

Periodic Functions

Periodic functions are functions which repeat: $f(t + P) = f(t)$ for all t . For example, if $f(t)$ is the amount of time between sunrise and sunset at a certain latitude, as a function of time t , and P is the length of the year, then $f(t + P) = f(t)$ for all t , since the Earth and Sun are in the same position after one full revolution of the Earth around the Sun.

We state this explicitly as the following definition: a function $f(t)$ is **periodic** with **period** $P > 0$ if

$$f(t + P) = f(t) \quad \text{for all } t.$$

Example. $f(t) = \sin(2t)$ is periodic with period $P = \pi$. This is true because, for all t ,

$$f(t + \pi) = \sin(2(t + \pi)) = \sin(2t + 2\pi) = \sin(2t) = f(t).$$

Notice, though, that in the example above $f(t) = \sin(2t)$ also has period $P = 2\pi$ and period $P = 3\pi$. In fact, it has period $P = n\pi$ for any integer $n = 1, 2, 3, \dots$

Graphically, a function with period P is one whose graph stays the same if it is shifted P to the left or right.

Base Period

Most periodic functions have a *minimal* period, which is often called either *the* period or the **base period**. For example, $\sin t$ has minimal period is 2π . It follows from this that the minimal period for $\sin(2t)$ is π .

The only exception is the constant function. Every value of $P > 0$ is a period and so it has no minimal period. (We don't allow $P = 0$ to be a period because then every function would be periodic with period $P = 0$.)

Windows

To fully describe a periodic function you only need to specify the period and the value of the function over one full period. We call an interval containing one full period a **window**. Typical choices for windows are $[-P/2, P/2)$ and $[0, P)$, but any interval of length P will work.

Frequency Terminology

Angular frequency, also called **circular frequency** has units of radians/unit time.

Frequency has units of cycles/unit time.

Since one cycle is 2π radians the relationship is

angular frequency = $2\pi \times$ frequency.

The above is the official terminology, but in actual practice many people say frequency when they mean angular frequency. In fact, that has been the general usage earlier in this course where we have called ω the frequency of $\cos(\omega t)$. You will have to use the context to decide exactly which frequency is being used.

For a function with period P the **base angular frequency** ω (also called the **fundamental** angular frequency) means the angular frequency corresponding to the base (or minimal) period P that is

$$\omega = \frac{2\pi}{P}.$$

Fourier Series

We will see that a periodic function with base frequency ω can be written as a sum of sines and cosines whose frequencies are *integer multiples* of ω . This is called the **Fourier series** for the function. That is, sines and cosines, the simplest periodic functions, are the "building blocks" for more general periodic functions.

Later in this session we will see exactly how to compute the Fourier series for a periodic function.

MIT OpenCourseWare
<http://ocw.mit.edu>

18.03SC Differential Equations
Fall 2011

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.