Product of Linear Approximations

Suppose we have two complicated functions and we need an estimate of the value of their product. We could multiply the functions out and then approximate the result, or we could approximate each function separately and then find the product of the two (simple) approximations. Does it matter which we do? Not very much.

Prove that the linear approximation of $f(x) \cdot g(x)$ equals the (linear part of the) product of the linear approximations of $f(x)$ and $g(x)$.

Solution

Let $f(x)$ and $g(x)$ be two functions of x that are defined and differentiable near some value x_0. We know that:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad \text{and} \quad g(x) \approx g(x_0) + g'(x_0)(x - x_0).$$

The product of these two linear approximations is:

$$(f(x_0) + f'(x_0)(x - x_0)) \cdot (g(x_0) + g'(x_0)(x - x_0))$$

$$= f(x_0)g(x_0) + f'(x_0)(x - x_0)g(x_0) + f(x_0)g'(x_0)(x - x_0) + f'(x_0)g'(x_0)(x - x_0)^2$$

$$= f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) + f'(x_0)g'(x_0)(x - x_0)^2.$$

On the other hand, the linear approximation of the product $(f \cdot g)(x)$ is:

$$(f \cdot g)(x_0) + (f \cdot g)'(x_0) \cdot (x - x_0)$$

$$= f(x_0)g(x_0) + (f'(x_o)g(x_0) + g'(x_0)f(x_0))(x - x_0)$$

The difference between the two approximations is $f'(x_0)g'(x_0)(x - x_0)^2$, which is a negligible amount when x is sufficiently close to x_0. For example, if $x - x_0 = 0.01$ then $(x - x_0)^2 = 0.0001$