wise linear. So I can write down what the function is, but I don't want to. I'm just going to draw a picture. So f_n of x from 0, 1 to $\mathbb{R}$.

This is how it looks. So there's 1, 1. And what I do is, I go to the point, $1/n$. And the function f_n of x is 0 up until then. And then, it's a linear function connecting $1/n$ to $2/n$ here. And then it connects this point to n/n , so or I should say, $1/n$, $2/n$, connects that to the origin. So that's f_n of x . It's just piece-wise linear.

So for example, if I want to draw f_1 , f_1 would look like there's 1, 0, 2, $1/2$. And let's say I wanted to draw f_{100} . What does that look like? So maybe I should make this one a little bit bigger. There's 1, $1/100$, and then that should be $1/200$.

And then if I go up to 200, is this piece-wise linear function, which is getting-- it's 0 from 1 up to $1/100$. So it's 0 most of the time. And it's 0 at the origin. But in between, it's very tall and very slim.

And so, my claim is that for all x in $(0, 1)$, limit as n goes to infinity of f_n of x equals 0. So this sequence of functions converges point-wise to 0.

So why is this? Well, let's just give a full proof of this rather than me talking it out. I mean, I'll talk it out and give a full proof. So let's look at the easiest spot first.

And I don't even need the formula for these guys. I just need to know that they have this basic characteristic that their point-wise linear, I mean, that they're piece-wise linear connecting 0 to 1 over $2n$ and $2n$ here, and then down to 1 over n 0. And then, there's 0 between that and 1.

So first off, if x equals 0, then all these functions are 0 at the origin. So they equals 0. So that equals 0. So that's fine. So now suppose x is in $(0, 1)$. And so, what do we want to show? I want to show limit as n goes to infinity of f_n of x equals 0.

And here, so what's the point here? Now, there's 1, there's x . So I have to give a-- well, I'm not even going to do an epsilon delta epsilon M argument. I'm just going to show you what happens. So there's x between 0 and 1. Now, let's choose a very large integer so that $1/M$ is less than x .

So here's $1/M$ strictly to the left of x . Now, what is the graph of f_n of x look like for n bigger than or equal to M ? It's 0 from x equals 1 to x equal $1/M$, and then shoots up, and then comes back down over here to 0. But here's the point. It's 0 all the way from 1 to $1/M$. So in particular, at x , f_n of x is 0.

So if I look at this sequence, f_n of x , which I'm trying to show converges to 0, it is 0-- no, it is a f_1 of x , f_2 of x , up to f_{M-1} of x . And then, at f_M of x , so at the-- now, so this is f_M of x spot, it's 0. And this point is only going to the left. So for all n bigger than or equal to M , now this will be $1/n$ will be to the left of x . And therefore, f_n of x will be 0. So this is 0, 0, 0, 0, and so on.

So I have the sequence is eventually 0 for all n bigger than or equal to capital M . So and therefore, the limit, I mean, it's pretty easy to take a limit of a constant sequence. And which proves that this sequence of functions converges point-wise to 0.

Now, I didn't come up with this fancy example for just any old reason. It'll come back in a minute when we start answering some of these questions, or asking them, again, within the context of these two convergence-- ways of converging.

So, so far, I've only given you one definition of convergence-- point-wise convergence of a sequence of functions. And now, I'm going to give a slightly-- I'm going to give a stronger-- it's not slightly, it's much stronger-- definition of convergence of a sequence of functions.

So we have a sequence of functions and a given function from S to R . S is a non-empty subset of R . Then we say the sequence f_n converges point-wise or uniformly to f -- uniformly to f -- uniformly to-- it's the end of the day.

If I start mixing up some of my words, I'm always going to correct it. But the first word out of my mouth may not be the correct one-- converges uniformly to f of x to f if-- now we have an epsilon in statement. For all epsilon positive, there exists an M natural number such that for all n bigger than or equal to M , for all x in S , f_n of x minus f of x is less than epsilon.

Now, I want to make a brief comment. This looks suspiciously like point-wise convergence, if you just wrote down what it means for the limit as n goes to infinity of f_n of x goes to f of x . Except, there's a very subtle and important point. And that is, where does for all x and S appear? For point-wise convergence, you can state point-wise convergence as this being at the start of the line, for all x in S , for all ϵ positive, blah, blah, blah.

Here, it appears at the end of the line of the quantifiers. And this makes a very important difference between point-wise convergence and uniform convergence. Point-wise convergence means, I take a point x , I stick it into f_n of x . That gives me a sequence of numbers. And point-wise convergence says that sequence of numbers converges to f of x . For each x , I get a sequence of numbers, which converges to f of x .

Now, uniform convergence is actually saying something stronger. And I'll say that. So let me, in fact, let me draw a picture that goes with this definition. Let's make S to be an interval. Let's say my limiting function is f is given by this graph.

And so, what I'm going to do is basically shift the graph up and down by ϵ , meaning this length is ϵ , and so is this length, all the way across. So let me shade this in and re-outline-- oh.

So I get this little, if you like, shaded area around my function f . So this is f , f of x . That's the graph. And the shaded part, this is the set of all x and y , such that f of x minus y is less than ϵ . So I get a little tube snaking with f .

Now, what uniform convergence says, is that for all n bigger than or equal to some M , so given ϵ , for all n bigger than or equal to M , if I were to draw the graph of f_n of x , it better fall inside this tube across all of a, b . See, this tube is defined for all x between a, b . So it's making a statement about how close f_n of x is to f of x across the entire set.

Point-wise convergence just says, if I put an x into f_n of x , then eventually those numbers are getting close to f of x . Uniform convergence is a global property. It's saying, across the entire set, as n is getting large, the graph of this f_n is getting very close to the graph of f of x . Not just if I fix an x , the f_n of x at that point are converging to f of x .

So I've said a couple of times that uniform convergence is stronger than point-wise convergence. Let me actually prove this now. So let me prove that following theorem.

So if I have a sequence of functions from S to $\mathbb{R}$, and f_n , rather than write converges to f point-wise or uniformly, I'm going to put an arrow. And then, with the description afterwards, uniformly on S . And then this implies f_n converges to f point-wise on S .

So it's very simple. Again, what is the picture that's going on for uniform convergence is that f_n is getting close to f across the entire set that we're looking at. So certainly at one point, which is all you need for point-wise convergence. Each fixed point we should be getting close. So let ϵ be positive.

So first off, let's fix a number in the set S . So now we want to prove that the limit as n goes to infinity of f_n of c equals f of c . Let ϵ be positive. Since f_n converges to f uniformly, there exists a natural number M_0 such that for all n bigger than or equal to M_0 , for all x in S , f_n of x minus f of x is less than ϵ .

So choose M to be this M_0 , the M that is for this ϵ . Then for all n bigger than or equal to M , let's call this equation star, star with at the single point x equals at c , implies f_n of c minus f of c is less than ϵ . And thus, the limit as n goes to infinity of f of c , f_n of c equals f of c .

So I don't think I have enough time to do the example that I want to do. So I'm just going to leave you on the edge of your seat by stating the following theorem. That in fact, so this is a one-way street. Meaning point-wise convergence does not imply uniform convergence. So we just proved uniform implies point-wise. But the converse does not hold.

And what we'll prove next time is for in the setting of this simple example of x to the n . So-- and so, what we'll prove next time is the following. If I take any b between 0 and 1, then f_n convergence to f uniformly on the set $0, b$. So these functions are defined on $0, 1$. So they're certainly defined on $0, b$ for b less than 1.

But however, this sequence of functions does not converge uniformly to f on $0,1$. So here, this second part, since the f_n 's converge to point-wise on $0, 1$, the second part says that this is a one-way street. This is not a two-way street. These two modes of convergence-- uniform convergence and point-wise are not equivalent. All right. I think we'll stop there.