16.001 - Materials & Structures
Problem Set #10

Instructors:	Raúl Radovitzky
	Zachary Cordero
Teaching Assistants:	Grégoire Chomette
	Michelle Xu
	Daniel Pickard

Department of Aeronautics & Astronautics
M.I.T.

Question	Points
1	6
2	10
3	12
Total:	28

Problems M-10.1 [6 points]
A turbine blade of length $L = 0.5m$ in operation rotates around $\mathbf{e_2}$ with an angular velocity $\omega = 5000$rpm, as shown in Figure 1. The blade is constrained from extensional motion by a ring housing (i.e. the displacement $u(L) = 0$ at the extremity A), but there is frictionless sliding between the rotating blade and the ring surface. The blade is made of a titanium alloy with a mass density $\rho = 8470 kg \cdot m^{-3}$, a Young's modulus E = 80GPa, a cross-sectional area of $A_0 = 10cm^2$ and a yield stress of $\sigma_y = 400MPa$.

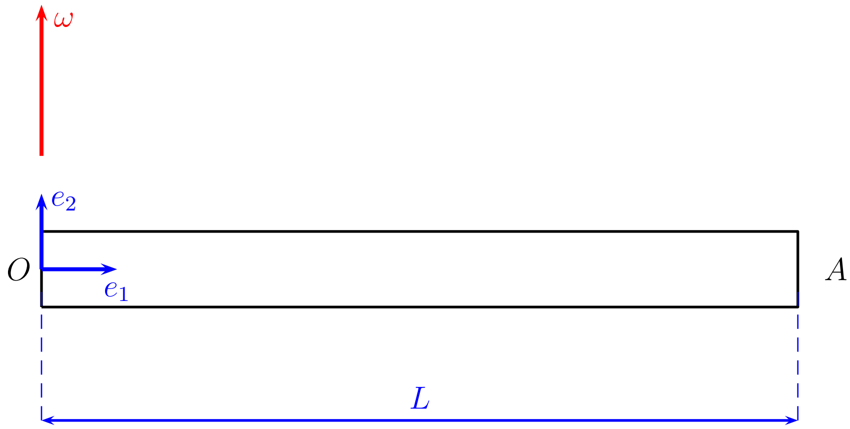


Figure 1: Rotating blade

1.1 (1 point) Write down the applicable governing equations and boundary conditions for this problem. What principles do they represent? Can this problem be solved by static considerations alone?

Solution: The governing equations and boundary conditions are given below. The first three equations represent the principles of equilibrium, compatibility and constitutive law respectively.

$$N' + P = 0 \tag{1}$$

$$\epsilon_{11} = \overline{u}_1' \tag{2}$$

$$N = EA(x)\overline{u}_1' \tag{3}$$

The boundary conditions are $u(0) = 0$ and $u(L) = 0$(zero displacement at both ends of the blade). The problem is statically indeterminate since (1) requires a boundary condition but we don't know the stress at $x = L$.

1.2 (3 points) Integrate the resulting equation(s) and apply the boundary conditions to obtain the following solution field distributions along the axis of the bar: displacement, strain, stress.

Solution: Using (3) in (1) we obtain

$$(EA_0\overline{u}_1')' + p = 0 \tag{4}$$

p is the centripetal force acting on the blade $p = \rho\omega^2 x\, A_0$. With this (4) simplifies to this second order ODE:

$$\overline{u}_1'' + \frac{\rho\omega^2}{E}x = 0 \tag{5}$$

Integrating twice we get

$$u(x) = -\frac{\rho\omega^2 x^3}{6E} + Bx + C \tag{6}$$

where B and C are arbitrary constants determined from the boundary conditions ($u(0) = 0$ and $u(L) = 0$) $B = \frac{\rho\omega^2}{E}\frac{L^2}{6}$ and $C = 0$. Now the displacement can be written in the following explicit form:

$$u(x) = -\frac{\rho\omega^2 x^3}{6E} + \frac{\rho\omega^2 L^2 x}{6E} \tag{7}$$

The strain can be written as:

$$\epsilon_{11} = \overline{u}_1' = -\frac{\rho\omega^2 x^2}{2E} + \frac{\rho\omega^2 L^2}{6E} \tag{8}$$

The stress can be written as:

$$\sigma_{11}(x) = E\epsilon_{11}(x) = -\frac{\rho\omega^2 x^2}{2} + \frac{\rho\omega^2 L^2}{6} \tag{9}$$

1.3 (1 point) What is the maximum stress and where does it happen? Will the material yield plastically for the data given?

Solution: The maximum stress happens at $x = 0$ and its value is:

$$\sigma_{11}^{\text{max}} = \frac{\rho\omega^2 L^2}{6} \tag{10}$$

For the data given, this gives $\sigma_{11}^{\text{max}} \approx 187MPa$. This stress clearly is less than the yield stress of the material which means that the material will not yield plastically.

1.4 (1 point) What is the maximum displacement and where does it happen?

Solution: To find the location of the maximum displacement, we set $\overline{u}_1'(x) = \epsilon_{11} = 0$ as shown below:

$$-\frac{\rho\omega^2 x^2}{2E} + \frac{\rho\omega^2 L^2}{6E} = 0 \rightarrow x = \frac{L}{\sqrt{3}}$$

The maximum displacement can be found using $x = \frac{L}{\sqrt{3}}$ in (7) and we obtain

$$u^{\text{max}} = \frac{(\rho w^2)L^3}{9\sqrt{3}E} = 0.000233\text{m} \tag{11}$$

Problems M-10.2 [10 points]
(M14, M15)

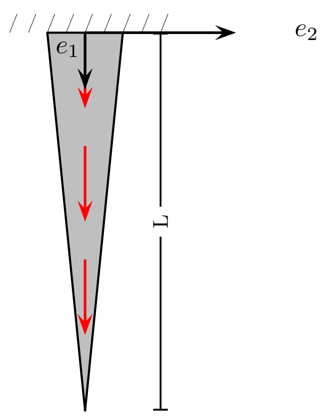
Consider the schematic below which depicts a nail being driven into a piece of wood. The nail has an elastic modulus E, length L, and a radius which varies linearly along its length according to the formula

$$R(x_1) = R_0 \left(1 - \frac{x_1}{L}\right) \tag{12}$$

Along its length, friction between the nail and the wood creates a distributed load per unit length $p_{dist}(x_1)$. The wood exerts a pressure normal to the surface of the nail proportional to the depth x_1 according to the following expression:

$$p(x_1) = \frac{p_0 x_1}{L}$$

The friction coefficient between the nail and wood is μ.



2.1 (2 points) Find an expression for $p_{dist}(x_1)$ in terms of the problems parameters.

Solution: First, lets find the quantity $\frac{dP}{da}$(the infinitesimal axial load over area) for the nail. This can be described as:

$$\frac{dP}{dA} = p(x_1)\mu = -\frac{p_0\mu}{L}x_1$$

integrate to get $p_{dist}(x_1)$.

$$\begin{aligned}
p_{dist}(x_1)dx_1 &= -\int_0^{2\pi} \frac{p_0\mu}{L} x_1 r d\theta dx_1 \\
p_{dist}(x_1) &= -\int_0^{2\pi} \frac{p_0\mu}{L} x_1 r d\theta \\
&= -2\pi \frac{p_0\mu}{L} r x_1 \\
&= -2\pi \frac{p_0\mu}{L} R(x_1) x_1 \\
&= -2\pi \frac{p_0\mu}{L} R_0 \left(1 - \frac{x_1}{L}\right) x_1 \\
&= -2\pi \frac{p_0\mu}{L} R_0 \left(x_1 - \frac{x_1^2}{L}\right)
\end{aligned}$$

The distributed force in the nail is then:

$$\boxed{p_{dist} = -2\pi \frac{p_0\mu}{L} R_0 \left(x_1 - \frac{x_1^2}{L}\right)} \tag{13}$$

2.2 (2 points) Integrate the equilibrium equation in closed form to obtain the load distribution $N(x_1)$. Determine the force N required to drive the nail into the ground farther.

Solution: Start with the equilibrium equation and integrate:

$$N'(x_1) = -p_{dist}(x_1) = 2\pi \frac{p_0 \mu}{L} R_0 \left(x_1 - \frac{x_1^2}{L} \right)$$
$$N(x_1) = \frac{2\pi p_0 \mu R_0}{L} \left(\frac{x_1^2}{2} - \frac{x_1^3}{3L} \right) + C_1$$

Apply the boundary condition $N(x_1 = L) = 0$.

$$\rightarrow \ C_1 = -\frac{\pi p_0 \mu R_0 L}{3}$$
$$N(x_1) = \frac{2\pi p_0 \mu R_0}{L} \left(\frac{x_1^2}{2} - \frac{x_1^3}{3L} \right) - \frac{\pi p_0 \mu R_0 L}{3} = \pi p_0 \mu R_0 \left(\frac{x_1^2}{L} - \frac{2x_1^3}{3L^2} - \frac{L}{3} \right)$$

The force required to drive the pillar into the ground farther can be defined as $N(x_1 = 0)$.

$$N(0) = -\frac{\pi p_0 \mu R_0 L}{3}$$

2.3 (1 point) Determine the stress field along the nail just before it moves farther.

Solution: The stress field can be found as follows:

$$\sigma_{11}(x_1) = \frac{N(x_1)}{A(x_1)} = \frac{N(x_1)}{\pi R(x_1)^2} = \frac{\pi p_0 \mu R_0 \left(\frac{x_1^2}{L} - \frac{2x_1^3}{3L^2} - \frac{L}{3}\right)}{\pi R_0^2 \left(1 - \frac{x_1}{L}\right)^2} \qquad = \frac{p_0\mu}{R_0} \frac{\left(\frac{x_1^2}{L} - \frac{2x_1^3}{3L^2} - \frac{L}{3}\right)}{\left(1 - \frac{x_1}{L}\right)^2}$$

Notice that the numerator may be factored as

$$\frac{x_1^2}{L} - \frac{2x_1^3}{3L^2} - \frac{L}{3} = -\frac{1}{3}\left(1 - \frac{x_1}{L}\right)^2 (L + 2x_1) \tag{14}$$

Simplifying then gives the final stress field:

$$\sigma_{11}(x_1) = \frac{p_0\mu}{R_0} \frac{-\frac{1}{3}\cancel{\left(1 - \frac{x_1}{L}\right)^2}(L + 2x_1)}{\cancel{\left(1 - \frac{x_1}{L}\right)^2}} \boxed{\sigma_{11}(x_1) = \frac{p_0\mu}{-3R_0}(L + 2x_1)} \tag{15}$$

2.4 (2 points) Determine the displacement field in the nail as it starts to move.

Solution: Start with

$$Eu'(x_1) = \sigma_{11}$$

and integrate

$$u'(x_1) = \frac{\sigma_{11}}{E} = \frac{p_0\mu}{-3R_0E}(L + 2x_1)$$
$$u(x_1) = \frac{p_0\mu}{-3R_0E}\left(Lx_1 + \frac{x_1^2}{2}\right) + c_2$$

Applying the boundary condition u(0) $= 0 \rightarrow c_2 = 0$. Thus the displacement field becomes

$$\boxed{u(x_1) = \frac{p_0\mu}{-3R_0E}\left(Lx_1 + \frac{x_1^2}{2}\right)}$$

2.5 (3 points) Find the maximum values of stress and displacement and their locations.

Solution: Our expressions for displacement and stress are:

$$u(x_1) = \frac{p_0\mu}{-3R_0E}\left(Lx_1 + \frac{x_1^2}{2}\right)$$
$$\sigma_{11}(x_1) = \frac{p_0\mu}{-3R_0}(L + 2x_1)$$

We can find the maxima of these functions by taking their derivatives and setting them equal to zero:

$$u'(x_1) = \frac{p_0\mu}{-3R_0E}(L + x_1)$$
$$\sigma'_{11}(x_1) = \frac{2p_0\mu}{-3R_0}$$

Unfortunately, neither of these functions have zero derivatives within our relevant region $x_1 \in [0, L]$. Thus, the maximum values must occur at one of the endpoints $x_1 = 0$ or $x_1 = L$. By inspection, we find that the maximum (magnitude) displacement occurs at $\boxed{x_1 = L}$ and is given by:

$$\boxed{|u(x_1)|_{\max} = \frac{p_0\mu L^2}{2R_0E}}$$

The maximum (magnitude) value of the stress also occurs at $\boxed{x_1 = L}$ and is (compressive):

$$\boxed{|\sigma_{11}(x_1)|_{\max} = -\frac{p_0\mu L}{R_0}}$$

Note: Recall that we imposed a boundary condition that $N(x_1 = L) = 0$. However, even though the force in the nail at $x_1 = L$ is zero, the stress $\sigma_{11}(x_1 = L)$ is non-zero (in fact the maximum compressive stress occurs here). This is because we have a singularity at $x_1 = L$ as both force $N(x_1)$ and cross-sectional area $A(x_1)$ of the nail go to zero here ($\sigma_{11}(x_1 = L) = \frac{N(x_1=L)}{A(x_1=L)} \rightarrow \frac{0}{0}$), leading to a strange and remarkable result of nonzero stress at this location!

○ **Problems M-10.3** [12 points]

The vertical rod shown in Figure 2 is made from an isotropic homogeneous linear elastic material (Young's modulus E, coefficient of thermal expansion α). It features a circular cross-section with varying radius $r(x_1)$ given by:

$$r(x_1) = R\sqrt{4 - 3\frac{x_1}{L}}$$

The rod is constrained at both its ends. It has the length L in the undeformed configuration at the reference temperature T_{ref}. It is subjected to a temperature change $\Delta T(x_1) = \Delta T_0 + kx_1$ which varies linearly with the spatial coordinate x_1.

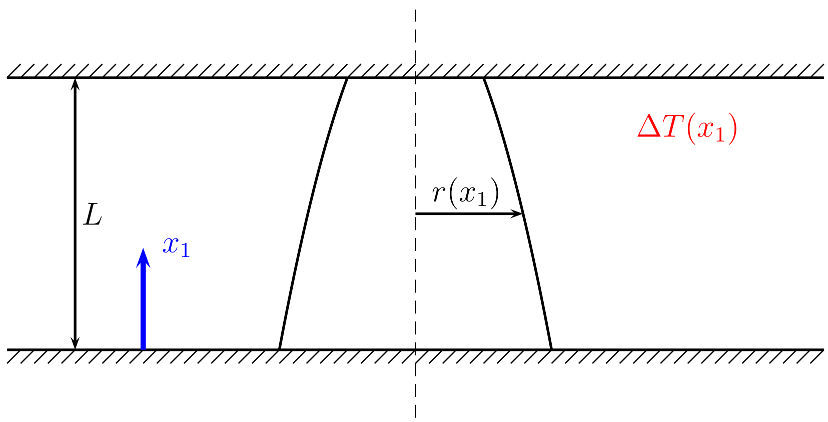


Figure 2: Constrained rod subjected to a temperature change $\Delta T(x_1)$.

3.1 (1 point) Is this system statically determinate?

Solution:
No, there are two reaction forces in the x_1-direction (at $x_1 = 0$ and at $x_1 = L$) but only one useful equilibrium equation (force balance in the x_1-direction) so the reaction forces cannot be determined from static equilibrium alone.

3.2 (3 points) State the equation governing the axial displacement $\bar{u}_1(x_1)$ of the rod.

Solution:
The general expression for the axial force $N_1(x_1)$ in a rod featuring a uniform Young's modulus E and a uniform coefficient of thermal expansion α is:

$$N_1(x_1) = EA(x_1)\left[\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} - \alpha\Delta T(x_1)\right] \tag{16}$$

In this problem, the temperature change is a function of x_1 given by

$$\Delta T(x_1) = \Delta T_0 + kx_1. \tag{17}$$

The cross-sectional area of the rod is linear in x_1 here:

$$A(x_1) = \pi(r(x_1))^2 = \pi R^2\left(4 - 3\frac{x_1}{L}\right) \tag{18}$$

The balance equation for the axial force $N_1(x_1)$ is

$$\frac{\mathrm{d}N_1}{\mathrm{d}x_1} + p_1(x_1) = 0 \tag{19}$$

where the distributed axial load $p_1(x_1)$ is zero here since no weight or other distributed loads were mentioned in the problem statement. It therefore follows that the axial force must be constant,

$$N_1(x_1) = N_0, \tag{20}$$

where N_0 is the yet unknown constant axial force. Inserting Eq. (16) into Eq. (20) yields the equation

$$EA(x_1)\left[\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} - \alpha\Delta T(x_1)\right] = N_0, \tag{21}$$

or inserting the expressions for $A(x_1)$ and $\Delta T(x_1)$,

$$E\pi R^2\left(4 - 3\frac{x_1}{L}\right)\left[\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} - \alpha\left(\Delta T_0 + kx_1\right)\right] = N_0. \tag{22}$$

The above equation can be readily solved for $\bar{u}_1(x_1)$ and N_0. Strictly speaking, however, it is not the governing equation for $\bar{u}_1(x_1)$ (alone) since it also contains the unknown axial force N_0. One could eliminate the latter by differentiating Eq. (21) with respect to x_1:

$$E\frac{\mathrm{d}A}{\mathrm{d}x_1}\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} - E\frac{\mathrm{d}A}{\mathrm{d}x_1}\alpha\Delta T(x_1) + EA(x_1)\frac{\mathrm{d}^2\bar{u}_1}{\mathrm{d}x_1^2} - EA(x_1)\alpha\frac{\mathrm{d}\Delta T}{\mathrm{d}x_1} = 0 \tag{23}$$

Inserting the expressions for $\Delta T(x_1)$ and $A(x_1)$ into Eq. (23) and rearranging the terms would then yield the equation governing the displacement $u_1(x_1)$ of the considered rod alone:

$$\left(\frac{4L}{3} - x_1\right)\frac{\mathrm{d}^2\bar{u}_1}{\mathrm{d}x_1^2} - \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} = \alpha k\left(\frac{4L}{3} - x_1\right) - \alpha\left(\Delta T_0 + kx_1\right) \tag{24}$$

3.3 (1 point) State the boundary conditions at $x_1 = 0$ and $x_1 = L$.

Solution:
The boundary conditions are:

$$\bar{u}_1(x_1 = 0) = 0 \tag{25}$$
$$\bar{u}_1(x_1 = L) = 0 \tag{26}$$

3.4 (2 points) Determine the general solution to your governing equation. (You can use mathematical software like Mathematica or MATLAB to do this if you wish.)

Solution:
The second-order ODE in Eq. (24) can be solved analytically or using mathematical software for the displacement field $\bar{u}_1(x_1)$. However, it is much easier to solve the first-order ODE in Eq. (22). Rearranging that equation immediately yields an expression for $\mathrm{d}\bar{u}_1/\mathrm{d}x_1$,

$$\frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} = \frac{N_0}{E\pi R^2\left(4 - 3\frac{x_1}{L}\right)} + \alpha\left(\Delta T_0 + kx_1\right), \tag{27}$$

in which N_0 is a yet unknown constant. Integrating that expression once with respect to x_1 yields

$$\bar{u}_1(x_1) = -\frac{L}{3}\frac{N_0}{E\pi R^2}\ln\left(4 - 3\frac{x_1}{L}\right) + \alpha x_1\left(\Delta T_0 + k\frac{x_1}{2}\right) + C \tag{28}$$

where C is a second unknown constant. Both N_0 and C will be determined from the boundary conditions in the next part.
As mentioned, the solution to Eq. (24) could have also been obtained using mathematical software such as the Symbolic Math Toolbox in MATLAB. This could have been done in the following way:

```
syms u(x) DeltaT0 k alpha L;

eqn = (4*L/3-x)*diff(diff(u,x),x)-diff(u,x) ...
      == ...
      (alpha*k)*(4*L/3-x) - alpha*(DeltaT0+k*x);

dsolve(eqn)
```

3.5 (2 points) Find the displacement field $\bar{u}_1(x_1)$ in the rod by specializing the general solution you found in the previous part to the boundary conditions.

Solution: Inserting the general solution (28) into the two boundary conditions stated above yields the values for the unknown constants N_0 and C:

$$\begin{aligned} N_0 &= -\frac{3\pi}{2\ln(4)} R^2 \alpha E \left(kL + 2\Delta T_0\right) \\ C &= -\frac{1}{2}\alpha L \left(kL + 2\Delta T_0\right) \end{aligned}$$

Therefore, the complete solution to the displacement $\bar{u}_1(x_1)$ is:

$$\begin{aligned} \bar{u}_1(x_1) &= \frac{\alpha L}{2}\left(kL + 2\Delta T_0\right)\left[\frac{1}{\ln(4)}\ln\left(4 - \frac{3x_1}{L}\right) - 1\right] + \frac{\alpha x_1}{2}\left(kx_1 + 2\Delta T_0\right) \\ &= \frac{\alpha L}{2\ln(4)}\left(kL + 2\Delta T_0\right)\ln\left(1 - \frac{3x_1}{4L}\right) + \frac{\alpha x_1}{2}\left(kx_1 + 2\Delta T_0\right) \end{aligned} \tag{29}$$

3.6 (3 points) Compute the axial force field $N_1(x_1)$, the stress field $\sigma_{11}(x_1)$, and the strain field $\varepsilon_{11}(x_1)$ in the rod.

Solution:
Strain field $\varepsilon_{11}(x_1)$:

$$\varepsilon_{11}(x_1) = \frac{\mathrm{d}\bar{u}_1}{\mathrm{d}x_1} = \alpha\left(\Delta T_0 + kx_1\right) - \frac{\alpha(kL + 2\Delta T_0)}{2\ln(4)}\frac{1}{\frac{4}{3} - \frac{x_1}{L}} \tag{30}$$

Axial force field $N_1(x_1)$:

$$N_1(x_1) = N_0 = -\frac{3\pi}{2\ln(4)}R^2\alpha E\left(kL + 2\Delta T_0\right) = \text{const.} \tag{31}$$

Respective stress field $\sigma_{11}(x_1)$:

$$\sigma_{11}(x_1) = \frac{N_1(x_1)}{A(x_1)} = -\frac{1}{2\ln(4)}\alpha E\left(kL + 2\Delta T_0\right)\frac{1}{\frac{4}{3} - \frac{x_1}{L}} \tag{32}$$