16.001 - Materials & Structures
Problem Set #11

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Question	Points
1	0
2	0
3	0
4	9
5	3
Total:	12

Problems M-11.1 [0 points]
Consider the cantilever beam shown in Figure 1. A linearly changing distributed force acts along the length L of the beam. The equation describing the distributed force is:

$$q(x) = q_0 \frac{x}{L}$$

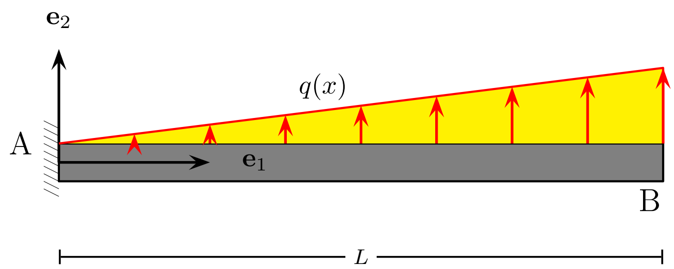


Figure 1: Loaded cantilever beam.

Please do the following:

1.1 (1 point) Write down the governing equations for force and moment equilibrium.

Solution:
The governing equations for force and moment equilibrium in beam theory are

$$\boxed{S'(x) + p_2(x) = 0}$$
$$\boxed{M'(x) + S(x) = 0}$$

The equations can be combined to give

$$M''(x) = p_2(x)$$

1.2 (1 point) Write down the boundary conditions for this problem. Can you find the internal forces and moments in the beam by equilibrium considerations alone?

Solution:
We have two boundary conditions at each end $x = 0$ and $x = L$. At the clamped end $x = 0$, the boundary conditions are:

$$u(0) = 0$$
$$u'(0) = 0$$

and the boundary conditions at the free end $x = L$ are:

$$S(L) = 0$$
$$M(L) = 0$$

Since we have two boundary conditions involving forces/moments ($S(0) = 0$ and $M(0) = 0$) we can solve the shear and moment equilibrium equations to obtain $S(x)$ and $M(x)$. Thus, we can obtain forces/moments using equilibrium considerations alone.

1.3 (1 point) Determine the moment and shear distributions $M(x)$ and $S(x)$.

Solution: To solve for the shear distribution

$$\begin{aligned} S'(x) + p_2(x) &= 0 \\ S'(x) &= -p_2(x) \\ S'(x) &= -q_0 \frac{x}{L} \end{aligned}$$

Integrating:

$$S(x) = -q_0 \frac{x^2}{2L} + c_1$$

Applying the boundary condition $S(L) = 0 \rightarrow c_1 = q_0 \frac{L}{2}$

$$\boxed{S(x) = q_0 \left(\frac{L}{2} - \frac{x^2}{2L} \right)}$$

To solve for the moment distribution

$$\begin{aligned} M'(x) + S(x) &= 0 \\ M'(x) &= -S(x) \\ M'(x) &= q_0 \left(\frac{x^2}{2L} - \frac{L}{2} \right) \end{aligned}$$

Integrating:

$$M(x) = q_0 \left(\frac{x^3}{6L} - \frac{Lx}{2} \right) + c_2$$

Applying the boundary condition $M(L) = 0 \rightarrow c_2 = q_0 \frac{L^2}{3}$

$$\boxed{M(x) = q_0 \left(\frac{x^3}{6L} - \frac{Lx}{2} + \frac{L^2}{3} \right)}$$

1.4 (1 point) Write down the governing equilibrium equation for deflection, and solve for the deflection distribution $u(x)$.

Solution: The moment-curvature relation is

$$EIu^{''}(x) = M(x) = q_0\left(\frac{x^3}{6L} - \frac{Lx}{2} + \frac{L^2}{3}\right)$$

Dividing by EI and integrating:

$$u^{'}(x) = \frac{q_0}{EI}\left(\frac{x^4}{24L} - \frac{Lx^2}{4} + \frac{L^2x}{3}\right) + c_3$$

Applying the boundary condition $u'(0) = 0 \rightarrow c_3 = 0$:

$$u^{'}(x) = \frac{q_0}{EI}\left(\frac{x^4}{24L} - \frac{Lx^2}{4} + \frac{L^2x}{3}\right)$$

Integrating

$$u(x) = \frac{q_0}{EI}\left(\frac{x^5}{120L} - \frac{Lx^3}{12} + \frac{L^2x^2}{6}\right) + c_4$$

Applying the boundary condition $u(0) = 0 \rightarrow c_4 = 0$

$$\boxed{u(x) = \frac{q_0}{EI}\left(\frac{x^5}{120L} - \frac{Lx^3}{12} + \frac{L^2x^2}{6}\right)}$$

Exercise: Check that the deflection distribution statisfies all of the boundary conditions.

○ **Problems M-11.2** [0 points]

A pile of sand has been placed on a simply-supported bridge as shown in Figure 2. The pile has the total mass M and features a sinusoidal mass distribution.

The bridge has the total length L and a rectangular cross-section of width w (along the x_3-direction) and thickness t (along the x_2-direction). It is made from an isotropic linear elastic material with Young's modulus E, moment of inertia I, and can be considered massless.

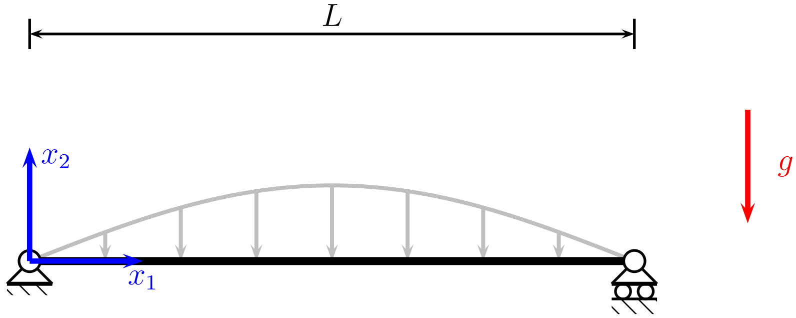


Figure 2: A simply-supported bridge loaded by a pile of sand.

2.1 (1 point) Is this system statically determinate?

Solution: Yes, the system is statically determinate. The two unknown vertical reaction forces at $x_1 = 0$ and at $x_1 = L$ can be computed from the force equilibrium in the x_2-direction and the moment equilibrium. (Each reaction force corresponds to half the weight of the pile of sand.)

2.2 (1 point) Find an expression for the distributed load $p_2(x_1)$ that acts on the bridge.

Solution:
As described in the problem statement, the distribution $m(x_1)$ of the mass of the pile of sand per unit length is of the form

$$m(x_1) \;=\; A \sin\left(\pi \frac{x_1}{L}\right) \tag{1}$$

where the constant A can be determined from the total mass M of the pile as follows:

$$\begin{aligned} M \;&=\; \int_{x_1=0}^{x_1=L} m(x_1) \mathrm{d}x_1 \\ &=\; \int_{x_1=0}^{x_1=L} A \sin\left(\pi \frac{x_1}{L}\right) \mathrm{d}x_1 \\ &=\; \frac{2AL}{\pi} \\ \Rightarrow A \;&=\; \frac{\pi}{2} \frac{M}{L} \end{aligned}$$

Therefore, the distribution of the mass of the pile of sand per unit length is

$$m(x_1) \;=\; \frac{\pi}{2} \frac{M}{L} \sin\left(\pi \frac{x_1}{L}\right) \tag{2}$$

so that the distributed load $p_2(x_1)$ acting on the beam becomes

$$p_2(x_1) \;=\; -g\, m(x_1) \;=\; -\frac{\pi}{2} \frac{Mg}{L} \sin\left(\pi \frac{x_1}{L}\right). \tag{3}$$

2.3 (1 point) Determine the bending moment distribution $M_3(x_1)$ in the bridge resulting from the weight of the pile of sand. Determine its maximum value and the location x at which it occurs.

Solution:

The following equilibrium equations for a beam were derived in class:

$$\begin{aligned}\frac{\mathrm{d}S_2}{\mathrm{d}x_1} + p_2(x_1) &= 0\\ \frac{\mathrm{d}M_3}{\mathrm{d}x_1} + S_2(x_1) &= 0\end{aligned}$$

Combining them yields:

$$\begin{aligned}\frac{\mathrm{d}^2M_3}{\mathrm{d}x_1^2} &= p_2(x_1)\\ &= -\frac{\pi}{2}\frac{Mg}{L}\sin\left(\pi\frac{x_1}{L}\right) \qquad (4)\end{aligned}$$

Integrating the above expression twice yields

$$M_3(x_1) = \frac{\pi}{2}\frac{Mg}{L}\left(\frac{L}{\pi}\right)^2\sin\left(\pi\frac{x_1}{L}\right) + C_1x_1 + C_2 \qquad (5)$$

where C_1 and C_2 are two unknown integration constants that are to be determined from boundary conditions. Since the bridge is simply-supported, the boundary conditions on $M_3(x_1)$ are:

$$\begin{aligned}M_3(x_1 = 0) &= 0\\ M_3(x_1 = L) &= 0\end{aligned}$$

Inserting (5) into the boundary conditions reveals $C_1 = C_2 = 0$ so that

$$M_3(x_1) = \frac{LMg}{2\pi}\sin\left(\pi\frac{x_1}{L}\right). \qquad (6)$$

Trivially, the maximum bending moment and the location at which it occurs are:

$$M_{max} = \frac{LMg}{2\pi} \qquad x_{M,max} = \frac{L}{2} \qquad (7)$$

2.4 (1 point) Compute the resulting deflection $\bar{u}_2(x_1)$ of the bridge, find its maximum value and the location x at which it occurs.

Solution:
The equation relating the beam deflection $\bar{u}_2(x_1)$ to the bending moment $M_3(x_1)$ derived in class is:

$$M_3(x_1) = EI\frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} \qquad (8)$$

Since the bridge has a constant rectangular cross-section, we have

$$I = \frac{wt^3}{12} \tag{9}$$

here. Rearranging Eq. (8) yields:

$$\begin{aligned} \frac{\mathrm{d}^2\bar{u}_2}{\mathrm{d}x_1^2} &= \frac{1}{EI} M_3(x_1) \\ &= \frac{1}{2\pi}\frac{LMg}{EI}\sin\left(\pi\frac{x_1}{L}\right) \end{aligned} \tag{10}$$

Integrating the above expression twice yields

$$\bar{u}_2(x_1) = -\frac{1}{2\pi}\frac{LMg}{EI}\left(\frac{L}{\pi}\right)^2 \sin\left(\pi\frac{x_1}{L}\right) + C_3x_1 + C_4 \tag{11}$$

where C_3 and C_4 are two unknown integration constants that are to be determined from boundary conditions. Again, since the bridge is simply-supported, the boundary conditions on $\bar{u}_2(x_1)$ are:

$$\begin{aligned} \bar{u}_2(x_1 = 0) &= 0 \\ \bar{u}_2(x_1 = L) &= 0 \end{aligned}$$

Inserting (11) into the boundary conditions reveals $C_3 = C_4 = 0$ so that

$$\begin{aligned} \bar{u}_2(x_1) &= -\frac{1}{2\pi^3}\frac{L^3Mg}{EI}\sin\left(\pi\frac{x_1}{L}\right) \\ &= -\frac{6}{\pi^3}\frac{L^3Mg}{Ewt^3}\sin\left(\pi\frac{x_1}{L}\right). \end{aligned}$$

Trivially, the maximum displacement (in absolute value) and the location at which it occurs are:

$$u_{2,max} = \frac{L^3Mg}{2EI\pi^3} \qquad x_{u_2,max} = \frac{L}{2} \tag{12}$$

2.5 (1 point) Finally, plot $M_3(x)$ and $\bar{u}_2(x)$.

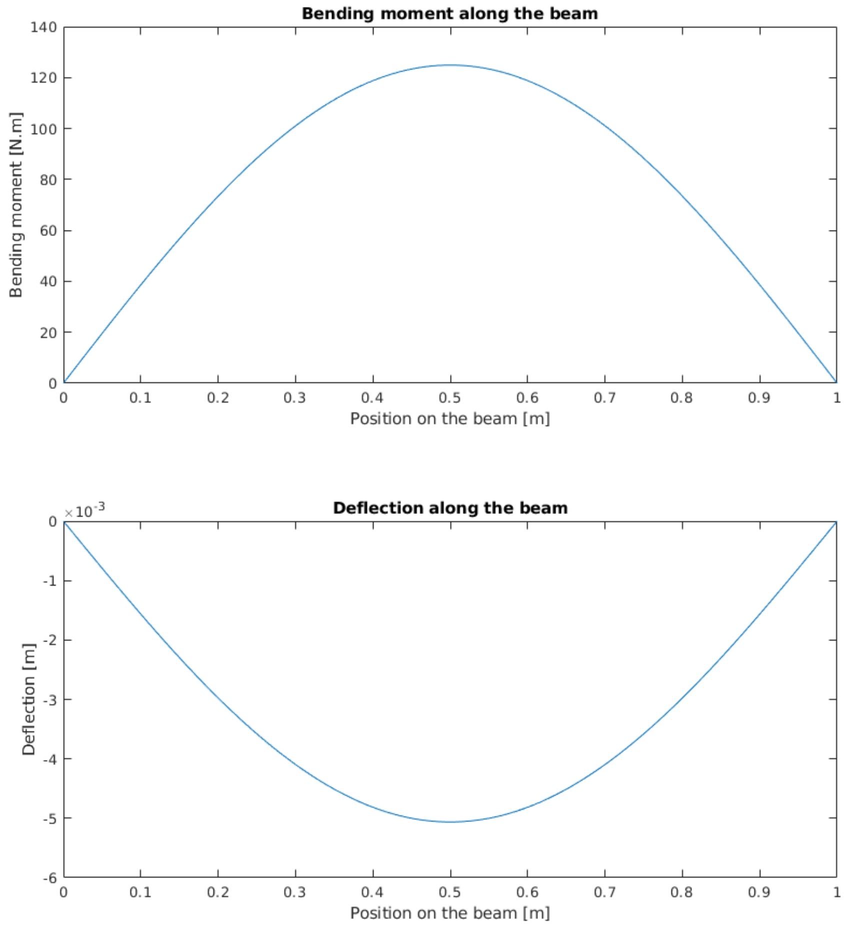
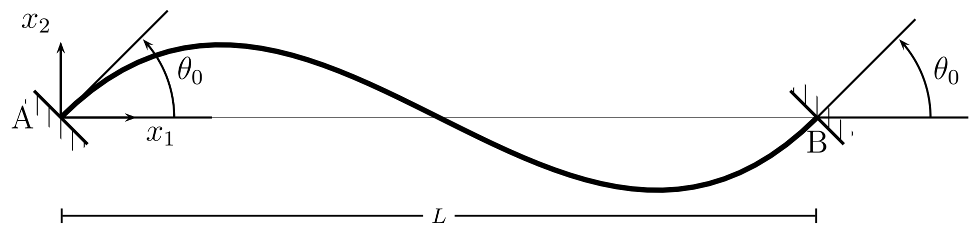


Figure 3: Bending moment and deflection along the beam

Problems M-11.3 [0 points]
The built-in beam shown in Figure 4 has a length L, and bending stiffness EI. The beam is subject to a rotation at both ends by a small angle θ_0 but is not allowed to deflect at either end.



3.1 (1 point) Write down the equations governing the distribution of the following functions: deflection $u(x)$, bending moment $M(x)$ and shear $S(x)$. Indicate what principle each equation represents. Show that you can combine these equations to obtain a single ordinary differential equation governing beam bending which reads as follows:

$$EIu^{(IV)}(x) = 0$$

Solution:

$$\text{equilibrium of moments: } M' + S = 0$$
$$\text{equilibrium of transverse forces: } S' + q = 0$$
$$\text{compatibility and constitutive law: } M = EIu''$$

Combine the three, use $q = 0$ to obtain sought result.

3.2 (1 point) Write down the boundary conditions for this problem.

Solution: The boundary conditions for this problem are

$$u(0) = u(L) = 0$$
$$u'(0) = u'(L) = \theta_0$$

3.3 (1 point) Based on the boundary conditions of the problem, explain if this problem is statically determinate or indeterminate? Provide a short but complete explanation of why that is the case.

Solution: The system is statically indeterminate. All the boundary conditions are of the kinematic type. There are no boundary conditions for the moment and/or the shear that would allow to integrate the equilibrium equations independently.

3.4 (1 point) Using the boundary conditions you determined in part (**2**), find the solution for the deflection of the beam $u(x)$, the moment $M(x)$ and the shear $S(x)$. You should obtain the following result:

$$\begin{aligned}
u(x) &= \theta_0 L\left(\frac{x}{L}\right)\left[2\left(\frac{x}{L}\right)^2 - 3\left(\frac{x}{L}\right) + 1\right] \\
u'(x) &= \theta_0\left[6\left(\frac{x}{L}\right)^2 - 6\left(\frac{x}{L}\right) + 1\right] \\
M(x) &= EI\frac{\theta_0}{L}\left[12\left(\frac{x}{L}\right) - 6\right] \\
S(x) &= -12\theta_0\frac{EI}{L^2}
\end{aligned}$$

Solution: The general solution of the governing equation is a cubic polynomial:

$$u(x) = Ax^3 + Bx^2 + Cx + D$$

The boundary conditions imply:

$$\begin{aligned}
u(0) = 0 &\rightarrow \boxed{D = 0} \\
u'(0) = \theta_0 &\rightarrow \boxed{C = \theta_0} \\
u(L) = 0 &\rightarrow AL^3 + BL^2 + \theta_0 L = 0 \\
u'(L) = \theta_0 &\rightarrow 3L^2A + 2LB + \cancel{\theta_0} = \cancelto{0}{\theta_0}
\end{aligned}$$

From the last two we get:

$$\boxed{A = 2\frac{\theta_0}{L^2}}, \boxed{B = -3\frac{\theta_0}{L}}$$

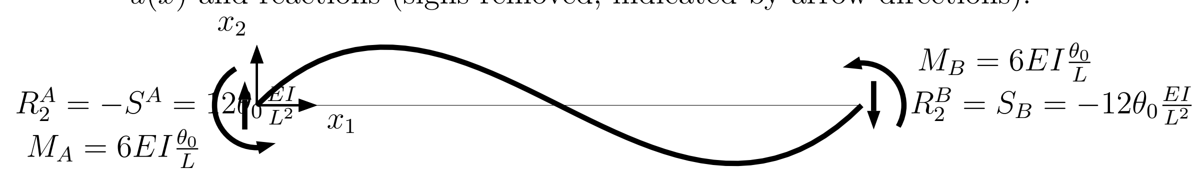
Substituting the coefficients back into the solution we obtain:

$$\begin{aligned}
u(x) &= \theta_0\left(\frac{2}{L^2}x^3 - \frac{3}{L}x^2 + x\right) \\
u'(x) &= \theta_0\left(\frac{6}{L^2}x^2 - \frac{6}{L}x + 1\right) \\
M(x) = EIu''[x] &= EI\frac{\theta_0}{L}\left(\frac{12}{L}x - 6\right) \\
S(x) = -M'(x) &= -12\theta_0\frac{EI}{L^2}
\end{aligned}$$

3.5 (1 point) Sketch all four functions and based on their shapes, provide an informed discussion of the behavior of the beam under this loading. The discussion should include: the moment and shear force reactions at the supports with their sign, the type of moment and shear distribution and how they are related and how they relate to the deformed shape of the beam, the extreme values of these quantities and where they take place.

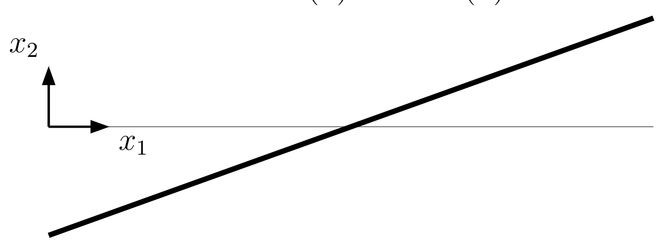
Solution: The functions are represented below:

$u(x)$ and reactions (signs removed, indicated by arrow directions):

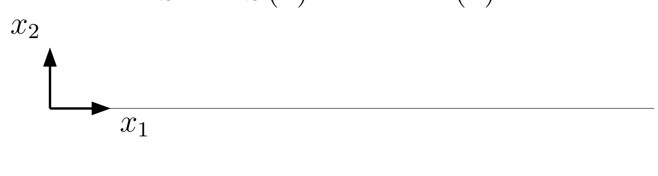


Rotation $u'(x)$:

Moment $M(x) = EIu''(x)$:



Shear $S(x) = -EIu'''(x)$:



It can be seen that the deformed shape is maintained by two moment reaction of the same sign which maintain the imposed rotation at both ends. The moment distribution is linear and zero at the middle. This is where the deflection has an inflection point. To the left, the concavity is negative and increases toward the left support, where the negative moment is maximum. To the right, the concavity is positive and increases toward the support, where the positive moment is maximum. The moment reactions require a counter-couple which is provided by the shear reactions as shown in the figure which implies a constant nagative shear distribution.

○ **Problems M-11.4** [9 points]

Consider the beams shown in Figures 5-7. The beams have a constant Young's Modulus E, moment of inertia I, width b, and height h.

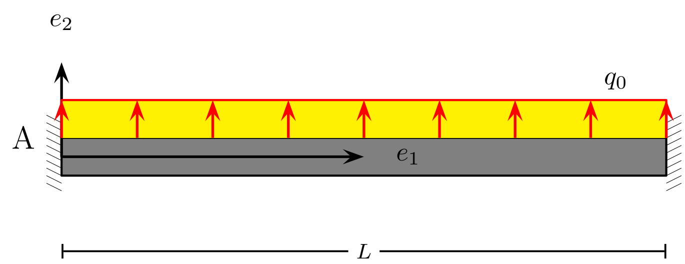


Figure 5: First configuration

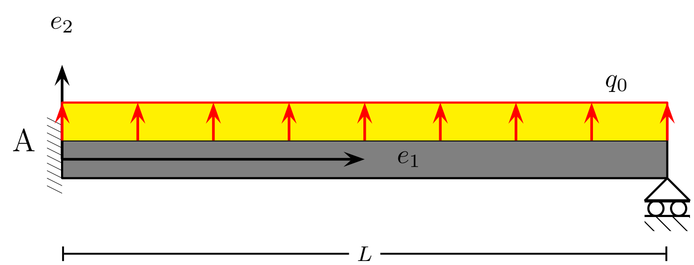


Figure 6: Second configuration

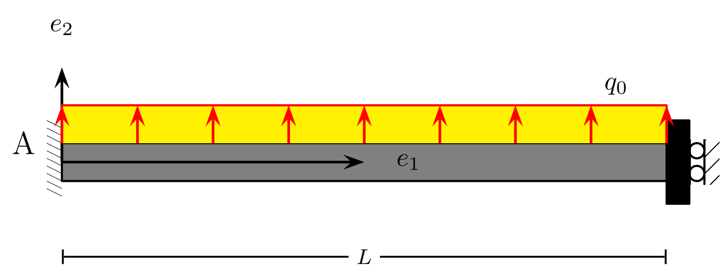


Figure 7: Third configuration

By integration of the governing equations, obtain the following:

- The beam deflection distribution $u(x)$
- The internal bending moment distribution $M(x)$
- The internal shear force distribution $S(x)$
- The maximum deflection, bending moment and shear force, together with the value of x at which they occur respectively
- Finally, plot $u(x)$, $M(x)$, and $S(x)$

4.1 (3 points) For the first beam

4.2 (3 points) For the second beam

4.3 (3 points) For the third beam

Problems M-11.5 [3 points]

In the figures below, two cantilever beams of length L that are rigidly clamped at $x_1 = 0$ are shown. Both beams were straight in their initial undeformed (stress-free) configuration. They were then deformed as follows: The end at $x_1 = L$ of the beam in Figure 8 was rotated by the angle α without being displaced; the end at $x_1 = L$ of the beam in Figure 9 was vertically displaced by the distance δ.

Both beams are made from a homogeneous linear elastic material with Young's modulus E, and their constant cross-sections possess the moment of inertia I.

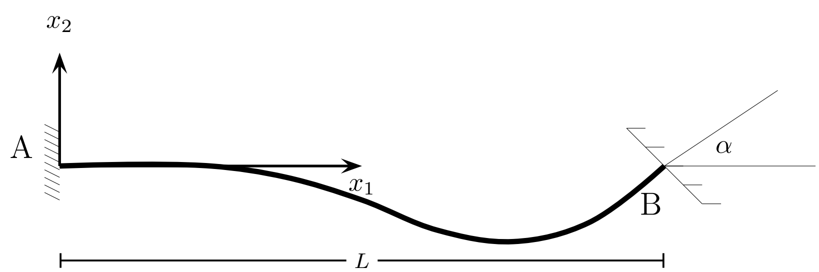


Figure 8: The end of this initially straight cantilever beam was rotated by the angle α without being displaced.

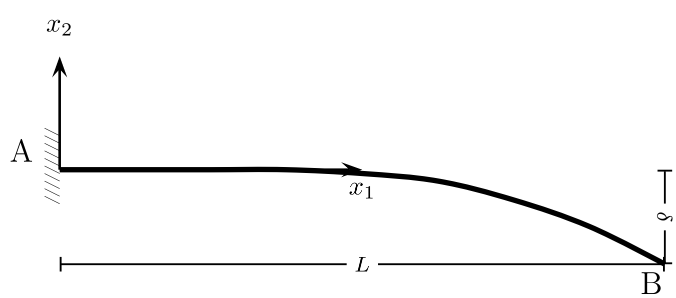


Figure 9: The end of this initially straight cantilever beam was displaced in the negative x_2-direction by the distance δ.

Please do the following for each of the beams:

5.1 (1 point) State the boundary conditions.

5.2 (1 point) State whether the beam system is statically determinate or statically indeterminate.

5.3 (1 point) Obtain the bending moment $M_3(x_1)$, the shear force $S_2(x_1)$, the deflection $\bar{u}_2(x_1)$, and the slope $\frac{\mathrm{d}\bar{u}_2}{\mathrm{d}x_1}$ along the length of the beam. Try to find the most efficient approach, and explain which equations to use and why. (Take into account whether or not you can solve for the shear force and the bending moment without the use of the moment-curvature relation $M_3(x_1) = EI\bar{u}_2''(x_1)$).