PROFESSOR: Convolution is a tricky concept and hard to understand. But it gives so much insight that it's worthwhile coming to grips with it. The *Convolution: Accumulation* mathlet provides one perspective on the convolution integral. I'll explain the applet using a story.

There's a lake in Iowa with a stream running into it and a stream running out of it. Farm runoff puts phosphate into the lake. The rate of deposition of the phosphate varies over the year. Most in the summer, none in the dead of winter. Suppose the farm is new and that t equals 0, there is no phosphate in the lake.

This is called rest initial conditions. The question is, how much phosphate is there in the lake at some later time, say t. This will be an accumulation of contributions from earlier times, but the earlier contributions will have decayed somewhat by time t.

We can visualize this process using the *Convolution: Accumulation* mathlet. I'll set it up to model what we're looking at. The signal f of t is the rate of input of phosphate into the lake. And I will model it by the menu item f of t is 1 plus cosine b*t. You can see it drawn in the lower graphing window here.

A certain proportion of the phosphate in the lake leaves it during each unit of time. This results in an exponential decay of each contribution. This decay profile is called the weight function, denoted here by g of t. And I will select the weight function, giving exponential decay e to the minus a*t.

Now let's see what happens if I let a small amount of time elapse. How small? Well I can select that by picking the step size. And I will select a step size of one eighth. So now let's let a little bit of time elapse. Say, amount of time of one eighth.

If I click on the time slider at one eighth, I get this picture. t equals 0 happens to occur at mid-summer, so that the signal has a value of 2 in kilograms per unit time.

This phosphate decays away as water comes into the lake and carries it off according to the weight function. And this is indicated by the exponential decay function here in the bottom window.

The weight function is a rate. It needs to get multiplied by the step size before it gives an actual amount of phosphate in the lake. The step size is one eighth, and the product is the signal, the value of the signal at t equals 0, that's 2, times the weight function, that's the exponential decay, times the step size, which is one eighth.

So that gives all together one quarter times that exponential decay. And you see that drawn in the top graphing window here, the value near t equals 0 is a quarter and it decays away exponentially.

Now let's move on to the next time interval. I'll click here, and you see what happens. The signal is now a little bit less and the time is a little bit later. So the weight function is scaled and shifted a little bit differently in this bottom graphing window.

It gets multiplied by the step size one eighth again, and laid down on top of the graph of the phosphate arising from the first time interval. So the sum of those two is a record of the phosphate in the lake arising from these first two time intervals as it decays away in later time.

This process continues. You can see the next few steps by clicking them up. Here's the effect of the third time interval as it decays away, but laid down on top of the preceding contributions to the lake, and so on.

Or I can animate the entire process and watch the thing evolve as time increases. So you can see, the effect is a steady state is reached after a while, you get a sinusoidal total amount of phosphate in the lake. It's delayed a little bit from the peak. The maximum amount of phosphate in the lake occurs a little after mid-summer.

Now imagine shrinking the step size. In the limit, as the step size goes to 0, this process is described by the convolution integral. At time t, all the contributions from time 0 to time t are present. And so we're going to take an integral from u equals 0 to u equals t.

The contribution from time u decays by the factor of g of t minus u by time t. So the contribution from between time u and time u plus du is given by the product f of u, the signal of time u, times g of t minus u, the weight function evaluated at t minus u, times du.

And the integral of this differential, from u equals 0 to u equals t, is the amount of phosphate in the lake at time t. And it's given by the convolution integral.

This system is modeled by a differential equation, namely dx/dt plus a*x equals f of t. The weight function of the operator dx/dt plus a*x is e to the minus a*t.

And the solution to the differential equation with rest initial conditions is given by the convolution integral of the input signal with the weight function. This statement is a general fact for LTI operators, not just first-order operators.