17.810/17.811 – Game Theory

Lecture 8: Social Choice

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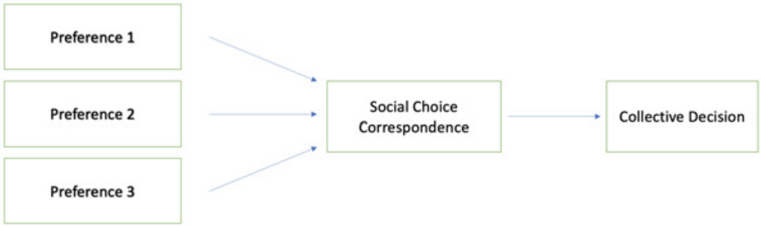
Social Choice

The study of collective decisionmaking.

- How do the preferences of individuals aggregate up to the preferences of groups?



Social Choice

A brief history of social choice:

- Begins with Nicolas de Condorcet (1743-1794) and Jean-Charles de Borda (1733-1799)
- Continues with Charles Dodgson (a.k.a. Lewis Carroll) (1832-1898)
- Takes off in the 20th century: Kenneth Arrow, William Riker, Amartya Sen, Duncan Black
- Becomes a victim of its own success?
  - A multiplicity of "impossibility" theorems leaves the literature unsure of where to go.

An Example

Andrew, Bonnie, and Chuck are three friends deciding how to spend an afternoon together: Museum of Fine Arts (MFA), Walden Pond (WP), or a Red Sox Game (RS).

Their preference orderings are as follows:

Andrew	**Bonnie**	**Chuck**
MFA	WP	RS
WP	RS	WP
RS	MFA	MFA

- Poll the group on their first choice and pick the majority?
- Round-robin tournament?

MFA vs. **WP**, MFA vs. **RS**, **WP** vs. RS $\rightarrow$ **WP wins**.

An Example

We assumed that everyone sincerely reported their preferences. Does anyone have an incentive to vote strategically, that is, not to simply vote their preferred choice?

Andrew	**Bonnie**	**Chuck**
MFA	WP	RS
WP	RS	WP
RS	MFA	MFA

MFA vs. **WP**, MFA vs. **RS**, **WP** vs. RS → **WP wins**.

Can anyone change the outcome, and would they want to?

- **Bonnie**: no – she's getting her top choice.
- **Andrew**: no – pivotal only in WP vs. RS, and he prefers WP
- **Chuck**: yes – choosing MFA in MFA vs. WP → no winner.

An Example

So, would Chuck want to vote strategically? That depends on the back-up rule.

What have we demonstrated with this exercise?

- There are many ways for groups to choose by voting — even by "majority."
- There are multiple majorities rather than "the majority."
- There are multiple forms of preference revelation — sincere and strategic.
- To be complete a voting rule must consider what decision (or alternative decision rule) we revert to if Plan A fails to yield a winner.

Intransitivity of Social Preferences

Consider a slightly different set of preference orderings:

Andrew	**Bonnie**	**Chuck**
MFA	WP	RS
WP	RS	MFA
RS	MFA	WP

(where preferences are the same as before, except Chuck changed his mind and now prefers MFA to WP)

Let's run the round-robin again:

MFA vs. RS $\rightarrow$ RS

RS vs. WP $\rightarrow$ WP

WP vs. MFA $\rightarrow$ MFA

We have a preference cycle: RS P_G MFA P_G WP P_G RS

Agenda Setting

Now consider a two-stage process:

1. Somebody (the "agenda setter") specifies the order of the round-robin tournament
2. Then we do round-robin voting with elimination: winner of first contest faces off against the next challenger, last one standing wins.

Suppose players know each others' preferences and voting is sincere. How should Andrew set the agenda?
WP vs. RS $\rightarrow$ WP, WP vs. MFA $\rightarrow$ MFA

How should Bonnie set the agenda?
MFA vs. RS $\rightarrow$ RS, RS vs. WP $\rightarrow$ WP

Because of the preference cycle, there is a way to set the agenda to get any winner. The agenda setter is very powerful!

Condorcet's Paradox

1	**2**	**3**
a	b	c
b	c	a
c	a	b

When a group $G = \{1, 2, 3\}$ must choose by majority rule from three alternatives $\{a, b, c\}$, the majority preference exhibits a cycle:

$$aP_G b$$

$$bP_G c$$

$$cP_G a$$

yielding an *intransitive* group ordering from individually *transitive* preferences.

Condorcet's Paradox

So what? How bad is this? A back-of-the-envelope calculation:

There are six possible (strict) orderings of alternatives a, b, c:

1	**2**	**3**	**4**	**5**	**6**
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

There are $6x6x6 = 216$ possible "societies" of three people.

How many of these societies exhibit preference cycles?

- There are 6 ($3x2x1$) ways to produce the "forward cycle" $aP_G bP_G cP_G a$ and 6 ways to produce the "backward cycle" $cP_G bP_G aP_G c$.
- Thus 12/216 possible societies (about 6%) have a cycling problem; the others all have a Condorcet winner.

Condorcet's Paradox

What happens if we increase the number of people/alternatives?

More generally, probability of a cycle given m alternatives and n individuals is given by:

$$Pr(m, n) = \frac{\#\text{ of problematic configurations}}{m!}$$

Condorcet's Paradox

Table 4.1
Probability of a Cyclical Majority, $Pr(m,n)$

Number of Alternatives (m)	Number of Voters (n): 3	5	7	9	11	limit
3	.056	.069	.075	.078	.080	.088
4	.111	.139	.150	.156	.160	.176
5	.160	.200	.215			.251
6	.202					.315
limit	≈1.000	≈1.000	≈1.000	≈1.000	≈1.000	≈1.000

SOURCE: William H. Riker, *Liberalism Against Populism* (San Francisco: Freeman, 1982), p. 122

Caveat: in all this we are assuming that any "society" has an equal probability of occurring.

- Of course for many real political problems it's better to think in terms of social groups.

Cycling in Distributive Politics

The "divide the dollar" game produces cyclical majorities.

- Consider three districts of a city, each represented by a city council member, debating how to allocate $1000
- Allocations are approved by majority rule

DISPLAY 4.4
MAJORITY CYCLE IN "DIVIDE THE DOLLARS" GAME

Distribution 1	Distribution 2	Majority Coalition Preferring 2 to 1
[333⅓, 333⅓, 333⅓]	[500, 500, 0]	{E,C}
[500, 500, 0]	[700, 0, 300]	{E,W}
[700, 0, 300]	[333⅓, 333⅓, 333⅓]	{C,W}

Arrow's Theorem

Is Condorcet's Paradox just an idiosyncratic feature of round-robin tournaments? of majority rule?

Can we overcome group-incoherence simply by using other preference aggregation mechanisms?

Arrow's Theorem tells us that Condorcet's Paradox is a problem for any reasonable method of aggregating individual preferences into collective preferences.

Arrow's Theorem

Setup

- There is a group of individuals $G = \{1, 2, ..., n\}$, where n is at least 3
- There is a set of alternatives $A = \{1, 2, ..., m\}$ where m is also at least 3
- The individuals in G are assumed to have preferences over the alternatives in A, called R_i for all $i \in G$

Question

What are the most minimal reasonable assumptions we want to put on individual preferences (R_i) and the social choice rule that aggregates them?

- We will argue that these are absolutely minimal requirements for a "reasonable" rule, but that nonetheless no rule satisfies them all

Some Minimal Reasonable Conditions

Definition (Condition U: Universal Domain)

Each $i \in G$ may adopt any strong or weak complete and transitive preference ordering over the alternatives in A.

Definition (Condition P: Pareto Optimality or Unanimity)

If every member of G prefers j to k (or is indifferent between them), then the group preference must reflect a preference for j over k (or an indifference between them).

Definition (Condition D: Nondictatorship)

There is no distinguished individual $i^* \in G$ whose own preferences dictate the group preference, independent of the other members of G.

Some Minimal Reasonable Conditions

Definition (Condition I: Independence of Irrelevant Alternatives)

If alternatives j and k stand in a particular relationship to one another in each group member's preferences, and this relationship does not change, then neither may the group preference between j and k. This is true even if individual preferences over other (irrelevant) alternatives in A change.

Arrow's Impossibility Theorem

There exists no mechanism for translating the preferences of rational individuals into a coherent group preference that simultaneously satisfies conditions **U**, **P**, **I**, *and* **D**.

Put more dramatically: *any scheme for producing a group choice that satisfies* **U**, **P**, *and* **I** *is either dictatorial or incoherent*.

- There is a trade-off between social rationality and the concentration of power.

Maybe we can tolerate a little incoherence in service of fairness?

- Sure. But there will usually be political entrepreneurs to exploit this vulnerability.

If we can't get coherent social preferences from *even these minimal conditions*, it gets even worse as you layer on more.

Arrow's Theorem and Majority Rule

Arrow's Theorem applies to the entire universe of possible preference aggregation mechanisms. Can we say something more narrow (and optimistic?) about the style of aggregation we care about in democratic politics — majority rule?

Definition (The Method of Majority Rule (MMR))

MMR requires that, for any pair of alternatives, j and k, j is preferred by the group to k (jP_Gk) if and only if the number of group members who prefer j to k exceeds the number of group members who prefer k to j.

Let's start from scratch. What are the reasonable conditions we want to put on an aggregation method that satisfies MMR?

Basic Fairness Conditions for MMR

Definition (Condition A: Anonymity)

Social preferences depend only on the *collection* of individual preferences, not on who has which preference.

Definition (Condition N: Neutrality)

Interchanging the ranks of alternatives j and k in each group member's preference ordering has the effect of interchanging the ranks of j and k in the group preference ordering.

Definition (Condition M: Monotonicity)

If an alternative j beats or ties another alternative k ($jR_G k$) and j rises in some group member's preferences from below k to the same or a higher rank than k, then j now strictly beats k ($jP_G k$).

May's Theorem

A method of preference aggregation over a pair of alternatives satisfies conditions **U**, **A**, **N**, *and* **M** *if and only if it is* **MMR**.

Thus satisfying these conditions defines the method of majority rule. Consequently, if you think MMR is inappropriate for some decision, it's probably to do with one of these conditions.

- What I should have for breakfast decided by a majority-rule vote by everyone who lives on my street?
- Constitutional amendments decided by majority rule?
- Law enforcement officers elected by majority rule?

May's Corollary

It turns out that the May's conditions are just special cases of the Arrow conditions.

May		**Arrow**
U (Universal Domain)	$\rightarrow$	U (Universal Domain)
A (Anonymity)	$\rightarrow$	D (Nondictatorship)
N (Neutrality)	$\rightarrow$	I (IIA)
M (Monotonicity)	$\rightarrow$	P (Unanimity)

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M (Monotonicity)	$\rightarrow$	P (Unanimity)

A: ignore the identity of voters, only count votes

D: no dictators

May's Corollary

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U (Universal Domain)	$\rightarrow$	U (Universal Domain)
A (Anonymity)	$\rightarrow$	D (Nondictatorship)
N (Neutrality)	$\rightarrow$	I (IIA)
M (Monotonicity)	$\rightarrow$	P (Unanimity)

N: interchanging j and k in every member's ordering interchanges j and k in group ordering

I: independence of irrelevant alternatives

May's Corollary

It turns out that the May's conditions are just special cases of the Arrow conditions.

May		**Arrow**
U (Universal Domain)	$\rightarrow$	U (Universal Domain)
A (Anonymity)	$\rightarrow$	D (Nondictatorship)
N (Neutrality)	$\rightarrow$	I (IIA)
M (Monotonicity)	$\rightarrow$	P (Unanimity)

M: if jR_Gk and j rises in one person's ordering, then jP_Gk
P: if everyone member of the group has jPk (jRk), then jP_Gk (jR_Gk)

Arrow's Theorem and May's Theorem

If you're interested in seeing proofs of these implications, see: *Kenneth O. May, "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision," Econometrica 20 (1952): 680–84.*

But the upshot is:

$$MMR \iff \{U, A, N, M\} \text{ (\textbf{May's Theorem})} \tag{1}$$

$$\{U, A, N, M\} \implies \{U, I, P, D\} \text{ (\textbf{May's Corollary})} \tag{2}$$

$$\{U, I, P, D\} \implies P_G \text{ is irrational (\textbf{Arrow's Theorem})} \tag{3}$$

Are these REALLY minimal conditions of fairness?

Let's look at them just one more time.

P (Unanimity): if everyone prefers j to k, then jP_Gk

D (Nondictatorship): seems prima facie unfair

I (IIA): more of a sensibleness condition than a fairness condition, but we probably want it

What about **U**?

- **U** is neither about fairness nor sensibleness. It's a domain requirement — expressing the desire to apply our results to a broad range of contexts.

So let's consider a domain restriction.

Black's Single-Peakedness Theorem

Consider a set A of alternatives from which a group G of individuals must make a choice. If, for every subset of three alternatives in A, one of these alternatives is never worst among the three for any group member, then this is sufficient consensus so that the method of majority rule yields group preferences P_G that are transitive.

Sen's Value-Restriction Theorem

What's so special about being "not worst"?

Take the set of alternatives $A = \{a, b, c, d, e\}$. What if, for $\{a, b, c\}$, all the members of the group agreed that b was not best? What if they agreed c was not middling?

Actually, there's nothing special about "not worst" — any value restriction will do.

Sen's Value-Restriction Theorem

Definition (Condition of Value Restriction)

A group's preferences are value-restricted if, for every collection of three alternatives under consideration, all members of the group agree that one of the alternatives in this collection is either **not best**, **not worst**, or **not middling** (with all members agreeing on which quality the alternative in question was not).

For instance, in $\{a, b, c\}$, all agree that b is not the middle alternative, and in $\{b, d, e\}$ all agree that d is not the best alternative, and for...

Sen's Value-Restriction Theorem

Theorem (Sen's Value-Restriction Theorem)

The method of majority rule yields coherent group preferences if individual preferences are value-restricted.

Some Single-Peaked Preferences

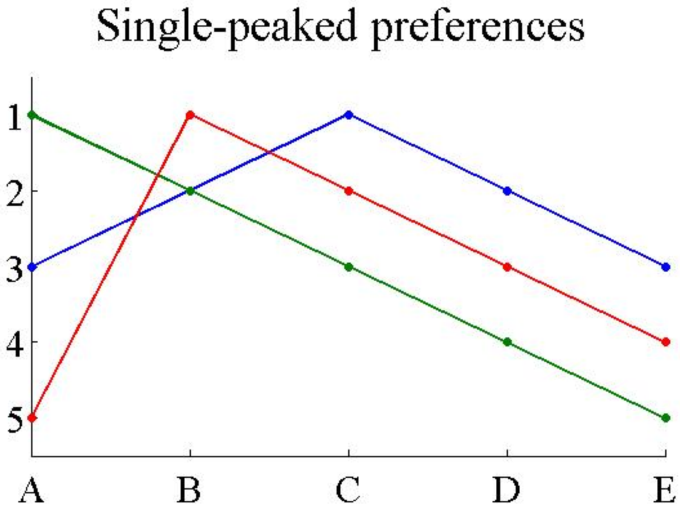


Image by El otro borges. This image is in the public domain. Source: Wikimedia Commons.

Where to Go from Here?

1. **Impose Single-Peaked Preferences!**
   - This is why we've been leaning so hard on quadratic utilities throughout the course.

$$u_i(x) = -(\bar{x}_i - x)^2$$

   - (Linear utilities are single-peaked too.)

$$u_i(x) = -|\bar{x}_i - x|$$

(And this is what I mean when I say Social Choice Theory brought about its own demise.)

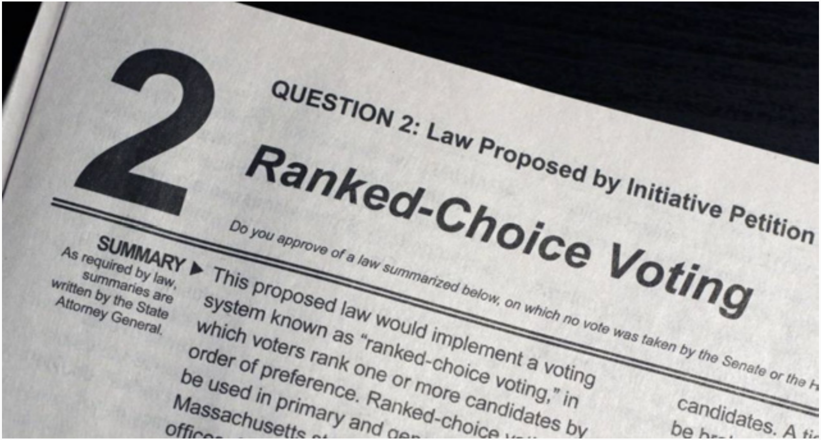
Where to Go from Here?

2. **Structure-Induced Equilibrium**
   - Remember the agenda setter from the first example? There usually is one.
   - In the U.S., we have party leaders, interest groups, the President, Congressional committees...
   - The study of American politics has mostly abandoned social choice in favor of more structured settings like we've been studying throughout the class.
   - Question: how did those institutions come into being?

Where to Go from Here?

3. **The Study of Voting Rules**
   - An extension of social choice that remains highly relevant is the study of different voting rules.
   - On the ballot in Massachusetts in the most recent election: Ranked Choice Voting (defeated by 10 percentage points)
   - (Cambridge is one of a handful of cities in America to have ranked-choice voting for city council.)
   - Another contested one: at-large vs. district elections

Ranked Choice Voting



Ranked Choice Voting: How it Works

1. Voters rank the candidates for a given office by preference on their ballots
2. If a candidate wins an outright majority of first-preference votes (i.e., more than 50 percent), they win
3. If no one wins a majority of first-preference votes, the candidate with the fewest first-preference votes is eliminated
4. All first-preference votes for the failed candidate are eliminated, lifting the second-preference choices indicated on those ballots
5. A new tally is conducted to determine whether any candidate has won an outright majority of the adjusted voters
6. The process is repeated until a candidate wins

Ranked Choice Voting and the Majority Criterion

Definition (The Majority Criterion)

The majority criterion states that if a candidate has a majority of first choice votes, then that candidate should be the winner of the election.

RCV satisfies the majority criterion due to step 2, a desirable feature.

Ranked Choice Voting: Advantages over Simple Majority

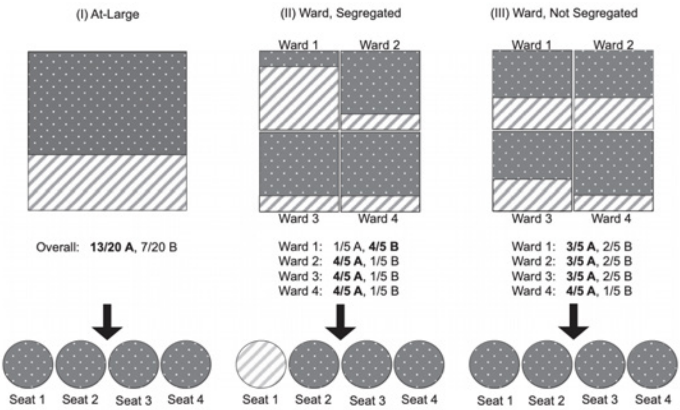
- If there is a clear first-choice preference, the outcome is the same as simple majority rule
- There are no "wasted votes"; RCV empowers people to vote their sincere preference
- Thus it empowers independent and third-party candidates, as well as a greater diversity of viewpoints within the parties $\rightarrow$ potential to alleviate polarization
- Elections are also more competitive when every vote counts $\rightarrow$ more accountability in office

Downsides (?): more complicated and difficult for voters to make sense of.

At-Large vs. District Elections

Another thing to consider is how different forms of majority rule protect some minimal degree of minority representation.

FIGURE 1 Conversion of Votes to Seats, Wards vs. At-Large Districts



Where to Go from Here?

4. **Multidimensional Choice Problems in Applied Survey Research**
   - Conjoint experiments & the Borda Count

Borda Count elections

Another social choice rule is the Borda Count, which works as follows:

1. Voters rank the alternatives
2. The alternatives are assigned points according to their rankings, and these points are summed across voters
3. The candidate with the most points wins

Points	**Rank**	**Voter 1**	**Voter 2**	**Voter 3**	**Voter 4**
4	1st	A	D	C	B
3	2nd	B	B	D	C
2	3rd	D	C	B	D
1	4th	C	A	A	A

Calculate the Borda points for A: $4 + 1 + 1 + 1 = 7$.

Borda Count elections

Interestingly, the Borda Count violates the Majority Criterion: there could be a majority whose first choice is A, but candidate B wins!

- How? If everyone picks B as their second choice.

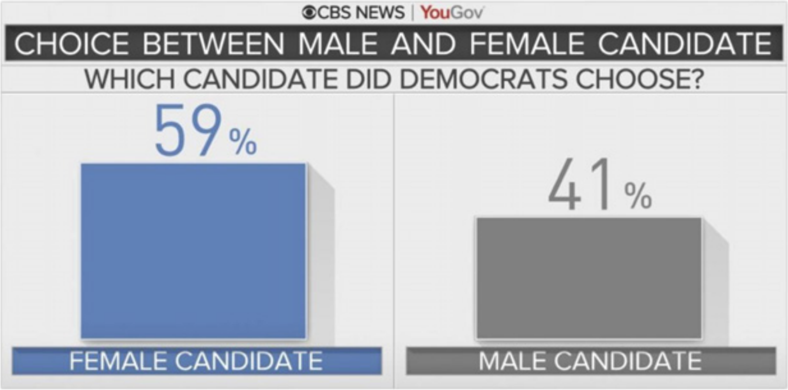
The Connection to Survey Research: Conjoint Experiments

Often survey researchers want to know: what kind of candidate should the Democratic party nominate? What kind of immigration do citizens support? These choice problems are fundamentally multidimensional. So researchers came up with conjoint experiments:

Candidate A	Candidate B
60	35
Black	White
Female	Male
State Attorney General	U.S. Representative
Middle class background	Working class background
From Northeast	From Midwest

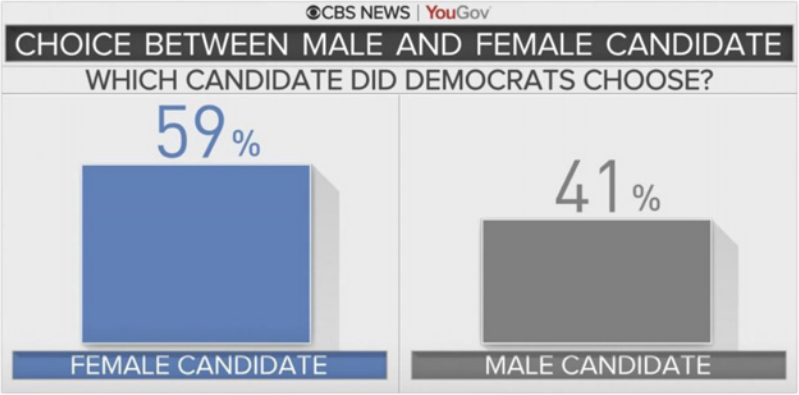
Conjoint Experiments

Voters showed a clear preference for females, all else equal. When given one male and one female, voters selected the female 59 percent of the time. Men and women both preferred female candidates, but women were especially likely to pick females over males - by over 20 percentage points.



Conjoint Experiments & the Borda Count

Except it turns out that running a conjoint experiment is like running a Borda Count election, which violates the majority criterion. So this result doesn't necessarily mean the majority of voters prefer women!



Beyond Conjoint Experiments

Survey tools use preference aggregation rules all the time, but most survey researchers probably never encountered social choice theory.

What other areas would benefit from these insights?